Lec 1 Friday Sept 2

Fall 2011

Math 846 Topics in Combinatorics B105 VV

11:00 AM MWF

Theme: the Double Affine Hecke algebra "DAHA"

- Defined by Cherednik in 1992
- Related to a class of multivariable orthogonal polynomials called the MacDonald/Koornwinder polys.
- In the rank 1 case these are the Askey-Wilson polynomials

Strategy

I Rank 1 case

- investigate structure of the algebra: automorphisms, basis, center, ...
- representation theory
- connection to AW polynomials

II Rank n case

Work thru book

MacDonald: Affine Hecke algebras and orthogonal polynomials

Cambridge U. press 2003

Summary of topics covered (written Dec 16 2011)

We investigated the double affine Hecke algebra (DAHA) of type $(C_1^\vee, C_1)$. This algebra is denoted $\hat{H}_q$

$\hat{H}_q$ is defined by generators $\{t_i^{\pm 1}\}_{i=0}^{3}$ and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \qquad i = 0,1,2,3$$

$$t_i + t_i^{-1} \text{ is central}$$

$$t_0 t_1 t_2 t_3 = q^{-1}$$

Part I Ring theory of $\hat{H}_q$

Basic facts

Automorphisms and anti automorphisms

An action of the braid group B_3 on $\hat{H}_q$

The elements $X = t_3 t_0$ and $Y = t_0 t_1$

The element $A = t_0 t_1 + (t_0 t_1)^{-1}$, $B = t_0 t_3 + (t_0 t_3)^{-1}$

$C = t_0 t_2 + (t_0 t_2)^{-1}$

A, B, C satisfy the $\mathbb{Z}_3$-symmetric Askey-Wilson relations

The universal Askey-Wilson algebra Δ_q

A homomorphism $\Delta_q \to \hat{H}_q$

A linear basis for $\hat{H}_q$

A presentation of $\hat{H}_q$ by gens and relations that involves X, Y

A linear basis for Δ_q

The homomorphism $\Delta_q \to \hat{H}_q$ is injective

the spaces
$$\hat{H}_q^+ = \{ h \in \hat{H}_q \mid ht_0 = t_0 h \},$$
$$\hat{H}_q^- = \{ h \in \hat{H}_q \mid ht_0 = t_0^{-1} h \}$$

A linear basis for $\hat{H}_q^+$

A presentation of $\hat{H}_q^+$ by generators and relations

A linear basis for $\hat{H}_q^-$

the center of $\hat{H}_q$

Some 2-sided ideals of $\hat{H}_q$ and $\hat{H}_q^+$

Part II Representation theory of $\hat{H}_q$

Basic facts

The elements $G_0 = t_0 - t_3 t_0 t_3^{-1}$ $\quad G_1 = t_1 - t_0 t_1 t_0^{-1}$

$G_2 = t_2 - t_1 t_2 t_1^{-1}$ $\quad G_3 = t_3 - t_2 t_3 t_2^{-1}$

How G_0, G_2 swap eigenspaces of X and G_1, G_3 swap eigenspaces of Y

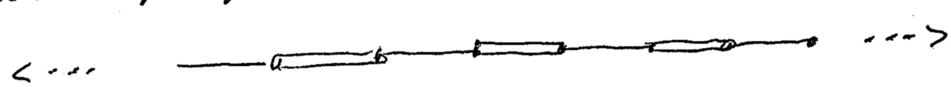
Description of an $\hat{H}_q$-module; the staircase picture

the actions of $\{t_i^{\pm 1}\}_{i=0}^{3}$ on the eigenspaces of X and Y

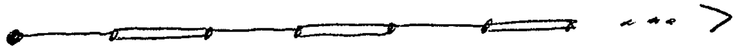
the action of X, Y on each others eigenspaces

the X-diagram and the Y-diagram

Description of an irreducible $\hat{H}_q$-module whose X-diagram is a doubly infinite path

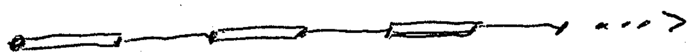


Description of an irreducible $\hat{H}_q$-module whose X-diagram is a semi-infinite path



Connection to the Askey-Wilson polynomials

Description of an irreducible $\hat{H}_q$-module whose X-diagram is a semi-infinite path



Connection to the Askey-Wilson polynomials

The Askey-Wilson polynomials: 3-term recurrence, parameter arrays, the Askey-Wilson q-difference operator, the Askey-Wilson relations, the split basis

An $\hat{H}_q$-module structure on the Laurent polynomials $\mathbb{F}[y, y^{-1}]$

A linear basis for $\mathbb{F}[y, y^{-1}]$ that makes X upper triangular and Y lower triangular

The actions of $\{t_i^{\pm 1}\}_{i=0}^{3}$ on the above basis

The elements X, Y satisfy the nonsymmetric tri-diagonal relations

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I: Rank 1 DAHA

Conventions

- An algebra is meant to be associative and have a 1
- A subalgebra has same 1 as parent algebra
- Fix a field $\mathbb{F}$
- Fix $0 \neq q \in \mathbb{F}$

[Often we will restrict to case $q^4 \neq 1$]

Def 1 Let $\hat{H}_q$ denote the $\mathbb{F}$-algebra defined by generators

$\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ $\qquad \mathbb{I} = \{0, 1, 2, 3\}$

and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \qquad i \in \mathbb{I} \qquad (01)$$

$$t_i + t_i^{-1} \text{ is central} \qquad \text{''} \qquad (02)$$

$$t_0 t_1 t_2 t_3 = q^{-1} \qquad (03)$$

We call $\hat{H}_q$ the <u>universal</u> <u>DAHA</u> <u>of type</u> <u>$(C_1^\vee, C_1)$</u>

$\hat{H}_q$ is our main object of study

Related algebras

Def 2 Fix nonzero $k_i \in \mathbb{F}$ $i \in \mathbb{I}$

let $H(k_0, k_1, k_2, k_3; q)$ denote the $\mathbb{F}$-algebra defined

by generators

$$\{t_i\}_{i \in \mathbb{I}}$$

and rels

$$(t_i - k_i)(t_i - k_i^{-1}) = 0 \qquad i \in \mathbb{I} \qquad (*)$$

$$t_0 t_1 t_2 t_3 = q^{-1} \qquad (**)$$

This is the (ordinary) DAHA of type $(C_1^\vee, C_1)$

LEM 3 $\exists$ unique $\mathbb{F}$-algebra hom $\hat{H}_q \to H(k_0, k_1, k_2, k_3; q)$

that sends "evaluation homomorphism"

$$t_i \longrightarrow t_i \qquad i \in \mathbb{I}$$

This hom is surjective.

pf (*), (**) imply each gen t_i of $H(k_0, k_1, k_2, k_3; q)$ is invertible and

$$t_i + t_i^{-1} = k_i + k_i^{-1}$$

is central

$\square$

Def 4 Let $\hat{H}$ denote the $\mathbb{F}$-algebra defined

by gens

$$\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$$

and rels

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \qquad i \in \mathbb{I}$$

$$t_i + t_i^{-1} \quad \text{central} \qquad "$$

$$t_0 t_1 t_2 t_3 \quad \text{central}$$

In an earlier paper I called $\hat{H}$ the universal DAHA of type $(C_1^\vee, C_1)$

LEM 5 $\exists$ unique $\mathbb{F}$-algebra hom $\hat{H} \to \hat{H}_q$

that sends

$$t_i \to t_i \qquad i \in \mathbb{I}$$

this hom is surj. $\square$

pf clear

I mention $\hat{H}$ for completeness, our focus will be $\hat{H}_q$

An $\hat{H}_q$-module

To motivate $\hat{H}_q$ we display a $\hat{H}_q$-module

let λ = indet

$\mathbb{F}[\lambda]$ = $\mathbb{F}$-algebra of polys in λ that have all coefs in $\mathbb{F}$

$\mathbb{F}[\lambda, \lambda^{-1}]$ = -- Laurent polys ...

LEM 6 Assume char $\mathbb{F} \neq 2$, $\exists \mathring{\imath} \in \mathbb{F}$ s.t. $\mathring{\imath}^2 = -1$

Assume q not a root of 1.

Then $\mathbb{F}[\lambda, \lambda^{-1}]$ is an $\hat{H}_q$-module with

$$t_0 . \lambda^i = \mathring{\imath} \lambda^{-i} \qquad i \in \mathbb{Z} \text{ (integers)}$$

$$t_1 . \lambda^i = -\mathring{\imath} q^{-2i} \lambda^{-i} \qquad ''$$

$$t_2 . \lambda^i = \mathring{\imath} q^{1-2i} \lambda^{1-i} \qquad ''$$

$$t_3 . \lambda^i = -\mathring{\imath} \lambda^{1-i}$$

On this module

$$t_j + t_j^{-1} = 0 \qquad j \in \mathbb{I}$$

pf One checks that for $j \in \mathbb{I}$

t_j^2 acts on $\mathbb{F}[\lambda, \lambda^{-1}]$ as -1

so $t_j + t_j^{-1} = 0$

Check $t_0 t_1 t_2 t_3 = q^{-1}$:

$$\lambda^i \xrightarrow[t_3]{} -\mathring{\imath} \lambda^{1-i} \xrightarrow[t_2]{} q^{2i-1} \lambda^i \xrightarrow[t_1]{} -\mathring{\imath} q^{-1} \lambda^{-i} \xrightarrow[t_0]{} q^{-1} \lambda^i$$

$\square$

Comments on LEM 6

Define $\quad X = t_3 t_0 \qquad Y = t_0 t_1$

Obs $\hat{H}_q$ is gen by $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$

Obs
$$X . \lambda^i = \lambda^{i+1} \qquad i \in \mathbb{Z}$$
$$Y . \lambda^i = q^{-2i} \lambda^i \qquad ''$$
$$t_0 . \lambda^i = i^{\circ} \lambda^{-i}$$

X: shift right $\Rightarrow$

Y: the λ^i are eigenvectors

t_0: swap $\Leftrightarrow$

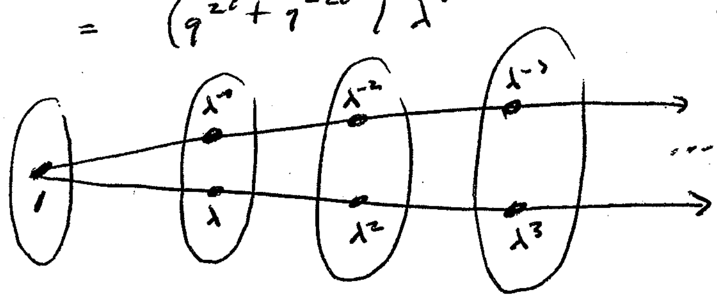
Consider $\quad X + X^{-1}, \qquad Y + Y^{-1}$

$$(X+X^{-1}) . \lambda^i = \lambda^{i-1} + \lambda^{i+1} \qquad i \in \mathbb{Z}$$

"adjacency operator" for ∞ path

$$(Y+Y^{-1}) . \lambda^i = (q^{2i} + q^{-2i}) \lambda^i \qquad i \in \mathbb{Z}$$

eigenspaces for $Y+Y^{-1}$:



eigenvalues of $Y+Y^{-1}$: $\quad 2 \qquad q^2+q^{-2} \qquad q^4+q^{-4} \qquad q^6+q^{-6} \qquad \cdots$

Let $T =$ subalgebra of $\hat{H}_q$ gen by

$$x + x^{-1}, \qquad y + y^{-1}$$

T acts on $V = \mathbb{F}[\lambda, \lambda^{-1}]$ as "subconstituent algebra" of ∞ path

Interpret t_0:

Obs t_0 commutes with $x + x^{-1}$, $y + y^{-1}$

$t_0^2 = -1$ on V so eigvals are $i, -i$

Find eigenspaces

Define $V_0 =$ subspace of V with basis

$$1, \ \lambda + \lambda^{-1}, \ \lambda^2 + \lambda^{-2}, \ \cdots \qquad *$$

Define $V_1 =$ subspace of V with basis

$$\lambda - \lambda^{-1}, \ \lambda^2 - \lambda^{-2}, \ \cdots \qquad **$$

Then

$$V = V_0 + V_1 \qquad (\text{dir sum})$$

$V_0 =$ eigenspace for t_0 with eigval i

$V_1 = \quad \ldots \quad -i$

Each of V_0, V_1 is irred. T-submodule of V

Call V_0 the <u>primary</u> T-submodule

Obs

$\dfrac{1-i t_0}{2}$ acts on V_0 as 1 and on V_1 as 0

$\dfrac{1+i t_0}{2}$ .. V_0 -- 0 -- V_1 -- 1

So $\dfrac{1 \pm i t_0}{2}$ act on V as the central idempotents for T.

Rel basis $(*)$ the matrix rep $X+X^{-1}$, $Y+Y^{-1}$ is

$$X+X^{-1}: \quad \left[\begin{array}{cccccc} 0 & 2 & & & & \\ 1 & 0 & 1 & & 0 & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & \ddots & \\ & 0 & & & & \ddots \end{array}\right] \quad \begin{pmatrix}\text{irred} \\ \text{tridiag}\end{pmatrix}$$

$$Y+Y^{-1}: \quad \mathrm{diag}(2, q^2+q^{-2}, q^4+q^{-4}, \ldots)$$

Above tridiag matrix describes the 3-term rec for a sequence of polynomials $\{p_i\}_{i=0}^{\infty}$:

$$p_0 = 1 \qquad\qquad p_1 = \lambda$$

$$\lambda p_1 = p_2 + 2p_0$$

$$\lambda p_i = p_{i+1} + p_{i-1} \qquad\qquad i = 2, 3 \ldots$$

One checks

$$p_i(X+X^{-1})\, v_0 = v_i \qquad\qquad i = 0, 1, 2 \ldots$$

where $\quad v_0 = 1 \qquad v_i = \lambda^i + \lambda^{-i} \qquad 1 \le i < \infty$

.... the Chebyshev polynomial of 1st kind.

$\mathbb{F}$ arb Fix $0 \neq q \in \mathbb{F}$

Recall $\hat{H}_q$ is $\mathbb{F}$-algebra defined by gens $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \qquad i \in \mathbb{I} \qquad D1$$

$$t_i + t_i^{-1} \text{ central} \qquad \text{"} \qquad D2$$

$$t_0 t_1 t_2 t_3 = q^{-1} \qquad D3$$

Obs q^{-1} is equal to each of

$t_0 t_1 t_2 t_3$, $t_1 t_2 t_3 t_0$, $t_2 t_3 t_0 t_1$, $t_3 t_0 t_1 t_2$

Aut / Antiauts of $\hat{H}_q$

Let A denote an $\mathbb{F}$-algebra

By an <u>automorphism</u> of A we mean an $\mathbb{F}$-alg iso $A \to A$

Set of auts of A form a group under composition, called $\mathrm{Aut}(A)$

LEM 7 $\exists$ automorphism of $\hat{H}_q$ that sends

$$t_0 \to t_1 \to t_2 \to t_3 \to t_0 \qquad \text{"}\mathbb{Z}_4\text{-symmetry"}$$

pf clear

LEM 8 Put

$$\varepsilon_i \in \{1, -1\} \qquad i \in I$$

such that

$$\varepsilon_0 \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$$

Then $\exists$ automorphism of $\hat{H}_q$ that sends

$$t_i \longrightarrow \varepsilon_i t_i \qquad i \in I$$

"$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ - sym"

pf clear.

We now consider some auts that are not so trivial

The Braid group B_3 is defined by generators ρ, σ

and rels $\rho^3 = \sigma^2$.

For convenience put

$$\tau = \rho^3 = \sigma^2$$

Aside: B_3 has another presentation by gens

u, v and rels

$$uvu = vuv$$

[As an ex, find the iso between the two presentations.]

LEM 9 $\quad B_3$ acts on $\hat{H}_q$ as a gp of auts such that

$$\tau(h) = t_0^{-1} h t_0 \qquad \forall h \in \hat{H}_q$$

and ρ, σ do the following:

h	t_0	t_1	t_2	t_3
$\rho(h)$	t_0	$t_0^{-1} t_3 t_0$	t_1	t_2
$\sigma(h)$	t_0	$t_0^{-1} t_3 t_0$	$t_1 t_2 t_1^{-1}$	t_1

pf $\quad \exists$ aut T of $\hat{H}_q$ that sends

$$h \longrightarrow t_0^{-1} h t_0 \qquad \forall h \in \hat{H}_q$$

Define

$$t_0^\vee = t_0 \qquad t_1^\vee = t_0^{-1} t_3 t_0 \qquad t_2^\vee = t_1 \qquad t_3^\vee = t_2$$

Show $t_0^\vee, t_1^\vee, t_2^\vee, t_3^\vee$ satisfy defining rels for $\hat{H}_q$

D1 ✓

D2:

$$t_0^\vee + t_0^{\vee -1} = t_0 + t_0^{-1} \quad \text{is central}$$

$$t_1^\vee + t_1^{\vee -1} = t_0^{-1}(t_3 + t_3^{-1}) t_0$$

$$= t_3 + t_3^{-1}$$

is central

$$t_2^\vee + t_2^{\vee -1} = t_1 + t_1^{-1}$$

is central

$t_3^\vee + t_3^{\vee -1} = t_2 + t_2^{-1}$ is central

D3 : $t_0^\vee t_1^\vee t_2^\vee t_3^\vee = t_0 \, t_0^{-1} t_3 t_0 \, t_1 t_2$

$= t_3 t_0 t_1 t_2$

$= q^{-1}$ ✓

Now $\exists$ $\mathbb{F}$-alg hom $\rho : \hat{H}_q \to \hat{H}_q$ that sends

$t_i \to t_i^\vee \qquad i \in \mathbb{I}$

Show $\rho^3 = T$

To do this show ρ^3, T agree at t_i for $i \in \mathbb{I}$

t_0 fixed by T

t_0 fixed by ρ hence ρ^3

ρ sends

$t_3 \to t_2 \to t_1 \to t_0^{-1} t_3 t_0 \to t_0^{-1} t_2 t_0 \to t_0^{-1} t_1 t_0$

ρ^3 sends

$t_i \to t_0^{-1} t_i t_0 \qquad i = 1, 2, 3$

So ρ^3, T agree at t_1, t_2, t_3

Now $\rho^3 = T$

T^{-1} exists $\Rightarrow$ ρ^{-1} exists $\Rightarrow$ ρ is aut of $\hat{H}_q$

Define

$$\hat{t}_0 = t_0, \qquad \hat{t}_1 = t_0^{-1} t_3 t_0, \qquad \hat{t}_2 = t_1 t_2 t_1^{-1}, \quad \hat{t}_3 = t_1$$

Show $\hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3$ sat defining rels for $\hat{H}_q$

D1 ✓

D2: $\hat{t}_0 + \hat{t}_0^{-1} = t_0 + t_0^{-1}$ $\qquad$ $\hat{t}_1 + \hat{t}_1^{-1} = t_0^{-1}(t_3 + t_3^{-1}) t_0 = t_3 + t_3^{-1}$

$\hat{t}_2 + \hat{t}_2^{-1} = t_1(t_2 + t_2^{-1}) t_1^{-1} = t_2 + t_2^{-1}$ $\qquad$ $\hat{t}_3 + \hat{t}_3^{-1} = t_1 + t_1^{-1}$

all central ✓

D3:

$$\hat{t}_0 \hat{t}_1 \hat{t}_2 \hat{t}_3 = \cancel{t_0}\, \cancel{t_0^{-1}} t_3 t_0\, t_1 t_2 \cancel{t_1^{-1}}\, \cancel{t_1}$$

$$= t_3 t_0 t_1 t_2$$

$$= q^{-1} \quad ✓$$

Now $\exists$ $\mathbb{F}$-alg hom $S : \hat{H}_q \to \hat{H}_q$ that sends

$$t_i \to \hat{t}_i \qquad \forall i \in \mathbb{I}$$

show $S^2 = T$

t_0 fixed by T, S^2

$$t_1 \xrightarrow[S]{} t_0^{-1} t_3 t_0 \xrightarrow[S]{} t_0^{-1} t_1 t_0$$

S^2, T agree at t_1 ✓

$$t_3 \xrightarrow[S]{} t_1 \xrightarrow[S]{} t_0^{-1} t_3 t_0$$

S^2, T agree at t_3 ✓

$$t_2 \xrightarrow[S]{} t_1 t_2 t_1^{-1} \xrightarrow[S]{} \underbrace{t_0^{-1} \underbrace{t_3 t_0 t_1 t_2}_{q^{-1}} \underbrace{t_1^{-1} t_0^{-1} t_3^{-1}}_{= t_1^{-1} t_0^{-1} t_3^{-1} t_2^{-1} t_2 = (t_2 t_3 t_0 t_1)^{-1} t_2 = q t_2} t_0}_{= t_0^{-1} t_2 t_0}$$

Shows S^2, T agree at t_2 ✓

Now $S^2 = T$

Now S^{-1} exists $\Rightarrow$ S is aut of $\hat{H}_q$

Result follows. $\square$

We record a fact from L9 and its pf.

LEM 10 For the B_3-action on $\hat{H}_q$ from L9,

τ fixes every central element

and ρ, σ do the following

h	$t_0+t_0^{-1}$	$t_1+t_1^{-1}$	$t_2+t_2^{-1}$	$t_3+t_3^{-1}$
$\rho(h)$	$t_0+t_0^{-1}$	$t_3+t_3^{-1}$	$t_1+t_1^{-1}$	$t_2+t_2^{-1}$
$\sigma(h)$	$t_0+t_0^{-1}$	$t_3+t_3^{-1}$	$t_2+t_2^{-1}$	$t_1+t_1^{-1}$

pf clear from pf of L9. $\square$

let $\mathcal{A}$ denote an $\mathbb{F}$-algebra

By an <u>anti automorphism</u> of $\mathcal{A}$ we mean an iso of $\mathbb{F}$-vector spaces $\gamma : \mathcal{A} \to \mathcal{A}$ s.t.

$$\gamma(ab) = \gamma(b)\gamma(a) \qquad \forall a, b \in \mathcal{A}$$

[For example the transpose map is an antiaut of the matrix algebra $\mathrm{Mat}_{n \times n}(\mathbb{F})$ $n = 1, 2, \ldots$]

Another view:

Define $\mathcal{A}^{op}$ to be the $\mathbb{F}$-vector space $\mathcal{A}$ together with mult

$$\underset{(\text{in } \mathcal{A}^{op})}{a\,b} = \underset{(\text{in } \mathcal{A})}{b\,a} \qquad \forall a, b \in \mathcal{A}$$

Then $\mathcal{A}^{op}$ is an $\mathbb{F}$-algebra.

For any map $\gamma : \mathcal{A} \to \mathcal{A}$ TFAE:

(i) γ is an antiaut of $\mathcal{A}$

(ii) γ is an $\mathbb{F}$-algebra iso $\mathcal{A} \to \mathcal{A}^{op}$

Under composition $\circ$ the auts / anti auts of A are related as follows

$\circ$	aut	antiaut
aut	aut	antiaut
antiaut	antiaut	aut

Set of auts and antiauts of A form a group under composition, called $AAut(A)$

Either

$$Aut(A) = AAut(A)$$

or

Aut(A) is a normal subgroup of AAut(A) that has index 2

LEM 11 $\exists$ antiaut $\dagger$ of $\hat{H}_q$ that sends

$$t_0 \to t_0, \quad t_1 \to t_3, \quad t_2 \to t_2, \quad t_3 \to t_1$$

Moreover $\dagger^2 = 1$.

pf Define some elts of $\hat{H}_q^{\,op}$:

$$t_0^\dagger = t_0 \qquad t_1^\dagger = t_3$$

$$t_2^\dagger = t_2 \qquad t_3^\dagger = t_1$$

Show $t_0^\dagger, t_1^\dagger, t_2^\dagger, t_3^\dagger$ sat def rels for $\hat{H}_q$ (in the alg $\hat{H}_q^{\,op}$)

D1 ✓

D2 ✓

D3: $\underset{(\text{in } \hat{H}_q^{op})}{t_0^\dagger t_1^\dagger t_2^\dagger t_3^\dagger} \overset{?}{=} q^{-1}$

$$\begin{aligned} LHS &= t_3^\dagger t_2^\dagger t_1^\dagger t_0^\dagger \quad (\text{in } \hat{H}_q) \\ &= t_1 t_2 t_3 t_0 \quad (\;\cdot\cdot\;) \\ &= q^{-1} \end{aligned}$$ ✓

So $\exists$ $\mathbb{F}$-alg hom $\dagger: \hat{H}_q \to \hat{H}_q^{\,op}$ that sends

$$t_i \to t_i^\dagger \qquad \forall i \in \mathbb{I}$$

By constr $\dagger^2 = 1$ so $\dagger$ is invertible hence bijection.

So $\dagger: \hat{H}_q \to \hat{H}_q^{\,op}$ is $\mathbb{F}$-alg iso, hence anti aut of $\hat{H}_q$ $\square$

LEM 12 $\exists$ iso of $\mathbb{F}$-algebras

$$\zeta : \hat{H}_q \longrightarrow \hat{H}_{q^{-1}}$$

that sends

$$t_0 \longrightarrow t_0^{-1} \qquad t_1 \to t_3^{-1}$$

$$t_2 \to t_2^{-1} \qquad t_3 \to t_1^{-1}$$

[caution: ζ does not send $q \to q^{-1}$. It fixes q along with every scalar]

pf (Write capital for elements in $\hat{H}_{q^{-1}}$)

So
$$T_0 T_1 T_2 T_3 = \varphi^{-1} = q \quad (\text{in } \hat{H}_{q^{-1}}) \qquad t_0 t_1 t_2 t_3 = q^{-1} \quad (\text{in } \hat{H}_q)$$

Define
$$t_0^\zeta = T_0^{-1} \qquad t_1^\zeta = T_3^{-1}$$

$$t_2^\zeta = T_2^{-1} \qquad t_3^\zeta = T_1^{-1}$$

show $t_0^\zeta, t_1^\zeta, t_2^\zeta, t_3^\zeta$ sat def rels for $\hat{H}_q$

D1 ✓

D2 ✓

D3:
$$t_0^\zeta t_1^\zeta t_2^\zeta t_3^\zeta \overset{?}{=} q^{-1}$$
$$\| $$
$$T_0^{-1} T_3^{-1} T_2^{-1} T_1^{-1} \qquad \text{ok.}$$
$$\| $$
$$(T_1 T_2 T_3 T_0)^{-1}$$
$$\| $$
$$q^{-1}$$

So $\exists$ $\mathbb{F}$-alg hom $\zeta : \hat{H}_q \to \hat{H}_{q^{-1}}$ that sends

$$t_i \to t_i^\zeta \quad \text{for } i \in \mathbb{I}$$

Obs ζ is bij, hence iso. $\square$

The elements X, Y

DEF 13 Let X, Y denote the following elements of $\hat{H}_q$:

$$X = t_3 t_0 \qquad\qquad Y = t_0 t_1$$

Obs X^{-1}, Y^{-1} exist.

LEM 14 We have

$$t_1 = t_0^{-1} Y,$$

$$t_2 = q^{-1} Y^{-1} t_0 X^{-1}$$

$$t_3 = X t_0^{-1}$$

Moreover $\hat{H}_q$ is generated by $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$

pf ex □

LEM 15 Recall aut of $\hat{H}_q$ from L 7 ($\mathbb{Z}_4$-sym)

this aut sends

$$X \to Y \to q^{-1}X^{-1} \to q^{-1}Y^{-1} \to X$$

pf

$$X = t_3 t_0 \to t_0 t_1 = Y$$

$$t_0 t_1 \to t_1 t_2 = \underbrace{t_1 t_2 t_3 t_0}_{q^{-1}} \underbrace{(t_3 t_0)^{-1}}_{X^{-1}}$$

$\square$

LEM 16 For the B_3 action on $\hat{H}_q$ the gens ρ, σ act on X, Y as follows.

σ sends

$$X \to t_0^{-1} Y t_0$$

$$Y \to X$$

ρ sends

$$X \to q^{-1} Y^{-1} t_0 X^{-1} t_0$$

$$Y \to X$$

pf routine

$\square$

LEM 17 The antiaut $\dagger$ of $\hat{H}_q$ sends

$$X \to Y$$
$$Y \to X$$

pf

$$\begin{array}{ccc} X & \quad & Y \\ \| & & \| \\ t_3 t_0 & & t_0 t_1 \\ \downarrow & & \downarrow \\ t_0 t_1 & & t_3 t_0 \\ \| & & \| \\ Y & & X \end{array}$$

LEM 18 The $\mathbb{F}$-alg iso $\gamma : \hat{H}_q \to \hat{H}_{q^{-1}}$ sends

$$X \to Y^{-1}$$
$$Y \to X^{-1}$$

pf

$$X = t_3 t_0 \to t_1^{-1} t_0^{-1} = Y^{-1}$$
$$Y = t_0 t_1 \to t_0^{-1} t_3^{-1} = X^{-1}$$

$\square$

LEC 3 Friday Sept 1 9/9/11

$\mathbb{F}$ arb $0 \neq q \in \mathbb{F}$

Continue to discuss univ DAHA $\hat{H}_q$ of type $(C_1^\vee, C_1)$

Recall

$$X = t_3 t_0 \qquad Y = t_0 t_1$$

We will consider

$$X + X^{-1}, \qquad Y + Y^{-1}$$

LEM 19 Let A denote any $\mathbb{F}$-algebra. Given invertible $u, v \in A$ s.t. each of

$$u + u^{-1}, \qquad v + v^{-1}$$

is central in A. Then

(i) $uv + (uv)^{-1} = vu + (vu)^{-1}$

(ii) $uv + (uv)^{-1}$ commutes with each of u, v

pf (i) Write

$$U = u + u^{-1} \qquad V = v + v^{-1} \qquad \text{(Central els)}$$

$$\begin{aligned} uv + (uv)^{-1} &= uv + (U-u)(V-v) \\ &= uv + vu - vU - uV + UV \end{aligned}$$

Sim

$$\begin{aligned} vu + (vu)^{-1} &= vu + (U-u)(V-v) \\ &= vu + uv - uV - Vu + UV \end{aligned}$$

(ii) $$u^{-1}\left(uv + v^{-1}u^{-1}\right)u = vu + u^{-1}v^{-1}$$
$$= uv + v^{-1}u^{-1} \quad \text{by (i)}$$

So $uv + v^{-1}u^{-1}$ commutes with u.

Case of v is sim. □

Back to $\hat{H}_q$

Cor 20 For dist $i, j \in I$

(i) $t_i t_j + (t_i t_j)^{-1} = t_j t_i + (t_j t_i)^{-1}$

(ii) $t_i t_j + (t_i t_j)^{-1}$ commutes with each of t_i, t_j.

Next goal: Find nice formula for

$$q Y^{\pm 1} X^{\pm 1} - q^{-1} X^{\pm 1} Y^{\pm 1}$$

DEF 21 Let

$$T_i = t_i + t_i^{-1} \qquad i \in \mathbb{I}$$

So T_i central in $\hat{H}_q$

Note: Often convenient to eliminate t_i^{-1} using Def 21.

LEM 22 The $\mathbb{F}$-algebra $\hat{H}_q$ has a presentation by generators

$$t_i, \quad T_i \qquad i \in \mathbb{I}$$

and relations

$$t_i^2 = t_i T_i - 1 \qquad i \in \mathbb{I}$$

$$T_i \text{ central}$$

$$t_0 t_1 t_2 t_3 = q^{-1}$$

pf ex

We mentioned

$$X^{\pm 1},\ Y^{\pm 1},\ t_0^{\pm 1}$$

generate $\hat{H}_q$. In terms of these generators the $\{T_i\}_{i \in \mathbb{I}}$ look as follows.

LEM 23

(i) $T_0 = t_0 + t_0^{-1}$

(ii) T_1 is equal to each of

$$t_0^{-1} Y + Y^{-1} t_0, \qquad Y t_0^{-1} + t_0 Y^{-1}$$

(iii) T_2 is equal to each of

$$q t_0^{-1} Y X + q^{-1} X^{-1} Y^{-1} t_0$$

$$q X t_0^{-1} Y + q^{-1} Y^{-1} t_0 X^{-1}$$

$$q Y X t_0^{-1} + q^{-1} t_0 X^{-1} Y^{-1}$$

(iv) T_3 is equal to each of

$$t_0^{-1} X + X^{-1} t_0, \qquad X t_0^{-1} + t_0 X^{-1}$$

pf (i) ✓

(ii)
$$t_0^{-1}Y + Y^{-1}t_0 = t_0^{-1}t_0t_1 + t_1^{-1}t_0^{-1}t_0$$
$$= t_1 + t_1^{-1}$$
$$= T_1$$

Also
$$T_1 = Y\,T_1\,Y^{-1}$$
$$= Y\left(t_0^{-1}Y + Y^{-1}t_0\right)Y^{-1}$$
$$= Yt_0^{-1} + t_0Y^{-1}$$

(iii) By L 14

$$qXt_0^{-1}Y + q^{-1}Y^{-1}t_0X^{-1} = t_2^{-1} + t_2$$
$$= T_2$$

Now
$$T_2 = Y\,T_2\,Y^{-1}$$
$$= Y\left(qXt_0^{-1}Y + q^{-1}Y^{-1}t_0X^{-1}\right)Y^{-1}$$
$$= q\,YXt_0^{-1} + q^{-1}t_0X^{-1}Y^{-1}$$

And
$$T_2 = X^{-1}\,T_2\,X$$
$$= X^{-1}\left(qXt_0^{-1}Y + q^{-1}Y^{-1}t_0X^{-1}\right)X$$
$$= q\,t_0^{-1}YX + q^{-1}X^{-1}Y^{-1}t_0$$

(iv): Sim to (ii)

$\square$

Since $T_0 = t_0 + t_0^{-1}$ the algebra $\hat{H}_q$ is gen by

$$X^{\pm 1}, \; Y^{\pm 1}, \; t_0, \; T_0 .$$

In terms of these gens the T_1, T_2, T_3 look as follows.

LEM 24

(i) T_1 is equal to each of

$$(T_0 - t_0) Y + Y^{-1} t_0, \qquad Y(T_0 - t_0) + t_0 Y^{-1}$$

(ii) T_2 is equal to each of

$$q(T_0 - t_0) YX + q^{-1} X^{-1} Y^{-1} t_0,$$

$$q X (T_0 - t_0) Y + q^{-1} Y^{-1} t_0 X^{-1},$$

$$q Y X (T_0 - t_0) + q^{-1} t_0 X^{-1} Y^{-1}$$

(iii) T_3 is equal to each of

$$(T_0 - t_0) X + X^{-1} t_0, \qquad X(T_0 - t_0) + t_0 X^{-1}$$

pf In Lem 23 elim t_0^{-1} using $t_0^{-1} = T_0 - t_0$ $\square$

We now show how the t_0 "commutes past" the $X^{\pm 1}$, $Y^{\pm 1}$

LEM 2.5

(i) $t_0 X = X^{-1} t_0 + X T_0 - T_3$

(ii) $t_0 X^{-1} = X t_0 - X T_0 + T_3$

(iii) $t_0 Y = Y^{-1} t_0 + Y T_0 - T_1$

(iv) $t_0 Y^{-1} = Y t_0 - Y T_0 + T_1$

pf (i), (ii) come from L24 (iii)

(iii), (iv) come from L24 (i) $\square$

Applying $\mathbb{Z}_4$-symmetry to L25

and using

$$X \to Y \to q^{-1}X^{-1} \to q^{-1}Y^{-1} \to X$$

we get equations that show how t_1, t_2, t_3

commute past $X^{\pm 1}, Y^{\pm 1}$. [I won't write down]

LEM 26 The following hold in $\hat{H}_q$:

(i) $t_0 t_2 = q^{-1} t_3^{-1} T_1 - q^{-1} Y X^{-1}$

(ii) $t_0^{-1} t_2^{-1} = q\, t_1 T_3 - q X^{-1} Y$

pf (i)
$$q^{-1} Y X^{-1} = q^{-1} t_0 t_1 t_0^{-1} t_3^{-1}$$
$$= t_0 t_1 \underbrace{t_0^{-1} t_3^{-1} t_3 t_0} t_1 t_2$$
$$= t_0 t_1^2 t_2$$
$$= t_0 (t_1 T_1 - 1) t_2$$
$$= t_0 t_1 t_2 \underset{t_3 t_3^{-1}}{\wedge} T_1 - t_0 t_2$$
$$= q^{-1} t_3^{-1} T_1 - t_0 t_2$$

(ii)
$$
\begin{aligned}
qX^{-1}Y &= q\, t_0^{-1} t_3^{-1}\, t_0 t_1 \\
&= t_0^{-1} t_3^{-1} \underbrace{t_0 t_1\; t_1^{-1} t_0^{-1}}\, t_1^{-1} t_2^{-1} \\
&= t_0^{-1} \left(t_3^{-1} T_3 - 1 \right) t_2^{-1} \\
&= t_0^{-1} t_3^{-1} t_2^{-1} \underset{t_1^{-1} t_1}{\wedge} T_3 - t_0^{-1} t_2^{-1} \\
&= q\, t_1\, T_3 - t_0^{-1} t_2^{-1}
\end{aligned}
$$
$\square$

Applying $\mathbb{Z}_4$-sym to L26 and using
$$X \to Y \to q^{-1}X^{-1} \to q^{-1}Y^{-1} \to X$$

We obtain

LEM 27

(i) $t_1 t_3 = q^{-1} t_0^{-1} T_2 - q^{-2} X^{-1} Y^{-1}$

(ii) $t_1^{-1} t_3^{-1} = q\, t_2 T_0 - Y^{-1} X^{-1}$

(iii) $t_2 t_0 = q^{-1} t_1^{-1} T_3 - q^{-1} Y^{-1} X$

(iv) $t_2^{-1} t_0^{-1} = q\, t_3 T_1 - q X Y^{-1}$

(v) $t_3 t_1 = q^{-1} t_2^{-1} T_0 - XY$

(vi) $t_3^{-1} t_1^{-1} = q\, t_0 T_2 - q^2 Y X$

DEF 28 let $\{C_i\}_{i \in \mathbb{I}}$ denote the following elements of $\hat{H}_q$:

$$C_0 = q\left(qYX - q^{-1}XY\right)$$

$$C_1 = -\left(q^{-1}YX^{-1} - qX^{-1}Y\right)$$

$$C_2 = q^{-1}\left(qY^{-1}X^{-1} - q^{-1}X^{-1}Y^{-1}\right)$$

$$C_3 = -\left(q^{-1}Y^{-1}X - qXY^{-1}\right)$$

LEM 29 Under $\mathbb{Z}_4$-symmetry

$$C_0 \to C_1 \to C_2 \to C_3 \to C_0$$

pf use

$$X \to Y \to q^{-1}X^{-1} \to q^{-1}Y^{-1} \to X$$ $\square$

Prop 30 The following hold in $\hat{H}_q$

	t_0T_2	t_1T_3	t_2T_0	t_3T_1	T_0T_2	T_1T_3
C_0	q	1	q^{-1}	1	$-q^{-1}$	-1
C_1	1	q	1	q^{-1}	-1	$-q^{-1}$
C_2	q^{-1}	1	q	1	$-q^{-1}$	-1
C_3	1	q^{-1}	1	q	-1	$-q^{-1}$

pf To get C_0 Compare LEM 27 (v), (vi)

Use
$$t_3^{-1}t_1^{-1} = (T_3-t_3)(T_1-t_1)$$
$$= T_1T_3 - t_1T_3 - t_3T_1 + t_3t_1$$

So
$$q t_0 T_2 - q^2 YX = T_1T_3 - t_1T_3 - t_3T_1 + q^{-1}\underbrace{t_2^{-1}}_{T_2-t_2} T_0 - XY$$

To get C_1, C_2, C_3 from C_0 apply $\mathbb{Z}_4$-symmetry $\square$

Next goal: Find B_3 action on

$X + X^{-1}$, $Y + Y^{-1}$.

DEF 31. Define

$$
\begin{aligned}
A &= Y + Y^{-1} \\
&= t_0 t_1 + (t_0 t_1)^{-1} \\
&= t_1 t_0 + (t_1 t_0)^{-1} \qquad \text{by Cor 20}
\end{aligned}
$$

$$
\begin{aligned}
B &= X + X^{-1} \\
&= t_3 t_0 + (t_3 t_0)^{-1} \\
&= t_0 t_3 + (t_0 t_3)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
C &= t_0 t_2 + (t_0 t_2)^{-1} \\
&= t_2 t_0 + (t_2 t_0)^{-1}
\end{aligned}
$$

$\mathbb{F}$ arb $\quad 0 \neq q \in \mathbb{F}$

Continue to discuss univ DAHA $\hat{H}_q$ of type $(C_1^\vee, C_1)$

Recall

$$\begin{aligned} A &= Y + Y^{-1} \\ &= t_0 t_1 + (t_0 t_1)^{-1} \\ &= t_1 t_0 + (t_1 t_0)^{-1} \end{aligned}$$

$$\begin{aligned} B &= X + X^{-1} \\ &= t_0 t_3 + (t_0 t_3)^{-1} \\ &= t_3 t_0 + (t_3 t_0)^{-1} \end{aligned}$$

$$\begin{aligned} C &= t_0 t_2 + (t_0 t_2)^{-1} \\ &= t_2 t_0 + (t_2 t_0)^{-1} \end{aligned}$$

Prop 32 $\quad$ the Braid group B_3 acts on A, B, C as follows:

(i) $\quad \tau$ fixes each of A, B, C

(ii) $\quad \rho$ sends $A \to B \to C \to A$

(iii) $\quad \sigma$ swaps $A \leftrightarrow B$ and sends $C \to C'$ where

$$\begin{aligned} qC + q^{-1} C' + AB &= q^{-1} C + q C' + BA \\ &= (q^{-1} t_0 + q t_0^{-1}) T_2 + T_1 T_3 \end{aligned}$$

pf

(i) Recall $\tau(h) = t_0^{-1} h t_0 \quad \forall h \in \hat{H}_q$

By Cor 20 (ii) t_0 commutes with each of A, B, C

(ii) Recall ρ fixes t_0 and sends

$$t_3 \to t_2 \to t_1 \to t_0^{-1} t_3 t_0$$

$$\rho \downarrow \qquad \begin{array}{ccc} t_0 t_1 + (t_0 t_1)^{-1} & & A \\ \downarrow & & \\ t_3 t_0 + (t_3 t_0)^{-1} & & B \\ \downarrow & & \\ t_2 t_0 + (t_2 t_0)^{-1} & & C \\ \downarrow & & \\ t_1 t_0 + (t_1 t_0)^{-1} & & A \end{array}$$

(iii) Recall σ:

h	t_0	t_1	t_2	t_3
$\sigma(h)$	t_0	$t_0^{-1} t_3 t_0$	$t_1 t_2 t_1^{-1}$	t_1

$$\sigma \downarrow \qquad \begin{array}{ccc} t_0 t_1 + (t_0 t_1)^{-1} & & A \\ \downarrow & & \\ t_3 t_0 + (t_3 t_0)^{-1} & & B \\ \downarrow & & \\ t_1 t_0 + (t_1 t_0)^{-1} & & A \end{array}$$

Define $C' = \sigma(C)$

Show

$$qC + q^{-1}C' + AB = (q^{-1}t_0 + qt_0^{-1})T_2 + T_1T_3 \qquad (*)$$

Obs

$$C' = \sigma(C)$$
$$= \sigma(t_0t_2) + \sigma(t_2^{-1}t_0^{-1})$$
$$\sigma(t_0t_2) = t_0t_1t_2t_1^{-1} = q^{-1}t_3^{-1}t_1^{-1}$$
$$= q^{-1}t_3^{-1}t_1^{-1} + q\,t_1t_3$$

To get $*$ show

$$q\,t_0t_2 + q\,t_2^{-1}t_0^{-1} + t_1t_3 + q^{-2}t_3^{-1}t_1^{-1}$$

$$+ \quad YX + YX^{-1} + Y^{-1}X + Y^{-1}X^{-1}$$

$$\overset{?}{=} \left(q^{-1}t_0 + q(T_0 - t_0)\right)T_2 + T_1T_3$$

To verify this, elim t_0t_2, $t_2^{-1}t_0^{-1}$, t_1t_3, $t_3^{-1}t_1^{-1}$ using LEM 26, 27 : simplify using Prop 30 (ex)

This gives $*$

We now show

$$q^{-1}C + qC' + BA = (q^{-1}t_0 + qt_0^{-1})\, T_2 + T_1T_3$$

To get this apply σ to each side of $*$

We saw σ swaps A, B and $C \to C'$

also $\sigma(C') = \sigma^2(C) = \tau(C) = C$

Also $\sigma T_2 = T_2$ $\quad \sigma T_1 = T_3$ $\quad \sigma T_3 = T_1$

$\square$

Thm 33 Assume $q^4 \neq 1$. Then

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{(q^{-1}t_0 + qt_0^{-1})T_1 + T_2T_3}{q+q^{-1}}$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{(q^{-1}t_0 + qt_0^{-1})T_3 + T_1T_2}{}$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{(q^{-1}t_0 + qt_0^{-1})T_2 + T_1T_3}{q+q^{-1}} \qquad \star$$

" $\mathbb{Z}_3$-symmetric Askey-Wilson relations "

pf: Get last equation $\star$

By Prop 32

$$qC + q^{-1}C' + AB = R$$
$$q^{-1}C + qC' + BA = R$$
$$R = (q^{-1}t_0 + qt_0^{-1})T_2 + T_1T_3$$

Elim C':

$$q(qC + AB - R) = q^{-1}(q^{-1}C + BA - R)$$

So

$$C(q^2 - q^{-2}) + qAB - q^{-1}BA = R(q - q^{-1})$$

So

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{R}{q+q^{-1}}$$

$\star$ proved

Apply ρ twice to $\star$ to get other two equations $\square$

COR 34 Assume $q^4 \neq 1$ Consider the subalgebra of $\hat{H}_q$ generated by A, B, C. In this subalgebra each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central.

pf By th 33 and since to commutes with A, B, C □

Motivated by Cor 34 we make a def.

DEF 35 Assume $q^4 \neq 1$. We define an $\mathbb{F}$-algebra Δ_q by generators and relations as follows.

The generators are A, B, C [view as abstract symbols, not as elts of $\hat{H}_q$]

Relations assert that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is central in Δ_q

Δ_q called the <u>Universal</u> <u>Askey-Wilson</u> <u>algebra</u>

We are going to show:

$\exists$ injection of $\mathbb{F}$-algebras

$$\Delta_q \longrightarrow \hat{H}_q$$

that sends

$$A \to t_0 t_1 + (t_0 t_1)^{-1}$$

$$B \to t_0 t_3 + (t_0 t_3)^{-1}$$

$$C \to t_0 t_2 + (t_0 t_2)^{-1}$$

In other words, the relations in Cor 34 are essentially the only ones relating A, B, C.

<u>Next immediate goal</u>: display a basis for $\mathbb{F}$-vectorspace $\hat{H}_q$

Going to show the following is a basis for $\hat{H}_q$:

$$Y^i X^j t_0^k T_1^r T_2^s T_3^t \qquad i, j, k \in \mathbb{Z}, \quad r, s, t \in \mathbb{N}$$

[$\mathbb{Z}$ integers
$\mathbb{N} = \{0, 1, 2 \ldots\}$]

The following is also a basis for $\hat{H}_q$:

$$Y^i X^j t_0^k T_0^l T_1^r T_2^s T_3^t \qquad i, j \in \mathbb{Z}, \ k \in \{0, 1\}, \ l, r, s, t \in \mathbb{N}$$

We will also obtain <u>reduction rules</u> that show how to write any given element of $\hat{H}_q$ as a linear comb of these basis vectors.

We mention some presentations for $\hat{H}_q$ that illuminate various aspects

LEM 36 The $\mathbb{F}$-algebra $\hat{H}_q$ is presented by generators and relations as follows.
The gens are $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$. The relations assert that each of

$$t_0 + t_0^{-1} \qquad t_0 X^{-1} + X t_0^{-1}$$

$$t_0 Y^{-1} + Y t_0^{-1} \qquad q^{-1} t_0 X^{-1} Y^{-1} + q Y X t_0^{-1}$$

is central and

$$X X^{-1} = 1, \quad X^{-1} X = 1, \quad Y Y^{-1} = 1, \quad Y^{-1} Y = 1$$

$$t_0 t_0^{-1} = 1, \quad t_0^{-1} t_0 = 1$$

pf let $\check{H}_q$ denote the $\mathbb{F}$-algebra with above presentation
show $\check{H}_q$ iso $\hat{H}_q$

claim: $\exists$ $\mathbb{F}$-alg hom

$$\check{H}_q \longrightarrow \hat{H}_q$$

$$t_0 \longrightarrow t_0$$

$$X \longrightarrow t_3 t_0$$

$$Y \longrightarrow t_0 t_1$$

pf cl the def rels for $\check{H}_q$ hold in $\hat{H}_q$ by LEM 23 claim $\exists$ $\mathbb{F}$-alg hom

$$\hat{H}_q \longrightarrow \check{H}_q$$

$$t_0 \longrightarrow t_0$$

$$t_1 \longrightarrow t_0^{-1} Y$$

$$t_2 \longrightarrow q^{-1} Y^{-1} t_0 X^{-1}$$

$$t_3 \longrightarrow X t_0^{-1}$$

pf cl show the def rels for $\hat{H}_q$ hold in $\check{H}_q$:

D1 ✓

D2 :

$t_0 + t_0^{-1}$ central in $\check{H}_q$? ✓

$t_0^{-1} Y + Y^{-1} t_0$ central in $\check{H}_q$?

$$\begin{aligned} &\parallel \\ &t_0^{-1}(t_0 Y^{-1} + Y t_0^{-1}) t_0 \\ &\parallel \\ &t_0 Y^{-1} + Y t_0^{-1} \\ &\text{"central" ✓} \end{aligned}$$

$\underbrace{q^{-1} Y^{-1} t_0 X^{-1} + q X t_0^{-1}}\, Y$ central in $\check{H}_q$?

$$\begin{aligned} &\parallel \\ &Y^{-1}\left(q^{-1} t_0 X^{-1} Y^{-1} + q\, Y X t_0^{-1}\right) Y \\ &\parallel \\ &q^{-1} t_0 X^{-1} Y^{-1} + q\, Y X t_0^{-1} \\ &\text{"central" ✓} \end{aligned}$$

$X t_0^{-1} + t_0 X^{-1}$ central in $\check{H}_q$? ✓

D3:

$$t_0(t_0^{-1}Y)(q^{-1}Y^{-1}t_0X^{-1})(Xt_0^{-1}) \overset{?}{=} q^{-1}$$

ok.

claim $\ast$, $\star$ are inverses:

$$\begin{array}{lllll}
\check{H}_q & \longrightarrow & \hat{H}_q & \longrightarrow & \check{H}_q \\
t_0 & \longrightarrow & t_0 & \longrightarrow & t_0 \quad ✓ \\
X & \longrightarrow & t_3t_0 & \longrightarrow & Xt_0^{-1}t_0 = X \quad ✓ \\
Y & \longrightarrow & t_0t_1 & \longrightarrow & t_0t_0^{-1}Y = Y \quad ✓
\end{array}$$

$$\begin{array}{lllll}
\hat{H}_q & \longrightarrow & \check{H}_q & \longrightarrow & \hat{H}_q \\
t_0 & \longrightarrow & t_0 & \longrightarrow & t_0 \\
t_1 & \longrightarrow & t_0^{-1}Y & \longrightarrow & t_0^{-1}t_0t_1 = t_1 \\
t_2 & \longrightarrow & q^{-1}Y^{-1}t_0X^{-1} & \longrightarrow & \underbrace{q^{-1}t_1^{-1}t_0^{-1}\cancel{t_0}\,\cancel{t_0^{-1}}t_3^{-1}} \\
 & & & & \quad \| \\
 & & & & q^{-1}\underbrace{t_1^{-1}t_0^{-1}t_3^{-1}t_2^{-1}}_{q}t_2 \quad = t_2 \quad ✓ \\
t_3 & \longrightarrow & Xt_0^{-1} & \longrightarrow & t_3t_0t_0^{-1} = t_3 \quad ✓
\end{array}$$

$\square$

$\mathbb{F}$ arb $\quad 0 \neq q \in \mathbb{F}$

Continue to discuss univ DAHA $\hat{H}_q$ of type $(C_1^\vee, C_1)$

Gens $\quad \{t_i^{\pm 1}\}_{i \in \mathbb{I}} \qquad \mathbb{I} = \{0,1,2,3\}$

Rels

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \qquad i \in \mathbb{I} \qquad (\rho 1)$$

$$t_i + t_i^{-1} \text{ is central} \qquad \text{"} \qquad (\rho 2)$$

$$t_0 t_1 t_2 t_3 = q^{-1} \qquad (\rho 3)$$

Write

$$T_i = t_i + t_i^{-1} \qquad i \in \mathbb{I}$$

$$X = t_3 t_0$$

$$Y = t_0 t_1$$

Next goal: show the following is a basis for $\mathbb{F}$-vector space $\hat{H}_q$:

$$Y^i X^j t_0^k \, T_1^r T_2^s T_3^t \qquad i,j,k \in \mathbb{Z}, \quad r,s,t \in \mathbb{N} = \{0,1,2,\ldots\} \qquad *$$

We will first show the following is a basis for $\hat{H}_q$

$$Y^i X^j t_0^k \, T_0^l T_1^r T_2^s T_3^t \qquad i,j \in \mathbb{Z}, \quad k \in \{0,1\}, \quad l,r,s,t \in \mathbb{N} \qquad **$$

Last time

In Lem 36 We saw a presentation of $\hat{H}_q$ involving the gens $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$

Here is a variation of that presentation

LEM 37 The $\mathbb{F}$-alg $\hat{H}_q$ has a presentation by gens $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}, \{T_i\}_{i \in \mathbb{I}}$

and rels

$$XX^{-1} = 1, \quad X^{-1}X = 1, \quad YY^{-1} = 1, \quad Y^{-1}Y = 1$$

$$t_0 t_0^{-1} = 1, \quad t_0^{-1} t_0 = 1$$

T_i central $\quad i \in \mathbb{I}$

$$T_0 = t_0 + t_0^{-1}$$

$$T_1 = t_0 Y^{-1} + Y t_0^{-1}$$

$$T_2 = q^{-1} t_0 X^{-1} Y^{-1} + q YX t_0^{-1}$$

$$T_3 = t_0 X^{-1} + X t_0^{-1}$$

pf use L36 $\square$

In Lem 37 lets remove t_0^{-1} using

$$t_0^{-1} = T_0 - t_0$$

this gives

LEM 38 The $\mathbb{F}$-alg $\hat{H}_q$ has a presentation

by generators

$$X^{\pm 1}, \quad Y^{\pm 1}, \quad t_0, \quad \{T_i\}_{i \in \mathbb{I}}$$

and rels

$$XX^{-1} = 1, \quad X^{-1}X = 1, \quad YY^{-1} = 1, \quad Y^{-1}Y = 1$$

$$T_i \text{ central} \qquad i \in \mathbb{I}$$

$$t_0^2 = t_0 T_0 - 1$$

$$T_1 = t_0 Y^{-1} + Y(T_0 - t_0)$$

$$T_2 = q^{-1} t_0 X^{-1} Y^{-1} + q YX(T_0 - t_0)$$

$$T_3 = t_0 X^{-1} + X(T_0 - t_0)$$

We now give one final pres of $\hat{H}_q$ by gens and rels. There are many rels.

These relations will become the "reduction rules" that allow us to write any given element of $\hat{H}_q$ in the basis $**$

We start by writing t_0, t_1, t_2, t_3 as a linear combination of terms of form $**$

The element t_0 is included in $**$ so focus on t_1, t_2, t_3

LEM 39 The following rels hold in $\hat{H}_q$:

$$t_1 = T_1 - Y^{-1} t_0$$

$$t_2 = q^{-1} Y^{-1} T_3 - q^{-1} Y^{-1} X T_0 + q^{-1} Y^{-1} X t_0$$

$$t_3 = X T_0 - X t_0$$

pf t_1: $T_1 - Y^{-1} t_0 = t_1 + t_1^{-1} - (t_0 t_1)^{-1} t_0$
$= t_1$ ✓

t_3: $X T_0 - X t_0 = X(T_0 - t_0)$
$= X t_0^{-1}$
$= t_3$ ✓

t_2: Recall

$$t_2 = q^{-1} Y^{-1} t_0 X^{-1}$$
$$= q^{-1} Y^{-1} \left(X t_0 - X T_0 + T_3 \right) \qquad \text{by } L25(ii)$$

$\square$

Thm 40. The $\mathbb{F}$-algebra $\hat{H}_q$ has a presentation by gens $X^{\pm 1}$, $Y^{\pm 1}$, t_0, $\{T_i\}_{i \in \mathbb{I}}$ and rels

$$\left.\begin{array}{l} XX^{-1}=1, \quad X^{-1}X=1, \quad YY^{-1}=1, \quad Y^{-1}Y=1 \\ T_i \text{ is central} \qquad i \in \mathbb{I} \\ t_0^2 = t_0 T_0 - 1 \end{array}\right\} \text{"tertiary"}$$

"secondary"

$$t_0 X = X^{-1} t_0 + X T_0 - T_3$$

$$t_0 X^{-1} = X t_0 - X T_0 + T_3$$

$$t_0 Y = Y^{-1} t_0 + Y T_0 - T_1$$

$$t_0 Y^{-1} = Y t_0 - Y T_0 + T_1$$

"primary"

$XY =$

	X^{-1}	1	X
Y^{-1}	0	$t_0 T_3 - q^{-2} T_0 T_3$	$q^{-2} T_0^2 - q^{-2} t_0 T_0$
1	0	$q^{-1} T_0 T_2 - q t_0 T_2$	$t_0 T_1 - T_0 T_1$
Y	0	0	q^2

$X^{-1}Y =$

	X^{-1}	1	X
Y^{-1}	0	$q^{-2}T_0T_3 - t_0T_3$	$q^{-2}t_0T_0 - q^{-2}T_0^2$
1	0	$(q-q^{-1})q^{-1}T_1T_3 - q^{-1}T_0T_2 + q^{-1}t_0T_2$	$q^{-2}T_0T_1 - q^{-2}t_0T_1$
Y	q^{-2}	0	0

$X^{-1}Y^{-1} =$

	X^{-1}	1	X
Y^{-1}	q^2	$q^2t_0T_3 - q^2T_0T_3$	$q^2T_0^2 - q^2t_0T_0$
1	0	$qT_0T_2 - qt_0T_2$	$q^2t_0T_1 - q^2T_0T_1$
Y	0	0	0

$XY^{-1} =$

	X^{-1}	1	X
Y^{-1}	0	$q^{-2}T_0T_3 - q^{-2}t_0T_3$	$q^{-2} - q^{-2}T_0^2 + q^{-2}t_0T_0$
1	0	$q^{-1}t_0T_2 - q^{-1}T_0T_2$	$T_0T_1 - t_0T_1$
Y	0	0	0

pf show all the rels in the theorem statement hold in $\hat{H}_q$:

primary: These are the rels in Prop 30, in disguise. In prop 30, elim t_1, t_2, t_3 using L39.

secondary: 1st is clear. Last 4 come from L25

tertiary: clear. ✓

Now show the rels in the theorem statement imply the defining rels for $\hat{H}_q$ given in LEM 38

All but one defining rel in LEM 38 already appears among the rels in thm statement.

Remaining rel is

$$T_2 = q^{-1} t_0 X^{-1} Y^{-1} + q Y X (T_0 - t_0)$$

To verify this, eval RHS using rels from the theorem statement as reduction rules

$$T_2 \overset{?}{=} q^{-1} t_0 X^{-1} Y^{-1} + q \underbrace{Y X (T_0 - t_0)}_{\text{proper form } \checkmark}$$

$$\underline{t_0} X^{-1} Y^{-1} = \left(X t_0 - X T_0 + T_3 \right) Y^{-1}$$

$$= X\, t_0 Y^{-1} - \underline{\underline{X Y^{-1}}}\, T_0 + \underbrace{Y^{-1} T_3}_{\checkmark}$$

elim this using
$XY^{-1} = \cdots$
red rule

$$X\, \underline{t_0 Y^{-1}} = X \left(Y t_0 - Y T_0 + T_1 \right)$$

$$= \underline{\underline{X Y}}\, t_0 - \underline{\underline{X Y}}\, T_0 + \underbrace{X T_1}_{\checkmark}$$

elim using
$XY = \cdots$
red rule

Get result after cancellation (ex) $\square$

DEF 41 The generators

$$X^{\pm 1}, \quad Y^{\pm 1}, \quad t_0, \quad \{T_i\}_{i \in \mathbb{I}}$$

for $\hat{H}_q$ are called <u>balanced</u>

Note 42 Ref to pres for $\hat{H}_q$ in Thm 40

Consider the rels that assert the $\{T_i\}_{i \in \mathbb{I}}$ are central

These rels can be expressed as

$$T_i X^{\pm 1} = X^{\pm 1} T_i \qquad i \in \mathbb{I}$$

$$T_i Y^{\pm 1} = Y^{\pm 1} T_i \qquad \text{''}$$

$$T_i t_0 = t_0 T_i \qquad \text{''}$$

$$T_i T_j = T_j T_i \qquad \forall\, i > j \qquad i, j \in \mathbb{I}$$

DEF 43 By a <u>reduction rule</u> in $\hat{H}_q$ we mean an equation that appears in Thm 40 or Note 42

3 kinds:
- primary
- secondary
- tertiary

DEF 44 For $n \in \mathbb{N}$ by a <u>word of length n</u> in $\hat{H}_q$ we mean a product $g_1 g_2 \cdots g_n$ such that g_i is a balanced gen of $\hat{H}_q$ for $1 \le i \le n$.

Interp word of length 0 as mult ident of $\hat{H}_q$

A word is called <u>forbidden</u> whenever it is the LHS of a reduction rule. Each forbidden word has length 2

3 kinds of forbidden words: primary
secondary
tert.

DEF 45 Let w denote a forbidden word in $\hat{H}_q$ and consider the corresp reduction rule.

By a <u>descendent</u> of w we mean a word that appears on the RHS of that reduction rule.

ex $w = t_0 X$ is forb

corresp. red rule is

$$t_0 X = X^{-1} t_0 + X T_0 - T_3$$

Descendents of w are

$$X^{-1} t_0, \quad X T_0, \quad T_3$$

thm 46 The following is a basis for $\mathbb{F}$-vector space $\hat{H}_q$:

$$Y^i X^j t_0^k T_0^l T_1^r T_2^s T_3^t \qquad i,j \in \mathbb{Z} \qquad *$$

$$k \in \{0,1\} \qquad l,r,s,t \in \mathbb{N}$$

pf We invoke the

Bergman Diamond Lemma

Bergman:
The Diamond Lemma in Ring Theory
Adv. in Math. 29 (1978) 178–218

[Available thru "Find It" on Math Sci Net]

let $g_1 g_2 \cdots g_n$ denote a word in $\hat{H}_q$.

Call it <u>reducible</u> whenever $\exists\, i$ $(2 \le i \le n)$ s.t. $g_{i-1} g_i$ is forbidden. Word is <u>irreducible</u> iff not reducible

List $*$ consists of the irred words

let $w = g_1 g_2 \cdots g_n$ denote a word in $\hat{H}_q$

An <u>Inversion</u> in w is an ordered pair of integers (i,j) s.t. $1 \le i < j \le n$ and $g_i g_j$ is forbidden

3 kinds of inversions : primary
secondary
tert.

Let W denote the set of all words in $\hat{H}_q$.

We define a partial order $<$ on W.

Pick $w, w' \in W$ and write $w = g_1 g_2 \cdots g_n$

We say w <u>dominates</u> w' whenever $\exists\, i$ $(2 \leq i \leq n)$

s.t. $(i-1, i)$ is an inversion of w and w' is obtained

from w by replacing $g_{i-1} g_i$ by one of its

descendents. In this case either

(i) w has more primary inversions than w'

or

(ii) w, w' have same number of primary inv, but

w has more secondary inv than w'

or

(iii) w, w' have same number of primary inv

" " " secondary "

w has more tert inv than w'

Therefore: transitive closure of dominance relation on W

is a partial order on W which we denote by $<$

Properties of $<$:

- $\nexists$ an ∞ sequence of words "descending chain condition"

  $w_1 > w_2 > w_3 > \cdots$

  [since the number of inversions is finite]

- Given words w, w', w_1, w_2 in $\hat{H_q}$

  $w > w' \Rightarrow w_1 w w_2 > w_1 w' w_2$

  concatenation

  "semi group partial order"

- Given reducible word $w = g_1 g_2 \cdots g_n$ in $\hat{H_q}$

  So $\exists$ i $(2 \le i \le n)$ s.t. $g_{i-1} g_i$ is forbidden.

  $\exists$ red rule with $g_{i-1} g_i$ on LHS; in w replace $g_{i-1} g_i$ by RHS of this red rule and express w as a lin comb of words, each less than w with respect to $<$

  "red rules are compatible with $<$"

Ambiguities

Consider word $t_0^2 X$

this word is reducible:

$\underbrace{t_0 t_0}_{\text{Forb}} X$ $\qquad\qquad$ $t_0 \underbrace{t_0 X}_{\text{forb}}$

We can reduce $t_0^2 X$ 2 ways; elim t_0^2 first using red rule $t_0^2 = t_0 T_0 - 1$, or elim $t_0 X$ first using $t_0 X = X^{-1} t_0 + X T_0 - T_3$

Either way, after 3 steps get same thing:

$$X^{-1} t_0 T_0 + X T_0^2 - X - T_0 T_3$$

(ex)

Therefore the <u>overlap ambiguity</u> $t_0^2 X$ is <u>resolvable</u>

[there is another type of ambiguity called <u>inclusion ambiguity</u> but for this problem there are none]

To invoke the BDL, we must show that all the ambiguities are resolvable.

the nontrivial ambiguities are:

does not involve central elemt T_i

$t_0^2 X \qquad t_0^2 X^{-1} \qquad t_0^2 Y \qquad t_0^2 Y^{-1}$

$t_0 XY \qquad t_0 X^{-1}Y \qquad t_0 XY^{-1} \qquad t_0 X^{-1}Y^{-1}$

$XYY^{-1} \qquad XY^{-1}Y \qquad X^{-1}YY^{-1} \qquad X^{-1}Y^{-1}Y$

$XX^{-1}Y \qquad XX^{-1}Y^{-1} \qquad X^{-1}XY \qquad X^{-1}XY^{-1}$

$t_0 XX^{-1} \qquad t_0 X^{-1}X \qquad t_0 YY^{-1} \qquad t_0 Y^{-1}Y$

One checks each is resolvable (routine but tedious)

Now by the BDL the irred words form a basis for $\hat{H}_q$ $\square$

We have now shown $*$ is basis for $\hat{H}_q$

We now adjust the basis

let λ = indet

Recall $\mathbb{F}[\lambda, \lambda^{-1}]$ space of Laurent poly in λ

LEM 47 The following is a basis for $\mathbb{F}$-vector space $\mathbb{F}[\lambda, \lambda^{-1}]$:

$$\lambda^k (\lambda + \lambda^{-1})^l \qquad k \in \{0,1\}, \quad l \in \mathbb{N} \qquad \star$$

pf $\mathbb{F}[\lambda, \lambda^{-1}]$ has basis $\{\lambda^i\}_{i \in \mathbb{Z}}$

List the elements in order:

$$1, \lambda, \lambda^{-1}, \lambda^2, \lambda^{-2}, \ldots \qquad (1)$$

List $\star$ in order:

$$1, \lambda, \lambda + \lambda^{-1}, \lambda(\lambda + \lambda^{-1}), (\lambda + \lambda^{-1})^2, \lambda(\lambda + \lambda^{-1})^2 \ldots \qquad (2)$$

Write each el of (2) as a lin comb of (1).

Consider coeff matrix

This is upper triangular with all diag entries 1 $\to$ invertible.

Result follows. $\square$

Def 48 Let $\mathbb{T}$ denote the $\mathbb{F}$-subalgebra of $\hat{H}_q$ gen by

$$t_0^{\pm 1}, \quad T_1, \quad T_2, \quad T_3$$

Obs $\mathbb{T}$ is commutative. Note also $\mathbb{T}$ is gen by

$$t_0, \quad T_0, \quad T_1, \quad T_2, \quad T_3$$

Prop 49 (i) The following is a basis for $\mathbb{F}$-vector space $\mathbb{T}$:

$$t_0^k \, T_0^l \, T_1^r \, T_2^s \, T_3^t \qquad k \in \{0,1\}, \quad l, r, s, t \in \mathbb{N} \quad (*)$$

(ii) Another basis for $\mathbb{T}$ is

$$t_0^k \, T_1^r \, T_2^s \, T_3^t \qquad k \in \mathbb{Z}, \quad r, s, t \in \mathbb{N}$$

(iii) $\exists$ $\mathbb{F}$-alg iso

$$\mathbb{F}[\underbrace{\lambda_0^{\pm 1}, \lambda_1, \lambda_2, \lambda_3}_{\text{commuting indets}}] \longrightarrow \mathbb{T}$$

that sends

$$\lambda_0^{\pm 1} \to t_0^{\pm 1}, \qquad \lambda_1 \to T_1, \qquad \lambda_2 \to T_2, \qquad \lambda_3 \to T_3 \quad (**)$$

pf (ii).(iii) $\mathbb{F}$-alg $\mathbb{T}$ is commutative and gen by $t_0^{\pm 1}, T_1, T_2, T_3$. So $\exists$ surjective $\mathbb{F}$-alg

hom
$$\varphi : \mathbb{F}[\lambda_0^{\pm 1}, \lambda_1, \lambda_2, \lambda_3] \to \mathbb{T}$$
that satisfies (**).

By L47 the following is a basis for $\mathbb{F}$-vector space

$\mathbb{F}[\lambda_0^{\pm 1}, \lambda_1, \lambda_2, \lambda_3]$:

$$\lambda_0^k (\lambda_0 + \lambda_0^{-1})^l \lambda_1^r \lambda_2^s \lambda_3^t \qquad k \in \{0,1\}, \quad l, r, s, t \in \mathbb{N} \qquad (***)$$

The Hom φ sends (***) to (*), so (*) span $\mathbb{T}$.

Vectors (*) are lin indep by Th 46.

So (*) is basis for $\mathbb{T}$. Now φ is iso

$\square$

(ii) Clear from (iii)

Def 50 The elements
$$Y^i X^j \qquad i, j \in \mathbb{Z}$$
are lin indep by Th 46, so they form a basis for a subspace of $\hat{H}_q$ denoted $\mathbb{X}$

[caution: $\mathbb{X}$ not a subalgebra]

Prop 5[redacted] The map

$$\underline{X} \otimes \Pi \longrightarrow \hat{H}_q \qquad \otimes = \otimes_{\mathbb{F}}$$

$$u \quad v \longrightarrow uv$$

is an iso of $\mathbb{F}$-vector spaces

pf Compare basis for $\hat{H}_q$ in Thm 46, with basis for Π in Prop 49(i) $\square$

Thm 52 The following is a basis for the $\mathbb{F}$-vector space $\hat{H}_q$:

$$Y^i X^j t_0^k \; T_1^r \; T_2^s \; T_3^t \qquad i,j,k \in \mathbb{Z}, \quad r,s,t \in \mathbb{N}$$

pf Use Prop 51, def of $\underline{X}$, and the basis for Π in Prop 49 (ii) $\square$

Often useful to view $\hat{H}_q$ as (right) module for T

By Prop 51 each element $h \in \hat{H}_q$ can be written

uniquely as

$$h = \sum_{i,j \in \mathbb{Z}} Y^i X^j h_{ij} \qquad h_{ij} \in T$$

[f. many h_{ij} nonzero]

Call h_{ij} the <u>coefficient</u> of $Y^i X^j$ in h
The <u>coefficient matrix</u> for h is $(h_{ij})_{i,j \in \mathbb{Z}}$

View

$$h : \quad \begin{array}{c|ccccccc}
 & \cdots & x^{-2} & x^{-1} & 1 & x & x^2 & \cdots \\
\hline
\vdots & & & & \vdots & & & \\
y^{-2} & & & & & & & \\
y^{-1} & & & h_{-1-1} & h_{-10} & h_{-11} & & \\
1 & \cdots & & h_{0-1} & h_{00} & h_{01} & & \cdots \\
y & & & h_{1-1} & h_{10} & h_{11} & & \\
y^2 & & & & \vdots & & & \\
\vdots & & & & & & &
\end{array}$$

Ex. Back in th 40 we saw the coef matrices

for XY, XY^{-1}, $X^{-1}Y$, $X^{-1}Y^{-1}$

Recall A, B, C:

$$A = y + y^{-1}$$

$$B = X + X^{-1}$$

$$C = t_0 t_2 + (t_0 t_2)^{-1}$$

Find coef matrices

$$A: \quad \begin{array}{c|c} & \begin{array}{ccc} X^{-1} & 1 & X \end{array} \\ \begin{array}{c} y^{-1} \\ 1 \\ y \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$B: \quad \begin{array}{c|c} & \begin{array}{ccc} X^{-1} & 1 & X \end{array} \\ \begin{array}{c} y^{-1} \\ 1 \\ y \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

LEM 53 The coef matrix for C is

C:

	X^{-1}	1	X
y^{-1}	0	$-q^{-1} t_0^{-1} T_3$	$q^{-1} t_0^{-2}$
1	0	$t_0^{-1} T_2 + q^{-1} T_1 T_3$	$-q^{-1} t_0^{-1} T_1$
y	$-q^{-1}$	0	0

pf Apply red rules to C

$$C = t_0 t_2 + (t_0 t_2)^{-1}$$

$$t_0 t_2 = q^{-1} t_3^{-1} T_1 - q^{-1} Y X^{-1} \qquad \text{(by L26(ii))}$$

$\wedge$

$T_3 - t_3$

$\wedge$ L39

$X T_0 - X t_0$

$$= q^{-1} T_1 T_3 - q^{-1} X T_0 T_1 + q^{-1} X t_0 T_1 - q^{-1} Y X^{-1}$$

$$(t_0 t_2)^{-1} = q\, t_3 T_1 - q \underbrace{X Y^{-1}}_{\text{use red rule from Th 40}} \qquad \text{(by L27)}$$

$\wedge$

$X T_0 - X t_0$

Rest is routine (ex) $\square$

We adjust C in order to simplify the coeff matrix

Def 54 Put

$$\theta = -q t_0 \left(C - t_0^{-1} T_2 - q^{-1} T_1 T_3 \right)$$

$$= Y X^{-1} t_0 - Y^{-1} X t_0^{-1} + X T_1 + Y^{-1} T_3$$

Note that θ commutes with t_0 (since C does)

and

$$C = t_0^{-1} T_2 + q^{-1} T_1 T_3 - q^{-1} t_0^{-1} \theta$$

Coef matrix of θ is

	X^{-1}	1	X
Y^{-1}	0	T_3	$-t_0^{-1}$
1	0	0	T_1
Y	t_0	0	0

DEF 55 For any subset S of an $\mathbb{F}$-algebra A

let

$$\langle S \rangle_A = \mathbb{F}\text{-subalgebra of } A \text{ gen by } S$$

If identity of A is clear we often write $\langle S \rangle$

For example for $A = \overset{a}{H_q}$

$$\Pi = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle$$

— o —

By Th 52

$\langle X^{\pm 1} \rangle$ has a basis $\{X^i\}_{i \in \mathbb{Z}}$

$\langle Y^{\pm 1} \rangle$ " $\{Y^i\}_{i \in \mathbb{Z}}$

So each of these is iso $\mathbb{F}[\lambda, \lambda^{-1}]$

By Def 50 the map

$$\langle Y^{\pm 1} \rangle \otimes \langle X^{\pm 1} \rangle \to \underline{X}$$

$$u \otimes v \to uv$$

is iso of $\mathbb{F}$-vector spaces (not algebras!)

Combining this with Prop 51

the map

$$\langle Y^{\pm 1}\rangle \otimes \langle X^{\pm 1}\rangle \otimes \mathbb{T} \longrightarrow \hat{H}_q$$

$$u \otimes v \otimes w \longmapsto uvw$$

is iso of $\mathbb{F}$ vector spaces

Call the factorization

$$\hat{H}_q = \langle Y^{\pm 1}\rangle \langle X^{\pm 1}\rangle \mathbb{T}$$

the "YXT factorization"

Consider subalgebra

$$\langle t_0^{\pm 1}, t_1^{\pm 1} \rangle = \langle Y^{\pm 1}, t_0^{\pm 1} \rangle$$

LEM 56 The following is a basis for the $\mathbb{F}$-vector space $\langle t_0^{\pm 1}, t_1^{\pm 1} \rangle$

$$Y^i t_0^j T_1^k \qquad i, j \in \mathbb{Z}, \quad k \in \mathbb{N} \qquad *$$

pf Vectors $*$ contained in $\langle t_0^{\pm 1}, t_1^{\pm 1} \rangle$ ✓

vectors $*$ lin indep by Th 46

show $\underbrace{\mathrm{Span}(*)}_{\Psi} = \langle t_0^{\pm 1}, t_1^{\pm 1} \rangle$ $**$

Ψ contains 1 since this is included in $*$

$Y^{\pm 1} \Psi \subseteq \Psi$ ✓

show $t_0 \Psi \subseteq \Psi$

For $i \in \mathbb{N}$ one checks

$$t_0 Y^i \in \Psi \qquad t_0 Y^{-i} \in \Psi$$

by ind on i and red rules

$$t_0 Y = Y^{-1} t_0 + Y T_0 - T_1$$

$$t_0 Y^{-1} = Y t_0 - Y T_0 + T_1$$

Obs $T_0 \Psi = \Psi T_0 \subseteq \Psi$

Now $t_0^{-1} \Psi = (T_0 - t_0) \Psi \subseteq \Psi$

Ψ is left ideal of $\langle t_0^{\pm 1}, t_1^{\pm 1} \rangle$ that contains 1 so $**$ holds. $\square$

DEF 57 Let S denote the $\mathbb{F}$-algebra

defined by gens

$$s_i^{\pm 1} \qquad i \in \{0,1\}$$

and rels

$$s_i s_i^{-1} = s_i^{-1} s_i = 1 \qquad i = 0,1$$

$$s_i + s_i^{-1} \text{ is central} \qquad \text{''}$$

— o —

By cnstr $\exists$ $\mathbb{F}$-alg hom

$$S \longrightarrow \hat{H}_q$$

$$s_i^{\pm 1} \longrightarrow t_i^{\pm 1}$$

with image $\langle t_0^{\pm 1}, t_1^{\pm 1} \rangle$

Will show this is an injection, giving an iso of $\mathbb{F}$-algebras

$$S \to \langle t_0^{\pm 1}, t_1^{\pm 1} \rangle$$

Aut(S) contains:

- Fn $\varepsilon_0, \varepsilon_1 \in \{1, -1\}$ "$\mathbb{Z}_2 \times \mathbb{Z}_2$ sym"

  $$s_i \longrightarrow s_i^{\varepsilon_i} \qquad i = 0, 1$$

- $s_0 \leftrightarrow s_1$ "$\mathbb{Z}_2$ sym"

- Fn $\varepsilon_0, \varepsilon_1 \in \{1, -1\}$ "$\mathbb{Z}_2 \times \mathbb{Z}_2$ sym"

  $$s_i \longrightarrow \varepsilon_i s_i$$

AAaut(S) contains antiaut that fixes each of s_0, s_1

DEF 58 Put

$$S_i = s_i + s_i^{-1} \qquad i = 0, 1$$

$$R := s_0 s_1$$

Note by L19
$R + R^{-1}$ is central in S

Also $S = \langle R^{\pm 1}, s_0^{\pm 1} \rangle$
$= \langle R^{\mp 1}, s_0, S_0 \rangle$

Here is the analog of Th 40 for S:

Prop 59 The $\mathbb{F}$-alg S has a presentation

by gens $R^{\pm 1}$, s_0, S_0, S_1

and rels

$$RR^{-1} = 1, \qquad R^{-1}R = 1$$

$$S_i \text{ is central} \qquad i = 0,1$$

$$s_0^2 = s_0 S_0 - 1$$

$$s_0 R = R^{-1} s_0 + R S_0 - S_1$$

$$s_0 R^{-1} = R s_0 - R S_0 + S_1$$

pf Sim to Th 40 $\square$

Thm 60 The following is a basis for the $\mathbb{F}$-vector space S :

$$R^i s_0^j S_1^k \qquad i,j \in \mathbb{Z}, \quad k \in \mathbb{N} \qquad *$$

pf 1 Use Bergman Diamond Lemma (ex)

pf 2 One checks $*$ span S (same pf as for pf of L56)

To see that $*$ are lin indep, apply our $\mathbb{F}$-alg hom $S \to \hat{H}_q$:

$$R \to Y$$
$$s_0 \to t_0$$
$$S_0 \to T_0$$
$$S_1 \to T_1$$

Under this hom, image of $*$ is lin indep by L56, so $*$ is lin indep. $\square$

COR 61 The $\mathbb{F}$-alg hom

$$\begin{aligned} S &\to \hat{H}_q \\ s_i^{\pm 1} &\to t_i^{\pm 1} \end{aligned}$$

is an injection.

Pf The hom sends the basis for S from Th 60 to a lin indep set $\square$

By $\mathbb{Z}_4$-symmetry, for $j \in \mathbb{I}$ $\exists$

$\mathbb{F}$-alg injection

$$S \to \hat{H}_q$$

that sends

$$\begin{aligned} s_0^{\pm 1} &\to t_j^{\pm} \\ s_1^{\pm 1} &\to t_{j+1}^{\pm 1} \end{aligned} \qquad \left(\begin{array}{c} j+1 \text{ taken} \\ \text{mod } 4 \end{array}\right)$$

Prop 62 $\exists$ $\mathbb{F}$-alg injection $S \to \hat{H}_q$

that sends

$$s_0^{\pm 1} \longrightarrow t_0^{\pm 1}$$

$$s_1^{\pm 1} \longrightarrow t_2^{\pm 1}$$

pf the desired injection is the composition

$$\begin{array}{ccccc} S & \xrightarrow{\text{inj from Cor 61}} & \hat{H}_q & \xrightarrow{\rho^{-1}} & \hat{H}_q \\ s_0 & \longrightarrow & t_0 & \longrightarrow & t_0 \\ s_1 & \longrightarrow & t_1 & \longrightarrow & t_2 \end{array}$$

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$

Univ DAHA $\hat{H}_q$ type $(C_1^\vee, C_1)$

Related algebra S has gens $\{s_i^{\pm 1}\}_{i=0}^{1}$ and rels

$$s_i s_i^{-1} = s_i^{-1} s_i = 1 \qquad i = 0, 1$$

$$s_i + s_i^{-1} \text{ central} \qquad \text{"}$$

Write

$$S_i = s_i + s_i^{-1} \qquad i = 0, 1$$

$$R = s_0 s_1$$

Recall Thm 60: the $\mathbb{F}$-vector space S has basis

$$R^i s_0^j S_1^k \qquad i, j \in \mathbb{Z}, \quad k \in \mathbb{N}$$

We give some variations on Th 60

Th 63 Each of the following is a basis for the $\mathbb{F}$-vector space S:

(i) $s_0^j R^i S_1^k$ $\quad i,j \in \mathbb{Z},\ k \in \mathbb{N}$

(ii) $R^i s_1^j S_0^k$ $\quad$ " "

(iii) $s_1^j R^i S_0^k$ $\quad$ " "

pf (i) Apply the antiaut of S to the basis in Th 60

(ii) Apply the aut

$$s_0 \to s_1^{-1}$$
$$s_1 \to s_0^{-1}$$

to the basis in Th 60, and note $R \to R^{-1}$

(iii) Apply the antiaut of S to the basis (ii) $\square$

Th 64 The map

$$U \otimes V \to S$$
$$u \otimes v \to uv$$

is an iso of $\mathbb{F}$-vector spaces, where U, V is any of the following:

U	V
$\langle R^{\pm 1} \rangle$	$\langle A_0^{\pm 1}, S_1 \rangle$
$\langle R^{\pm 1} \rangle$	$\langle A_1^{\pm 1}, S_0 \rangle$
$\langle A_0^{\pm 1}, S_1 \rangle$	$\langle R^{\pm 1} \rangle$
$\langle A_1^{\pm 1}, S_0 \rangle$	$\langle R^{\pm 1} \rangle$

pf Use Th 60 and Th 63 $\square$

Th 65 The map

$$U \otimes V \otimes W \longrightarrow \hat{H}_q$$

$$u \otimes v \otimes w \longrightarrow uvw$$

is an iso of $\mathbb{F}$-vector spaces, where U, V, W is any permutation of

$$\langle X^{\pm 1} \rangle, \quad \langle Y^{\pm 1} \rangle, \quad \mathbb{T}$$

$$\mathbb{T} = \langle t_0^{\pm 1}, T_1, T_2, T_3 \rangle$$

pf. If above perm is identity, done by comments on $\mathbb{X}\mathbb{Y}\mathbb{T}$ factorization above L56

- We can swap the factors $\langle Y^{\pm 1} \rangle$, $\mathbb{T}$ by Th 64 and since

$$\langle Y^{\pm 1} \rangle \mathbb{T} = \underbrace{\langle t_0^{\pm 1}, t_1^{\pm 1} \rangle}_{\text{iso } S} \underbrace{\langle T_2, T_3 \rangle}_{\text{central}}$$

- We can swap factors $\langle X^{\pm 1} \rangle, \langle Y^{\pm 1} \rangle$ and fix $\mathbb{T}$ by applying the iso $\hat{H}_q \to \hat{H}_{q^{-1}}$ from L18

$$X \to Y^{-1}$$
$$Y \to X^{-1}$$
$$t_0 \to t_0^{-1}$$

and replacing q by q^{-1}

Result follows. $\square$

More on S

Find the elements in S that commute with s_0

Motivation

Consider

$$\underbrace{ad(s_0)}_{a} \; : \qquad S \longrightarrow S$$
$$s \longrightarrow s_0 s - s s_0$$

Find nullspace of a

Obs s_0, S_1 commute with s_0 so in this nullspace

Apply a to $R^{\pm 1}$

$$\begin{aligned} a: \; R \longrightarrow \quad & s_0 R - R s_0 \\ = \; & R^{-1} s_0 + R S_0 - S_1 - R s_0 \\ = \; & R^{-1} s_0 + R s_0^{-1} - S_1 \end{aligned}$$

$$\begin{aligned} R^{-1} \longrightarrow \quad & s_0 R^{-1} - R^{-1} s_0 \\ = \; & R s_0 - R S_0 + S_1 - R^{-1} s_0 \\ = \; & S_1 - R s_0^{-1} - R^{-1} s_0 \end{aligned}$$

View $\langle s_0^{\pm 1}, S_1 \rangle$ as scalars and consider

matrix rep a rel "basis" $1, R, R^{-1}$

$$a : \quad \begin{pmatrix} 0 & -S_1 & S_1 \\ 0 & s_0^{-1} & -s_0^{-1} \\ 0 & s_0 & -s_0 \end{pmatrix}$$

Eigenvalues are

$$0, \quad 0, \quad s_0^{-1} - s_0$$

eigenvectors are

$$1, \qquad R + R^{-1}, \qquad \underbrace{R s_0^{-1} + R^{-1} s_0 - S_1}_{\substack{\| \\ \underbrace{s_0 s_1 s_0^{-1} + \cancel{s_1} (s_0^{-1} s_0) - s_1 - \cancel{s_1^{-1}}}_{s_0 s_1 s_0^{-1} - s_1}}}$$

Put

$$G = s_0 s_1 s_0^{-1} - s_1$$

$$a(G) = G(s_0^{-1} - s_0)$$

$$\| $$

$$s_0 G - G s_0$$

Get

$$s_0 G = G s_0^{-1}$$

DEF 66 Put

$$S^+ = \{ s \in S \,/\, s_0 s = s s_0 \}$$

$$S^- = \{ s \in S \,/\, s_0 s = s s_0^{-1} \}$$

obs S^+ = $\mathbb{F}$-subalg of S

S^- = subspace of $\mathbb{F}$-vector space S

LEM 67

(i) Sum $S^+ + S^-$ is direct

(ii) $S^+S^- \subseteq S^-$

$S^-S^+ \subseteq S^-$

$S^-S^- \subseteq S^+$

(iii) $S^+ + S^-$ is an $\mathbb{F}$-subalgebra of S with $\mathbb{Z}_2$-grading.

pf routine $\square$

Find basis for $S^\pm$

LEM 68 In the $\mathbb{F}$-alg $\mathbb{F}[\lambda,\lambda^{-1}]$ consider the ideal

$$L = (\lambda-\lambda^{-1})\,\mathbb{F}[\lambda,\lambda^{-1}]$$

Then

$$\mathbb{F}[\lambda,\lambda^{-1}] = \mathbb{F}1 + \mathbb{F}\lambda + L \qquad \text{(dir sum)}$$

In other words

$$1,\ \lambda$$

is a basis for a complement of L in $\mathbb{F}[\lambda,\lambda^{-1}]$

pf One checks the following is a basis for $\mathbb{F}[\lambda,\lambda^{-1}]$

$$\dots\ \lambda^{-2}(\lambda-\lambda^{-1}),\ \lambda^{-1}(\lambda-\lambda^{-1}),\ \lambda-\lambda^{-1},\ 1,\ \lambda,\ \lambda(\lambda-\lambda^{-1}),\ \lambda^{2}(\lambda-\lambda^{-1}),\ \lambda^{3}(\lambda-\lambda^{-1}),\ \dots$$

From this basis we remove $1, \lambda$ to get a basis for L

Result follows. $\square$

LEM 69 For the $\mathbb{F}$-algebra S

(i) The subalg S^+ is gen by

$$R+R^{-1}, \quad s_0^{\pm 1}, \quad S_1$$

(ii) The $\mathbb{F}$-vector space S^+ has basis

$$(R+R^{-1})^i s_0^j S_1^k \qquad j \in \mathbb{Z}, \quad i,k \in \mathbb{N}$$

(iii) $$S = S^+ + RS^+ \qquad (\text{ds of vector spaces})$$

pf S has basis

$$R^i s_0^j S_1^k \qquad i,j \in \mathbb{Z}, \quad k \in \mathbb{N}$$

So S has basis

$$\left\{ \begin{array}{llll} (R+R^{-1})^i s_0^j S_1^k & j \in \mathbb{Z}, & i,k \in \mathbb{N} & (1) \\ (R+R^{-1})^i R s_0^j S_1^k & j \in \mathbb{Z}, & i,k \in \mathbb{N} & (2) \end{array} \right.$$

obs $\quad \mathrm{Span}(1) \subseteq S^+$

Suf to show

$$\mathrm{Span}(1) = S^+$$

Apply $a = ad(s_0)$ to $(1) \cup (2)$

$$(R+R^{-1})^i s_0^j S_1^k \longrightarrow 0$$

$$(R+R^{-1})^i R s_0^j S_1^k \longrightarrow (R+R^{-1})^i G s_0^j S_1^k$$

show the following are lin indep:

$$(R+R^{-1})^i G s_0^j S_1^k \qquad j \in \mathbb{Z}, \quad i,k \in \mathbb{N} \qquad \text{☆}$$

Easier to show stronger result that (1) ∪ ☆ are lin indep

Obs

$$G = R s_0^{-1} + R^{-1} s_0 - S_1$$

$$= \underbrace{(R+R^{-1}) s_0 - S_1}_{\text{terms in Span(1)}} - R(s_0 - s_0^{-1})$$

To show (1) ∪ ☆ is lin indep, suf to show (1) ∪ ✳ is lin indep where

$$(R+R^{-1})^i R (s_0 - s_0^{-1}) s_0^j S_1^k \qquad j \in \mathbb{Z}, \quad i,k \in \mathbb{N} \qquad ✳$$

(1) ∪ ✳ is lin indep since (1) ∪ (2) is lin indep □

LEM 69A For the $\mathbb{F}$-algebra S

(i) $S^- = GS^+ = S^+G$

(ii) the $\mathbb{F}$-vector space S^- has basis

$$(R+R^{-1})^i \, G \, s_0^j \, S_1^k \qquad i,j \in \mathbb{Z}, \; k \in \mathbb{N}$$

(iii) The $\mathbb{F}$-algebra $S^+ + S^-$ is gen by

$$R+R^{-1}, \quad G, \quad s_0^{\pm 1}, \quad S_1$$

and also by

$$R+R^{-1}, \quad R(s_0 - s_0^{-1}), \quad s_0^{\pm 1}, \quad S_1$$

(iv) The following is a basis for a complement of $S^+ + S^-$ in S

$$\left\{ \begin{array}{l} (R+R^{-1})^i \, R \, S_1^k \\ (R+R^{-1})^i \, R \, s_0 \, S_1^k \end{array} \right. \qquad i, k \in \mathbb{N}$$

pf Recall

$$S = S^+ + R S^+ \quad ds$$

S^+ has basis

$$(R+R^{-1})^i s_0^j \; S_1^k \qquad i,k \in \mathbb{N}, \; j \in \mathbb{Z}$$

Obs S^+ has basis

$$\star \begin{cases} (R+R^{-1})^i (s_0 - s_0^{-1}) \, s_0^j \; S_1^k & i,k \in \mathbb{N}, \; j \in \mathbb{Z} \\ (R+R^{-1})^i (s_0 - 1) \; S_1^k & i,k \in \mathbb{N} \\ (R+R^{-1})^i \; S_1^k & i,k \in \mathbb{N} \end{cases}$$

S has basis

$$\begin{cases} (R+R^{-1})^i \; s_0^j \; S_1^k & j \in \mathbb{Z} \quad i,k \in \mathbb{N} & (1) \\ (R+R^{-1})^i \, R \; (s_0 - s_0^{-1}) \, s_0^j \; S_1^k & j \in \mathbb{Z}, \; i,k \in \mathbb{N} & (2) \\ (R+R^{-1})^i \; R \; (s_0 - 1) \; S_1^k & i,k \in \mathbb{N} & (3) \\ (R+R^{-1})^i \, R \; S_1^k & i,k \in \mathbb{N} & (4) \end{cases}$$

We saw

$$G = \underbrace{(R+R^{-1}) s_0 - S_1}_{\text{terms in } S^+} - R(s_0 - s_0^{-1})$$

So WLOG in (2) replace $R(s_0 - s_0^{-1})$ by G

S has basis $\quad (1) \cup (2') \cup (3) \cup (4) \quad$ where ..

$$(R+R^{-1})^i G\, s_0^j\, S_1^k \qquad j \in \mathbb{Z}, \quad i,k \in \mathbb{N} \qquad (2')$$

Obs $\quad \operatorname{Span}(2') \subseteq S^-$

Suf to show

$$\operatorname{Span}(2') = S^-$$

Define map

$$b : \begin{array}{l} S \to S \\ s \to s_0 s - s s_0^{-1} \end{array}$$

S^- = null space of b

Apply b to above basis for S

b sends

$$\underline{1} \to s_0 - s_0^{-1}$$

$$G \to 0$$

$$R \to s_0 R - R s_0^{-1} = (R+R^{-1}) s_0 - S_1$$

$$= (R+R^{-1})(s_0 - 1) + R + R^{-1} - S_1$$

$$R(s_0 - 1) \to (R+R^{-1}) s_0 (s_0 - 1) - S_1 (s_0 - 1)$$

$$= (R+R^{-1})(s_0 - s_0^{-1}) s_0 - (R+R^{-1}+S_1)(s_0 - 1)$$

type	basis vector	image under b
(1)	$(R+R^{-1})^i s_0^j S_1^k$	$(R+R^{-1})^i (s_0 - s_0^{-1}) s_0^j S_1^k$
(2′)	$(R+R^{-1})^i G s_0^j S_1^k$	0
(3)	$(R+R^{-1})^i R (s_0-1) S_1^k$	$(R+R^{-1})^{i+1}(s_0-s_0^{-1}) s_0 S_1^k - (R+R^{-1})^{i+1}(s_0-1) S_1^k - (R+R^{-1})^i (s_0-1) S_1^{k+1}$
(4)	$(R+R^{-1})^i R\, S_1^k$	$(R+R^{-1})^{i+1}(s_0-1) S_1^k + (R+R^{-1})^{i+1} S_1^k - (R+R^{-1})^i S_1^{k+1}$

Obs for each vector in (1) ∪ (3) ∪ (4)

the image under b is in S^+

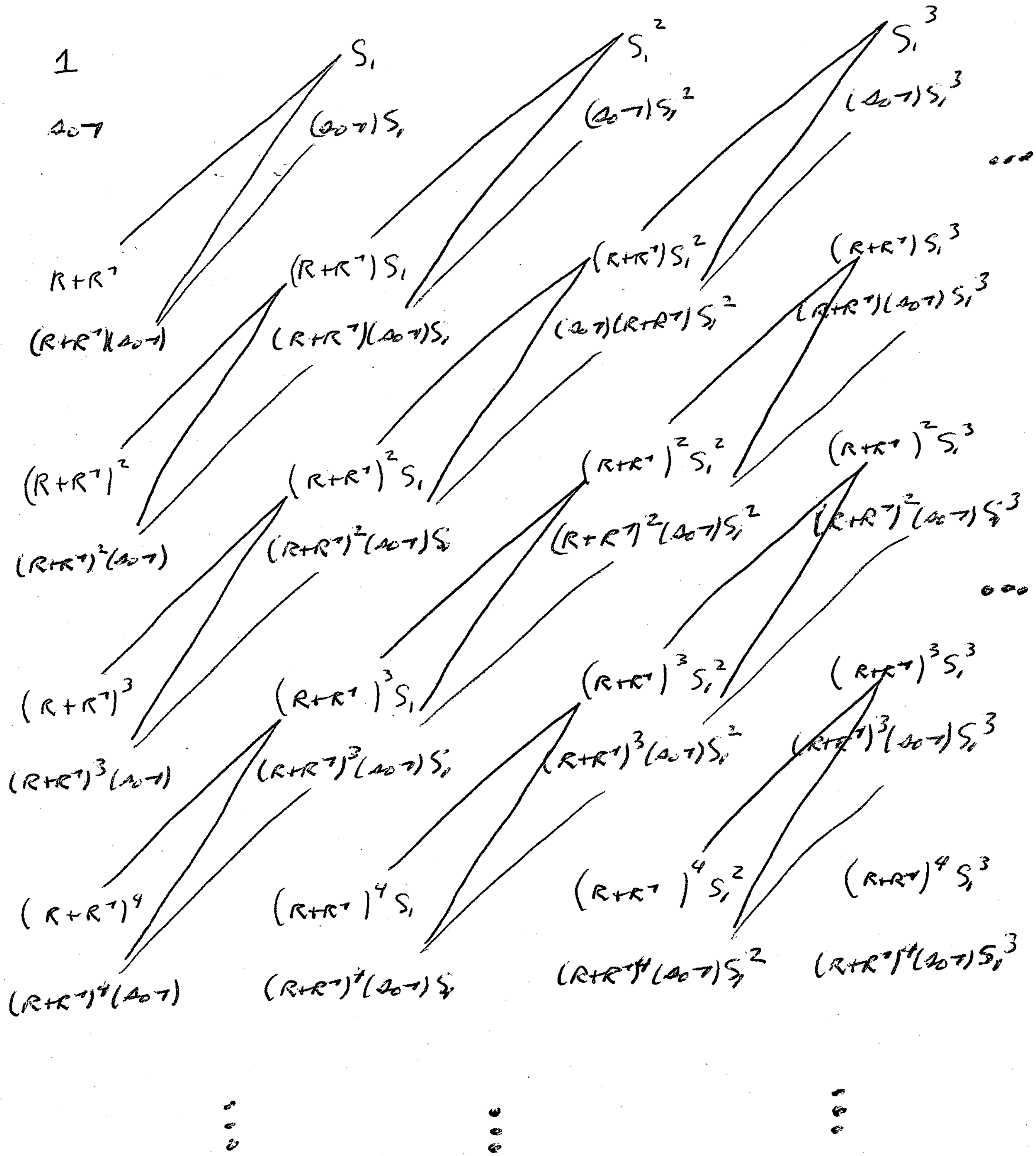
Write these images in terms of the basis for S^+

We find these images are lin indep.

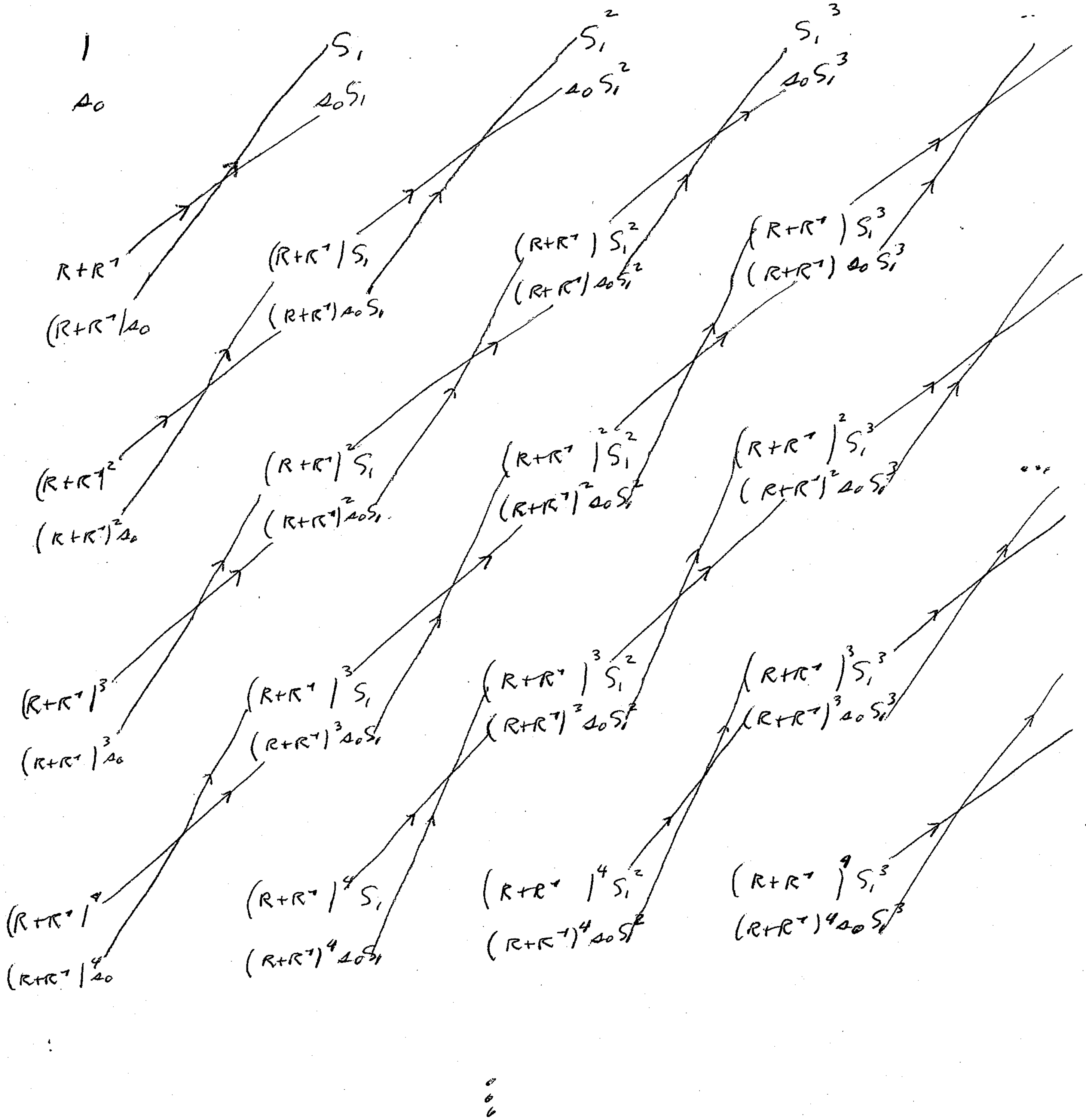
Therefore

$$S^- = \text{null sp of } b$$
$$= \text{Span}(2')$$

Result follows. □



1
S_1
S_1^2
S_1^3
$\Delta_0 - 1$
$(\Delta_0 - 1)S_1$
$(\Delta_0 - 1)S_1^2$
$(\Delta_0 - 1)S_1^3$
$\cdots$
$R+R^{-1}$
$(R+R^{-1})S_1$
$(R+R^{-1})S_1^2$
$(R+R^{-1})S_1^3$
$(R+R^{-1})(\Delta_0 - 1)$
$(R+R^{-1})(\Delta_0 - 1)S_1$
$(\Delta_0 - 1)(R+R^{-1})S_1^2$
$(R+R^{-1})(\Delta_0 - 1)S_1^3$
$(R+R^{-1})^2$
$(R+R^{-1})^2 S_1$
$(R+R^{-1})^2 S_1^2$
$(R+R^{-1})^2 S_1^3$
$(R+R^{-1})^2(\Delta_0 - 1)$
$(R+R^{-1})^2(\Delta_0 - 1)S_1$
$(R+R^{-1})^2(\Delta_0 - 1)S_1^2$
$(R+R^{-1})^2(\Delta_0 - 1)S_1^3$
$\cdots$
$(R+R^{-1})^3$
$(R+R^{-1})^3 S_1$
$(R+R^{-1})^3 S_1^2$
$(R+R^{-1})^3 S_1^3$
$(R+R^{-1})^3(\Delta_0 - 1)$
$(R+R^{-1})^3(\Delta_0 - 1)S_1$
$(R+R^{-1})^3(\Delta_0 - 1)S_1^2$
$(R+R^{-1})^3(\Delta_0 - 1)S_1^3$
$(R+R^{-1})^4$
$(R+R^{-1})^4 S_1$
$(R+R^{-1})^4 S_1^2$
$(R+R^{-1})^4 S_1^3$
$(R+R^{-1})^4(\Delta_0 - 1)$
$(R+R^{-1})^4(\Delta_0 - 1)S_1$
$(R+R^{-1})^4(\Delta_0 - 1)S_1^2$
$(R+R^{-1})^4(\Delta_0 - 1)S_1^3$
$\vdots$
$\vdots$
$\vdots$



1
Δ_0
S_1
$\Delta_0 S_1$
S_1^2
$\Delta_0 S_1^2$
S_1^3
$\Delta_0 S_1^3$
$R+R^{-1}$
$(R+R^{-1})\Delta_0$
$(R+R^{-1})S_1$
$(R+R^{-1})\Delta_0 S_1$
$(R+R^{-1})S_1^2$
$(R+R^{-1})\Delta_0 S_1^2$
$(R+R^{-1})S_1^3$
$(R+R^{-1})\Delta_0 S_1^3$
$(R+R^{-1})^2$
$(R+R^{-1})^2\Delta_0$
$(R+R^{-1})^2 S_1$
$(R+R^{-1})^2\Delta_0 S_1$
$(R+R^{-1})^2 S_1^2$
$(R+R^{-1})^2\Delta_0 S_1^2$
$(R+R^{-1})^2 S_1^3$
$(R+R^{-1})^2\Delta_0 S_1^3$
$(R+R^{-1})^3$
$(R+R^{-1})^3\Delta_0$
$(R+R^{-1})^3 S_1$
$(R+R^{-1})^3\Delta_0 S_1$
$(R+R^{-1})^3 S_1^2$
$(R+R^{-1})^3\Delta_0 S_1^2$
$(R+R^{-1})^3 S_1^3$
$(R+R^{-1})^3\Delta_0 S_1^3$
$(R+R^{-1})^4$
$(R+R^{-1})^4\Delta_0$
$(R+R^{-1})^4 S_1$
$(R+R^{-1})^4\Delta_0 S_1$
$(R+R^{-1})^4 S_1^2$
$(R+R^{-1})^4\Delta_0 S_1^2$
$(R+R^{-1})^4 S_1^3$
$(R+R^{-1})^4\Delta_0 S_1^3$

Things to show

- S is domain
- maps a, b sat $ab = ba = 0$
- S^- is image of S under a
- ✓ $S = S^+ + RS^+$ ds
- a sends $RS^+ \to GS^+ = S^-$

  $(R+R^{-1})^i R s_0^j S_1^k \to (R+R^{-1})^i G s_0^j S_1^k$