

Lec 1 Friday Sept 2

9/2/11

Fall 2011

Math 846 Topics in Combinatorics

BIOS IV

11:00 AM MWF

Theme: the Double Affine Hecke algebra
"DAHA"

- Defined by Cherednik in 1992
- Related to a class of multivariable orthogonal polynomials called the Mac Donald / Koornwinder polys.
- In the rank 1 case these are the Askey - Wilson polynomials

Strategy

I

Rank 1 case

- investigate structure of the algebra: automorphisms, basis, center, ...
- representation theory
- connection to AW polynomials

II

Rank n case

Work thru book

Mac Donald: Affine Hecke algebras and
orthogonal polynomials

Cambridge U. press 2003

Summary of topics covered (written Dec 16 2011)

We investigated the double affine Hecke algebra (DAHA) of type (C_1^v, C_1) . This algebra is denoted \hat{H}_q

\hat{H}_q is defined by generators $\{t_i^{\pm 1}\}_{i=0}^3$ and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad i=0,1,2,3$$

$t_i + t_i^{-1}$ is central

$$t_0 t_1 t_2 t_3 = q^{-1}$$

Part I Ring theory of \hat{H}_q

Basic facts

Automorphisms and anti automorphisms

An action of the braid group B_3 on \hat{H}_q

The elements $X = t_3 t_0$ and $Y = t_0 t_1$

The elements $A = t_0 t_1 + (t_0 t_1)^{-1}$ $B = t_0 t_2 + (t_0 t_2)^{-1}$

$$C = t_0 t_2 + (t_0 t_2)^{-1}$$

A, B, C satisfy the \mathbb{Z}_3 -symmetric Askey-Wilson relations

The universal Askey-Wilson algebra Δ_q

A homomorphism $\Delta_q \rightarrow \hat{H}_q$

A linear basis for \hat{H}_q

A presentation of \hat{H}_q by gens and relations mentioning X, Y

A linear basis for Δ_q

The homomorphism $\Delta_q \rightarrow \hat{H}_q$ is injective

The spaces $\hat{H}_9^+ = \{ h \in \hat{H}_9 \mid h t_0 = t_0 h \}$,
 $\hat{H}_9^- = \{ h \in \hat{H}_9 \mid h t_0 = t_0^{-1} h \}$

A linear basis for \hat{H}_9^+

A presentation of \hat{H}_9^+ by generators and relations

A linear basis for \hat{H}_9^-

The center of \hat{H}_9

Some 2-sided ideals of \hat{H}_9 and \hat{H}_9^+

Part II Representation theory of \hat{H}_9

Basic facts

The elements $G_0 = t_0 - t_3 t_0 t_3^{-1}$ $G_1 = t_1 - t_0 t_1 t_0^{-1}$
 $G_2 = t_2 - t_0 t_2 t_0^{-1}$ $G_3 = t_3 - t_2 t_3 t_2^{-1}$

How G_0, G_2 swap eigenspaces of X and G_1, G_3 swap eigenspaces of Y

Description of an \hat{H}_9 -module; the staircase picture

The actions of $\{ t_i^{\pm 1} \}_{i=0}^3$ on the eigenspaces of X and Y

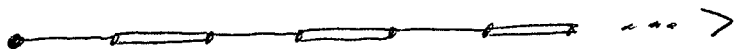
The action of X, Y on each others eigenspaces

The X -diagram and the Y -diagram

Description of an irreducible \hat{H}_9 -module whose X -diagram is a doubly infinite path

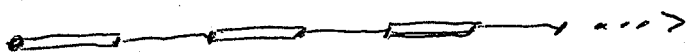


Description of an irreducible \hat{H}_q -module whose X -diagram is a semi-infinite path



Connection to the Askey-Wilson polynomials

Description of an irreducible \hat{H}_q -module whose X -diagram is a semi-infinite path



Connection to the Askey-Wilson polynomials

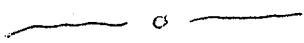
The Askey-Wilson polynomials: 3-term recurrence, parameter array, the Askey-Wilson q -difference operator, the Askey-Wilson relations, the explicit basis

An \hat{H}_q -module structure on the Laurent polynomials $\mathbb{F}[y, y^{-1}]$

A linear basis for $\mathbb{F}[y, y^{-1}]$ that makes X upper triangular and Y lower triangular

The actions of $\{t_i^{\pm 1}\}_{i=0}^3$ on the above basis

The elements X, Y satisfy the nonsymmetric tri-diagonal relations



I: Rank 1 DAHA

Conventions

- An algebra is meant to be associative and have a 1
- A subalgebra has same 1 as parent algebra
- Fix a field \mathbb{F}
- Fix $0 \neq q \in \mathbb{F}$

[Often we will restrict to case $q^4 \neq 1$]

Def 1 Let \hat{H}_q denote the \mathbb{F} -algebra defined by generators

$$\{t_i^{\pm 1}\}_{i \in I}$$

$$I = \{0, 1, 2, 3\}$$

and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1$$

$$i \in I$$

(01)

$$t_i + t_i^{-1} \text{ is central}$$

..

(02)

$$t_0 t_1 t_2 t_3 = q^{-1}$$

(03)

We call \hat{H}_q the universal DAHA of type
 $(C_1^1 | C_1)$

\hat{H}_q is our main object of study

Def 2 Fix nono $k_i \in \mathbb{F} \quad i \in \mathbb{I}$

let $H(k_0, k_1, k_2, k_3; q)$ denote the \mathbb{F} -algebra defined

by generators

$$\{t_i\}_{i \in \mathbb{I}}$$

and rels

$$(t_i - k_i)(t_i - k_i^{-1}) = 0 \quad i \in \mathbb{I} \quad (*)$$

$$t_0 t_1 t_2 t_3 = q^{-1} \quad (**)$$

This is the (ordinary) DAHA of type (C_1^+, C_1)

LEM 3 \exists unique \mathbb{F} -algebra hom $\hat{H}_q \rightarrow H(k_0, k_1, k_2, k_3; q)$

that sends

$$t_i \rightarrow \tilde{t}_i \quad i \in \mathbb{I}$$

"evaluation homomorphism"

This hom is surjective.

pf $(*)$, $(**)$ imply each gen \tilde{t}_i of $H(k_0, k_1, k_2, k_3; q)$ is invertible and

$$\tilde{t}_i + \tilde{t}_i^{-1} = k_i + k_i^{-1}$$

is central

□

Def 4 Let \hat{H} denote the \mathbb{F} -algebra defined

by gens

$$\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$$

and rels

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad i \in \mathbb{I}$$

$$t_i + t_i^{-1} \quad \text{central} \quad \dots$$

$$t_0 t_1 t_2 t_3 \quad \text{central}$$

In an earlier paper I called \hat{H} the universal OAMA of type (C_1^+, C_1)

LEM 5 \exists unique \mathbb{F} -algebra hom $\hat{H} \rightarrow \hat{H}_g$

that sends

$$t_i \rightarrow t_i \quad i \in \mathbb{I}$$

this hom is surj.

□

pf clear

I mention \hat{H} for completeness, our focus will be \hat{H}_g

An \hat{H}_q -module

To motivate \hat{H}_q we display a \hat{H}_q -module

Let $\lambda = \text{indet}$

$\mathbb{F}[\lambda] = \mathbb{F}$ -algebra of polys in λ that have all coeffs in \mathbb{F}

$\mathbb{F}[\lambda, \lambda^{-1}] = \dots$ Laurent polys \dots

LEM 6 Assume $\text{char } \mathbb{F} \neq 2, \exists i^0 \in \mathbb{F} \text{ s.t. } i^0{}^2 = -1$

Assume q not a root of 1.

Then $\mathbb{F}[\lambda, \lambda^{-1}]$ is an \hat{H}_q -module with

$t_0 \cdot \lambda^i = i^0 \lambda^{-i} \quad i \in \mathbb{Z} \text{ (integers)}$

$t_1 \cdot \lambda^i = -i^0 q^{-2i} \lambda^{-i} \quad \dots$

$t_2 \cdot \lambda^i = i^0 q^{1-2i} \lambda^{1-i} \quad \dots$

$t_3 \cdot \lambda^i = -i^0 \lambda^{1-i}$

On this module

$t_2 + t_2^{-1} = 0 \quad z \in \mathbb{Z}$

pf One checks that for $z \in \mathbb{Z}$

t_2^z acts on $\mathbb{F}[\lambda, \lambda^{-1}]$ as -1

so $t_2 + t_2^{-1} = 0$

Check $t_0 t_1 t_2 t_3 = q^{-1}$:

$\lambda^i \xrightarrow{t_3} -i^0 \lambda^{1-i} \xrightarrow{t_2} q^{2i-1} \lambda^i \xrightarrow{t_1} -i^0 q^{-1} \lambda^{-i} \xrightarrow{t_0} q^{-1} \lambda^i$

□

Comments on LEM 6

Define

$$X = t_3 t_0$$

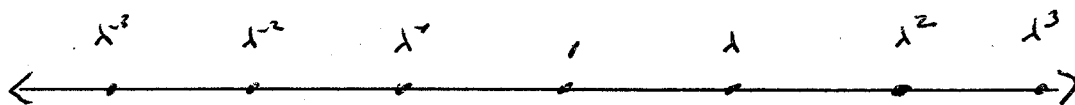
$$Y = t_0 t_1$$

obs \hat{H}_g is gen by $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$

obs $X \cdot \lambda^i = \lambda^{i+1} \quad i \in \mathbb{Z}$

$$Y \cdot \lambda^i = q^{-2i} \lambda^i$$

$$t_0 \cdot \lambda^i = q^0 \lambda^{-i}$$



X : shift right \Rightarrow

Y : the λ^i are eigenvectors

t_0 : swap \Leftrightarrow

Consider

$$X + X^{-1}$$

$$Y + Y^{-1}$$

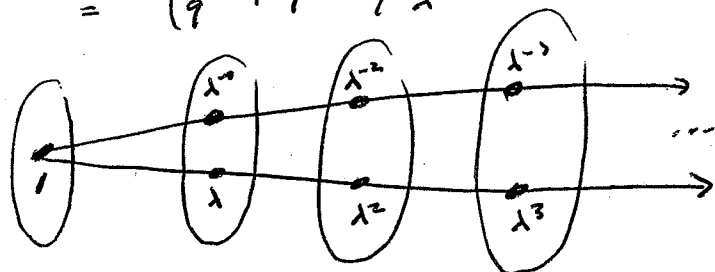
$$(X + X^{-1}) \cdot \lambda^i = \lambda^{i+1} + \lambda^{i-1} \quad i \in \mathbb{Z}$$

"adjacency operator" for ∞ path

$$(Y + Y^{-1}) \cdot \lambda^i = (q^{2i} + q^{-2i}) \lambda^i \quad i \in \mathbb{Z}$$

eigenspaces for $Y + Y^{-1}$:

eigenvalues $\lambda^i + \lambda^{-i}$:



eigenvalues $\lambda^i + \lambda^{-i}$: $2, q^2 + q^{-2}, q^4 + q^{-4}, q^6 + q^{-6}, \dots$

Let $T =$ subalgebra of \hat{H}_g gen by
 $x+x^{-1}, y+y^{-1}$

T acts on $V = \mathbb{F}[\lambda, \lambda^{-1}]$ as "subconstituent algebra"
of 20 paths

Interpret t_0 :

Obs t_0 commutes with $x+x^{-1}, y+y^{-1}$

$t_0^2 = -1$ on V so equals $i, -i$

Find eigenspaces

Define $V_0 =$ subspace of V with basis
 $1, \lambda+\lambda^{-1}, \lambda^2+\lambda^{-2}, \dots$

Define $V_1 =$ subspace of V with basis
 $\lambda-\lambda^{-1}, \lambda^2-\lambda^{-2}, \dots$

Then

$$V = V_0 + V_1 \quad (\text{dir sum})$$

$V_0 =$ eigenspace for t_0 with equal i

$V_1 = \dots -i$

Each of V_0, V_1 is T -submodule of V

Call V_0 the primary T -submodule

Obs

$$\frac{1 - \lambda^0 t_0}{2}$$

acts on V_0 as 1 and on V_1 as 0

$$\frac{1 + \lambda^0 t_0}{2}$$

... V_0 ... 0 ... V_1 ... 1

So

$$\frac{1 \pm \lambda^0 t_0}{2}$$

act on V as the central idempotents for T .

Rel basis (*) the matrix rep $X + X^{-1}$, $Y + Y^{-1}$ is

$X + X^{-1}$:

$$\begin{array}{cccc}
 0 & 2 & & \\
 1 & 0 & 1 & \\
 & 1 & 0 & 1 \\
 & & 1 & 0 & 1 \\
 & & & 1 & \\
 & & & & \ddots \\
 & & & & & \ddots \\
 & & & & & & \ddots \\
 & & & & & & & \ddots \\
 & & & & & & & & \ddots \\
 & & & & & & & & & \ddots
 \end{array}$$

(irred tridiag)

$Y + Y^{-1}$: diag (2, $q^2 + q^{-2}$, $q^4 + q^{-4}$, ...)

Above tridiag matrix describes the 3-term rec for a sequence of polynomials $\{p_i\}_{i=0}^{\infty}$:

$$p_0 = 1$$

$$p_1 = \lambda$$

$$\lambda p_i = p_{i+1} + 2p_{i-1}$$

$$\lambda p_i = p_{i+1} + p_{i-1}$$

$i = 2, 3, \dots$

One checks

$$p_i (X + X^{-1}) v_0 = v_i$$

$i = 0, 1, 2, \dots$

where

$$v_0 = 1$$

$$v_i = \lambda^i + \lambda^{-i}$$

$1 \leq i < \infty$

... Chebyshev polynomial of 1st kind.

//

\mathbb{F} arb Fix $0 \neq q \in \mathbb{F}$

Recall \hat{H}_q is \mathbb{F} -algebra defined by gens $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$
and relations

$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad i \in \mathbb{I} \quad D1$

$t_i + t_i^{-1}$ central .. D2

$t_0 t_1 t_2 t_3 = q^{-1} \quad D3$

Obs q^{-1} is equal to each of

$t_0 t_1 t_2 t_3, \quad t_1 t_2 t_3 t_0, \quad t_2 t_3 t_0 t_1, \quad t_3 t_0 t_1 t_2$

Auto / Antiauto of \hat{H}_q

Let A denote an \mathbb{F} -algebra

By an automorphism of A we mean an \mathbb{F} -alg iso
 $A \rightarrow A$

Set of auts of A form a group under composition, called $\text{Aut}(A)$

LEM 7 \exists automorphism of \hat{H}_q that sends

$t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow t_0 \quad \parallel \quad \mathbb{Z}_4\text{-symmetry} \quad \parallel$

pt clear

LEM 8 Puk

$$\varepsilon_i \in \{1, -1\} \quad i \in \mathbb{I}$$

such that

$$\varepsilon_0 \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$$

Then \exists automorphism of \hat{H}_9 that sends

$$t_i \rightarrow \varepsilon_i t_i \quad i \in \mathbb{I}$$

" $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -sym"

pf clear.

We now consider some auto that are not so trivial

The Braid group B_3 is defined by generators ρ, σ

and rels $\rho^3 = \sigma^2$

For convenience put

$$\gamma = \rho^3 = \sigma^2$$

Aside: B_3 has another presentation by gens

u, v and rels

$$u v u = v u v$$

[As an ex, find the iso between the two presentations.]

LEM 9 B_3 acts on \hat{H}_q as a gp of auts such that

$$\tau(h) = t_0^{-1} h t_0 \quad \forall h \in \hat{H}_q$$

and ρ, σ do the following:

| h | t_0 | t_1 | t_2 | t_3 |
|-------------|-------|--------------------|--------------------|-------|
| $\rho(h)$ | t_0 | $t_0^{-1} t_3 t_0$ | t_1 | t_2 |
| $\sigma(h)$ | t_0 | $t_0^{-1} t_3 t_0$ | $t_1 t_2 t_1^{-1}$ | t_2 |

pf \exists aut T of \hat{H}_q that sends

$$h \rightarrow t_0^{-1} h t_0 \quad \forall h \in \hat{H}_q$$

Define

$$t_0^v = t_0 \quad t_1^v = t_0^{-1} t_3 t_0 \quad t_2^v = t_1 \quad t_3^v = t_2$$

Show $t_0^v, t_1^v, t_2^v, t_3^v$ satisfy defining rels for \hat{H}_q

D1 \checkmark

D2:

$$t_0^v + t_0^{v^{-1}} = t_0 + t_0^{-1} \text{ is central}$$

$$t_1^v + t_1^{v^{-1}} = t_0^{-1} (t_3 + t_3^{-1}) t_0$$

$$= t_3 + t_3^{-1}$$

is central

$$t_2^v + t_2^{v^{-1}} = t_1 + t_1^{-1}$$

is central

$$t_3^v + t_3^{v^{-1}} = t_2 + t_2^{-1} \text{ is central}$$

D3:

$$\begin{aligned} t_0^v t_1^v t_2^v t_3^v &= t_0 t_0^{-1} t_3 t_0 t_1 t_2 \\ &= t_3 t_0 t_1 t_2 \\ &= q^{-1} \end{aligned}$$

Now \exists \mathbb{F} -alg hom $P: \hat{H}_q \rightarrow \hat{H}_q$ that sends
 $t_i \rightarrow t_i^v \quad i \in \mathbb{I}$

Show $P^3 = T$

To do this show P^3, T agree at t_i for $i \in \mathbb{I}$

t_0 fixed by T

t_0 fixed by P hence P^3

P sends

$$t_3 \rightarrow t_2 \rightarrow t_1 \rightarrow t_0^{-1} t_3 t_0 \rightarrow t_0^{-1} t_2 t_0 \rightarrow t_0^{-1} t_1 t_0$$

P^3 sends

$$t_i \rightarrow t_0^{-1} t_i t_0 \quad i=1,2,3$$

So P^3, T agree at t_1, t_2, t_3

Now $P^3 = T$

T^{-1} exists $\Rightarrow P^{-1}$ exists $\Rightarrow P$ is anti of \hat{H}_q

Define

$$\hat{t}_0 = t_0, \quad \hat{t}_1 = t_0^{-1} t_3 t_0, \quad \hat{t}_2 = t_1 t_2 t_1^{-1}, \quad \hat{t}_3 = t_2$$

Show $\hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3$ sat defining rels for \hat{H}_2

D1 ✓

D2: $\hat{t}_0 + \hat{t}_0^{-1} = t_0 + t_0^{-1}$

$$\begin{aligned} \hat{t}_1 + \hat{t}_1^{-1} &= t_0^{-1} (t_3 + t_3^{-1}) t_0 \\ &= t_3 + t_3^{-1} \end{aligned}$$

$$\begin{aligned} \hat{t}_2 + \hat{t}_2^{-1} &= t_1 (t_2 + t_2^{-1}) t_1^{-1} \\ &= t_2 + t_2^{-1} \end{aligned}$$

$$\hat{t}_3 + \hat{t}_3^{-1} = t_2 + t_2^{-1}$$

all central ✓

D3 :

$$\begin{aligned} \hat{t}_0 \hat{t}_1 \hat{t}_2 \hat{t}_3 &= \cancel{t_0} \cancel{t_0^{-1}} t_3 t_0 \cancel{t_1} \cancel{t_1^{-1}} t_2 \\ &= t_3 t_0 t_2 \\ &= z^{-1} \end{aligned}$$

Now \exists \mathbb{F} -alg hom $S: \hat{H}_2 \rightarrow \hat{H}_2$ that sends
 $t_i \rightarrow \hat{t}_i \quad \forall i \in \mathbb{I}$

show $S^2 = T$

t_0 fixed by T, S^2

$$t_1 \xrightarrow{S} t_0^{-1} t_3 t_0 \xrightarrow{S} t_0^{-1} t_1 t_0$$

S^2, T agree at t_1 ✓

$$t_3 \xrightarrow{S} t_1 \xrightarrow{S} t_0^{-1} t_3 t_0$$

S^2, T agree at t_3 ✓

$$t_2 \xrightarrow{S} t_1 t_2 t_1^{-1} \xrightarrow{S} \underbrace{t_0^{-1} t_3 t_0 t_1 t_2 t_1^{-1}}_{q^{-1}} \underbrace{t_0^{-1} t_3 t_0}_{\substack{\text{"} \\ t_0^{-1} t_0 t_3 t_0 \\ \text{"} \\ (t_2 t_3 t_0 t_1)^{-1} t_2 \\ \text{"} \\ t_2}}_{\substack{\text{"} \\ t_0^{-1} t_2 t_0}}$$

shows S^2, T agree at t_2 ✓

Now $S^2 = T$

Now S^{-1} exists $\Rightarrow S$ is ant of \hat{H}_g

Result follows. □

We record a fact from L9 and its pf.

LEM 10 For the B_3 -action on \hat{H}_9 from L9.

τ fixes every central element
and ρ, σ do the following

| h | $t_0 + t_0^{-1}$ | $t_1 + t_1^{-1}$ | $t_2 + t_2^{-1}$ | $t_3 + t_3^{-1}$ |
|-------------|------------------|------------------|------------------|------------------|
| $\rho(h)$ | $t_0 + t_0^{-1}$ | $t_3 + t_3^{-1}$ | $t_1 + t_1^{-1}$ | $t_2 + t_2^{-1}$ |
| $\sigma(h)$ | $t_0 + t_0^{-1}$ | $t_3 + t_3^{-1}$ | $t_2 + t_2^{-1}$ | $t_1 + t_1^{-1}$ |

pf clear from pf of L9.

□

Let A denote an \mathbb{F} -algebra

By an anti automorphism of A we mean an iso
of \mathbb{F} -vector spaces $\gamma: A \rightarrow A$ s.t.

$$\gamma(ab) = \gamma(b)\gamma(a) \quad \forall a, b \in A$$

For example the transpose map is an anti aut of
the matrix algebra $\text{Mat}_{n \times n}(\mathbb{F})$ $n=1, 2, \dots$

Another view:

Define A^{op} to be the \mathbb{F} -vector space A
together with mult

$$\begin{matrix} a & b & = & b & a & \forall a, b \in A \\ (\text{in } A^{op}) & & & (\text{in } A) & & \end{matrix}$$

Then A^{op} is an \mathbb{F} -algebra.

For any map $\gamma: A \rightarrow A$ TFAE:

(i) γ is an anti aut of A

(ii) γ is an \mathbb{F} -algebra iso $A \rightarrow A^{op}$

1
9

Under composition \circ the auto / anti auto of A are related as follows

| \circ | aut | anti aut |
|----------|----------|----------|
| aut | aut | anti aut |
| anti aut | anti aut | aut |

Set of auto and anti auto of A form a group under composition, called $AAut(A)$

Either

$$Aut(A) = AAut(A)$$

or

$Aut(A)$ is a normal subgroup of $AAut(A)$ that has index 2

LEM 11 \exists anti-autom \dagger of \hat{H}_9 that sends

$$t_0 \rightarrow t_0, \quad t_1 \rightarrow t_3, \quad t_2 \rightarrow t_2, \quad t_3 \rightarrow t_1$$

Moreover $\dagger^2 = 1$.

pf Define some els of \hat{H}_9^{op} :

$$\begin{aligned} t_0^\dagger &= t_0 & t_1^\dagger &= t_3 \\ t_2^\dagger &= t_2 & t_3^\dagger &= t_1 \end{aligned}$$

show $t_0^\dagger, t_1^\dagger, t_2^\dagger, t_3^\dagger$ sat defn for \hat{H}_9 (in the alg \hat{H}_9^{op})

D1 ✓

D2 ✓

D3: $t_0^\dagger t_1^\dagger t_2^\dagger t_3^\dagger \stackrel{?}{=} q^{-1}$
(in \hat{H}_9^{op})

$$\begin{aligned} \text{LHS} &= t_3^\dagger t_2^\dagger t_1^\dagger t_0^\dagger \quad (\text{in } \hat{H}_9) \\ &= t_1 t_2 t_3 t_0 \quad (\dots) \\ &= q^{-1} \quad \checkmark \end{aligned}$$

So \exists \mathbb{F} -alg hom $\dagger: \hat{H}_9 \rightarrow \hat{H}_9^{op}$ that sends

$$t_i \rightarrow t_i^\dagger \quad \forall i \in \mathbb{I}$$

By constr $\dagger^2 = 1$ so \dagger is invertible hence bijectm.

So $\dagger: \hat{H}_9 \rightarrow \hat{H}_9^{op}$ is \mathbb{F} -alg iso, hence anti-autom of \hat{H}_9 \square

LEM 12 \exists iso of F -algebras

$$\gamma: \hat{H}_q \rightarrow \hat{H}_{q^{-1}}$$

that sends

$$t_0 \rightarrow t_0^{-1} \qquad t_2 \rightarrow t_3^{-1}$$

[caution: γ does not send $q \rightarrow q^{-1}$. It fixes q along with every scalar]

pf (Write capital for elements in $\hat{H}_{q^{-1}}$)

so
$$T_0 T_1 T_2 T_3 = Q^{-1} = q \quad (in \hat{H}_{q^{-1}}) \qquad t_0 t_1 t_2 t_3 = q^{-1} \quad (in \hat{H}_q)$$

define
$$t_0^{-1} = T_0^{-1} \qquad t_1^{-1} = T_3^{-1}$$

$$t_2^{-1} = T_2^{-1} \qquad t_3^{-1} = T_1^{-1}$$

show $t_0^{-1}, t_1^{-1}, t_2^{-1}, t_3^{-1}$ sat def wls for \hat{H}_q

- D1 ✓
- D2 ✓
- D3:

$$t_0^{-1} t_1^{-1} t_2^{-1} t_3^{-1} = q^{-1} \quad ?$$

$$T_0^{-1} T_3^{-1} T_2^{-1} T_1^{-1} \quad OK.$$

$$(T_1 T_2 T_3 T_0)^{-1}$$

$$= q^{-1}$$

so \exists F -alg hom $\gamma: \hat{H}_q \rightarrow \hat{H}_{q^{-1}}$ that sends

$$t_i \rightarrow t_i^{-1} \quad \text{for } i \in \mathbb{I}$$

obs γ is big, hence iso.



The elements X, Y

DEF 13 Let X, Y denote the following elements of \hat{H}_2 :

$$X = t_3 t_0 \qquad Y = t_0 t_1$$

Obs X^{-1}, Y^{-1} exist.

LEM 14 We have

$$t_1 = t_0^{-1} Y$$

$$t_2 = Y^{-1} t_0 X^{-1}$$

$$t_3 = X t_0^{-1}$$

Moreover \hat{H}_2 is generated by $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$

pf ex

□

LEM 15 Recall aut of \hat{H}_9 from L 7 (Z₄-sym)

this aut sends

$$X \rightarrow Y \rightarrow g^{-1}X^{-1} \rightarrow g^{-1}Y^{-1} \rightarrow X$$

pf

$$X = t_0 t_1 \rightarrow t_0 t_1 = Y$$

$$t_0 t_1 \rightarrow t_1 t_2 = \underbrace{t_1 t_2 t_3 t_0}_{g^{-1}} \underbrace{(t_3 t_0)^{-1}}_{X^{-1}}$$

□

LEM 16 For the B₃ action on \hat{H}_9 the gens ρ, σ act on X, Y as follows.

σ sends

$$X \rightarrow t_0^{-1} Y t_0$$
$$Y \rightarrow X$$

ρ sends

$$X \rightarrow g^{-1} Y^{-1} t_0 X^{-1} t_0$$
$$Y \rightarrow X$$

pf routine

□

LEM 17 The antiaut τ of \hat{H}_7 sends

$$X \rightarrow Y$$

$$Y \rightarrow X$$

pf

$$\begin{array}{c}
 X \\
 \parallel \\
 t_3 t_0 \\
 \swarrow \searrow \\
 t_0 t_3 \\
 \parallel \\
 Y
 \end{array}$$

$$\begin{array}{c}
 Y \\
 \parallel \\
 t_0 t_3 \\
 \swarrow \searrow \\
 t_3 t_0 \\
 \parallel \\
 X
 \end{array}$$

LEM 18 The \mathbb{F} -alg iso $\gamma: \hat{H}_7 \rightarrow \hat{H}_9$ sends

$$X \rightarrow Y^{-1}$$

$$Y \rightarrow X^{-1}$$

pf

$$X = t_3 t_0 \rightarrow t_3^{-1} t_0^{-1} = Y^{-1}$$

$$Y = t_0 t_3 \rightarrow t_0^{-1} t_3^{-1} = X^{-1}$$

□

F arb $0 \neq q \in F$

Continue to discuss univ DHA \hat{H}_q of type $(C_{II}^V C_0)$

Recall

$$X = t_3 t_0 \quad Y = t_0 t_1$$

We will consider

$$X + X^{-1}, \quad Y + Y^{-1}$$

LEM 19 let A denote any F -algebra

Given invertible $u, v \in A$ s.t. each of

$$u + u^{-1}, \quad v + v^{-1}$$

is central in A . then

(i) $uv + (uv)^{-1} = vu + (vu)^{-1}$

(ii) $uv + (uv)^{-1}$ commutes with each of u, v

pf (i) Write

$$U = u + u^{-1}$$

$$V = v + v^{-1} \quad (\text{Central el})$$

$$\begin{aligned} uv + (uv)^{-1} &= uv + (U - u)(U - u)^{-1} \\ &= uv + vu - vU - uV + UV \end{aligned}$$

$$\begin{aligned} \text{Sim} \quad vu + (vu)^{-1} &= vu + (U - u)(V - v)^{-1} \\ &= vu + uv - uV - vU + UV \end{aligned}$$

(ii)

$$\begin{aligned}
 u^{-1} (uv + v^{-1}u^{-1})u &= vu + u^{-1}v^{-1} \\
 &= uv + v^{-1}u^{-1} \quad \text{by (i)}
 \end{aligned}$$

So $uv + v^{-1}u^{-1}$ commutes with u .

Case of v is sim.

□

Back to \hat{H}_g

Cor 20 For dist $i, j \in \mathbb{I}$

$$(i) \quad t_i t_j + (t_i t_j)^{-1} = t_j t_i + (t_j t_i)^{-1}$$

(ii) $t_i t_j + (t_i t_j)^{-1}$ commutes with each of t_i, t_j .

Next goal: Find nice formula for

$$q^{-1} Y X^{-1} - q^{-1} X Y^{-1}$$

DEF 21 Let

$$T_i = t_i + t_i^{-1} \quad i \in \mathbb{I}$$

So T_i central in \hat{H}_q

Note: Often convenient to eliminate t_i^{-1} using
def 21.

LEM 22 The \mathbb{F} -algebra \hat{H}_q has a presentation
by generators

$$t_i, T_i \quad i \in \mathbb{I}$$

and relations

$$t_i^2 = t_i T_i - 1 \quad i \in \mathbb{I}$$

T_i central

$$t_1 t_2 t_3 = q^{-1}$$

pf ex

We mentioned

$$X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$$

generate \hat{H}_q . In terms of these generators the

$\{T_i\}_{i \in \mathbb{Z}}$ look as follows.

LEM 23.

(i) $T_0 = t_0 + t_0^{-1}$

(ii) T_1 is equal to each of

$$t_0^{-1} Y + Y^{-1} t_0,$$

$$Y t_0^{-1} + t_0 Y^{-1}$$

(iii) T_2 is equal to each of

$$q t_0^{-1} Y X + q^{-1} X^{-1} Y^{-1} t_0$$

$$q X t_0^{-1} Y + q^{-1} Y^{-1} t_0 X^{-1}$$

$$q Y X t_0^{-1} + q^{-1} t_0 X^{-1} Y^{-1}$$

(iv) T_3 is equal to each of

$$t_0^{-1} X + X^{-1} t_0,$$

$$X t_0^{-1} + t_0 X^{-1}$$

pf (i) ✓

$$\begin{aligned}
\text{(ii)} \quad t_0^{-1}Y + Y^{-1}t_0 &= t_0^{-1}t_0t_1 + t_1^{-1}t_0^{-1}t_0 \\
&= t_1 + t_1^{-1} \\
&= T_1
\end{aligned}$$

Also

$$\begin{aligned}
T_1 &= Y T_1 Y^{-1} \\
&= Y (t_0^{-1}Y + Y^{-1}t_0) Y^{-1} \\
&= Y t_0^{-1} + t_0 Y^{-1}
\end{aligned}$$

(iii) By L 14

$$\begin{aligned}
q X t_0^{-1}Y + q^{-1} Y^{-1}t_0 X^{-1} &= t_2^{-1} + t_2 \\
&= T_2
\end{aligned}$$

Now

$$\begin{aligned}
T_2 &= Y T_2 Y^{-1} \\
&= Y (q X t_0^{-1}Y + q^{-1} Y^{-1}t_0 X^{-1}) Y^{-1} \\
&= q Y X t_0^{-1} + q^{-1} t_0 X^{-1} Y^{-1}
\end{aligned}$$

And

$$\begin{aligned}
T_2 &= X^{-1} T_2 X \\
&= X^{-1} (q X t_0^{-1}Y + q^{-1} Y^{-1}t_0 X^{-1}) X \\
&= q t_0^{-1}Y X + q^{-1} X^{-1} Y^{-1} t_0
\end{aligned}$$

(iv) sim to (ii)

□

Since $T_0 = t_0 + t_0^{-1}$ the algebra $\hat{\mathfrak{H}}_2$ is gen by

$$X^{\pm 1}, Y^{\pm 1}, t_0, T_0$$

In terms of these gens the T_1, T_2, T_3 look as follows.

LEM 24

(i) T_1 is equal to each of

$$(T_0 - t_0)Y + Y^{-1}t_0, \quad Y(T_0 - t_0) + t_0Y^{-1}$$

(ii) T_2 is equal to each of

$$q(T_0 - t_0)YX + q^{-1}X^{-1}Y^{-1}t_0,$$

$$qX(T_0 - t_0)Y + q^{-1}Y^{-1}t_0X^{-1},$$

$$qYX(T_0 - t_0) + q^{-1}t_0X^{-1}Y^{-1}$$

(iii) T_3 is equal to each of

$$(T_0 - t_0)X + X^{-1}t_0, \quad X(T_0 - t_0) + t_0X^{-1}$$

pt In Lem 23 elem t_0^{-1} using $t_0^{-1} = T_0 - t_0$ \square

We now show how the t_0 "commutes past"
the $X^{\pm 1}, Y^{\pm 1}$

7

LEM 2.5

$$(i) \quad t_0 X = X^{-1} t_0 + X T_0 - T_3$$

$$(ii) \quad t_0 X^{-1} = X t_0 - X T_0 + T_3$$

$$(iii) \quad t_0 Y = Y^{-1} t_0 + Y T_0 - T_1$$

$$(iv) \quad t_0 Y^{-1} = Y t_0 - Y T_0 + T_1$$

pf (i), (ii) Come from L24 (iii)

(iii), (iv) Come from L24 (i)

□

8

Applying Z_4 -symmetry to L2S

and using

$$X \rightarrow Y \rightarrow q^{-1}X^{-1} \rightarrow q^{-1}Y^{-1} \rightarrow X$$

we get equations that show how t_1, t_2, t_3

commute past $X^{\pm 1}, Y^{\pm 1}$. [I won't write down]

LEM 26 The following hold in H_7 :

$$(i) \quad t_0 t_2 = q^{-1} t_3^{-1} T_1 - q^{-1} Y X^{-1}$$

$$(ii) \quad t_0^{-1} t_2^{-1} = q t_1 T_3 - q X^{-1} Y$$

pf (i)

$$q^{-1} Y X^{-1} = q^{-1} t_0 t_1 t_0^{-1} t_3^{-1}$$

$$= t_0 t_1 \underbrace{t_0^{-1} t_3^{-1}}_{t_3 t_0 t_1 t_2}$$

$$= t_0 t_1^2 t_2$$

$$= t_0 (t_1 T_1 - 1) t_2$$

$$= t_0 t_1 t_2 T_1 - t_0 t_2$$

$$= q^{-1} t_3^{-1} T_1 - t_0 t_2$$

$$\begin{aligned}
 \text{(ii)} \quad qX^{-1}Y &= q t_0^{-1} t_3^{-1} t_0 t_1 \\
 &= t_0^{-1} t_3^{-1} \underbrace{t_0 t_1 t_1^{-1} t_0^{-1} t_1^{-1} t_2^{-1}} \\
 &= t_0^{-1} (t_3^{-1} T_3 - 1) t_2^{-1} \\
 &= t_0^{-1} t_3^{-1} t_2^{-1} T_3 - t_0^{-1} t_2^{-1} \\
 &= q t_1 T_3 - t_0^{-1} t_2^{-1}
 \end{aligned}$$

□

Applying Z_4 -sym to L26 and using
 $X \rightarrow Y \rightarrow q^{-1} X^{-1} \rightarrow q^{-2} Y^{-1} \rightarrow X$

We obtain

LEM 27

$$\text{(i)} \quad t_1 t_3 = q^{-1} t_0^{-1} T_2 - q^{-2} X^{-1} Y^{-1}$$

$$\text{(ii)} \quad t_1^{-1} t_3^{-1} = q t_2 T_0 - Y^{-1} X^{-1}$$

$$\text{(iii)} \quad t_2 t_0 = q^{-1} t_1^{-1} T_3 - q^{-1} Y^{-1} X$$

$$\text{(iv)} \quad t_2^{-1} t_0^{-1} = q t_3 T_1 - q X Y^{-1}$$

$$\text{(v)} \quad t_3 t_1 = q^{-1} t_2^{-1} T_0 - X Y$$

$$\text{(vi)} \quad t_3^{-1} t_1^{-1} = q t_0 T_2 - q^{-2} Y X$$

DEF 28 let $\{C_i\}_{i \in \mathbb{Z}_4}$ denote the following elements of \hat{H}_g :

$$C_0 = \frac{1}{2} (gYX - g^{-1}XY)$$

$$C_1 = - \frac{1}{2} (g^{-1}YX^{-1} - gX^{-1}Y)$$

$$C_2 = \frac{1}{2} (gY^{-1}X^{-1} - g^{-1}X^{-1}Y^{-1})$$

$$C_3 = - \frac{1}{2} (g^{-1}Y^{-1}X - gXY^{-1})$$

LEM 29 Under \mathbb{Z}_4 -symmetry

$$C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_0$$

pf use

$$X \rightarrow Y \rightarrow g^{-1}X^{-1} \rightarrow g^{-1}Y^{-1} \rightarrow X$$

□

Prop 30 The following hold in \hat{H}_q

| | $t_0 T_2$ | $t_1 T_3$ | $t_2 T_0$ | $t_3 T_1$ | $T_0 T_2$ | $T_1 T_3$ |
|-------|-----------|-----------|-----------|-----------|-----------|-----------|
| C_0 | q | 1 | q^{-1} | 1 | $-q^{-1}$ | -1 |
| C_1 | 1 | q | 1 | q^{-1} | -1 | $-q^{-1}$ |
| C_2 | q^{-1} | 1 | q | 1 | $-q^{-1}$ | -1 |
| C_3 | 1 | q^{-1} | 1 | q | -1 | $-q^{-1}$ |

pf To get C_0 Compare LEM 27 $(v_i, (v_i))$

Use
$$t_3^{-1} t_1^{-1} = (T_3 - t_3)(T_1 - t_1)$$

$$= T_1 T_3 - t_1 T_3 - t_3 T_1 + t_3 t_1$$

So
$$q t_0 T_2 - q^2 Y X = T_1 T_3 - t_1 T_3 - t_3 T_1 + \underbrace{q^2 t_2^{-1}}_{T_2 - t_2} T_0 - X Y$$

To get C_1, C_2, C_3 from C_0 apply Z_4 -symmetry □

Next goal: Find B_3 action on

$$X + X^{-1}, \quad Y + Y^{-1}$$

DEF 31. Define

$$\begin{aligned} A &= Y + Y^{-1} \\ &= t_0 t_1 + (t_0 t_1)^{-1} \\ &= t_1 t_0 + (t_1 t_0)^{-1} \end{aligned}$$

by Cor 20

$$\begin{aligned} B &= X + X^{-1} \\ &= t_2 t_0 + (t_2 t_0)^{-1} \\ &= t_0 t_2 + (t_0 t_2)^{-1} \end{aligned}$$

$$\begin{aligned} C &= t_0 t_2 + (t_0 t_2)^{-1} \\ &= t_2 t_0 + (t_2 t_0)^{-1} \end{aligned}$$

\mathbb{F} arb $0 \neq q \in \mathbb{F}$

Continue to discuss univ DAHA \hat{H}_q of type (C_1^v, C_1)

Recall

$$\begin{aligned} A &= Y + Y^{-1} \\ &= t_0 t_1 + (t_0 t_1)^{-1} \\ &= t_1 t_0 + (t_1 t_0)^{-1} \end{aligned}$$

$$\begin{aligned} B &= X + X^{-1} \\ &= t_0 t_3 + (t_0 t_3)^{-1} \\ &= t_3 t_0 + (t_3 t_0)^{-1} \end{aligned}$$

$$\begin{aligned} C &= t_0 t_2 + (t_0 t_2)^{-1} \\ &= t_2 t_0 + (t_2 t_0)^{-1} \end{aligned}$$

Prop 32 The Braid group B_3 acts on A, B, C as follows:

(i) τ fixes each of A, B, C

(ii) ρ sends $A \rightarrow B \rightarrow C \rightarrow A$

(iii) σ swaps $A \leftrightarrow B$ and sends $C \rightarrow C'$ where

$$\begin{aligned} qC + q^{-1}C' + AB &= q^{-1}C + qC' + BA \\ &= (q^{-1}t_0 + qt_0^{-1}) T_2 + T_1 T_3 \end{aligned}$$

pf

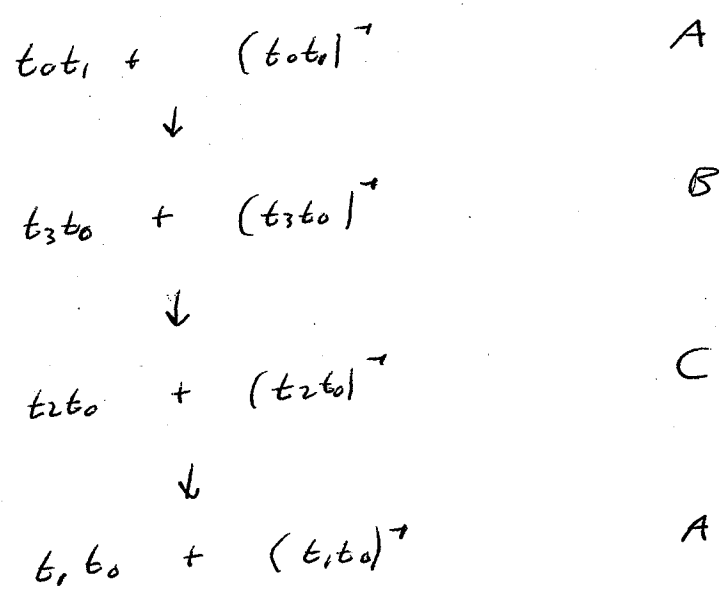
(i) Recall $\tau(h) = t_0^{-1} h t_0 \quad \forall h \in \hat{H}_g$

By Cor 20 (ii) t_0 commutes with each of A, B, C .

(ii) Recall ρ fixes t_0 and sends

$$t_3 \rightarrow t_2 \rightarrow t_1 \rightarrow t_0^{-1} t_3 t_0$$

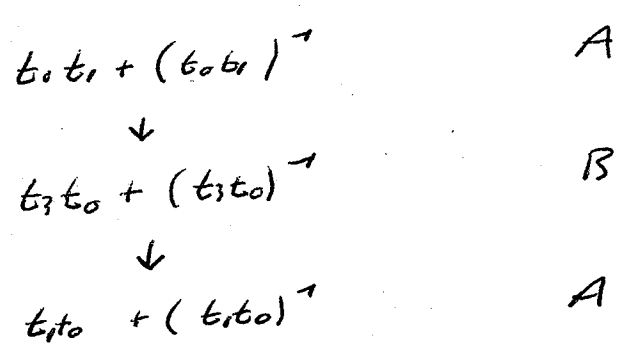
$\rho \downarrow$



(iii) Recall σ :

| | | | | |
|------|-----|------------------|------------------|-----|
| h | t_0 | t_1 | t_2 | t_3 |
| σ(h) | t_0 | t_0^{-1} t_3 t_0 | t_1 t_2 t_1^{-1} | t_1 |

$\sigma \downarrow$



Define $C' = \sigma(C)$

Show

$$qC + q^{-1}C' + AB = (q^{-1}t_0 + qt_0^{-1})T_2 + T_1T_3 \quad (*)$$

Obs

$$\begin{aligned} C' &= \sigma(C) \\ &= \sigma(t_0t_2) + \sigma(t_2^{-1}t_0^{-1}) \\ &\quad \parallel \\ &\quad t_0t_1t_2t_1^{-1} \\ &\quad \parallel \\ &\quad q^{-1}t_3^{-1}t_1^{-1} \\ &= q^{-1}t_3^{-1}t_1^{-1} + qt_1t_3 \end{aligned}$$

To get * show

$$qt_0t_2 + qt_2^{-1}t_0^{-1} + t_1t_3 + q^{-2}t_3^{-1}t_1^{-1}$$

+

$$YX + YX^{-1} + Y^{-1}X + Y^{-1}X^{-1}$$

=

$$(q^{-1}t_0 + q(t_0^{-1} - t_0)) | T_2 + T_1T_3$$

To verify this, elem $t_0t_2, t_2^{-1}t_0^{-1}, t_1t_3, t_3^{-1}t_1^{-1}$
using LEM 26, 27. simplify using Prop 30 (ex)

This gives *

We now show

$$q^{-1}C + qC' + BA = (q^{-1}b_0 + qb_0^{-1}) T_2 + T_1 T_3$$

To get this apply σ to each side of *

We saw σ swaps A and B and $C \rightarrow C'$

$$\text{also } \sigma(C) = \sigma^2(C) = \tau(C) = C$$

$$\text{Also } \sigma T_2 = T_2 \quad \sigma T_1 = T_3 \quad \sigma T_3 = T_1$$

□

Thm 33 Assume $q^4 \neq 1$. Then

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{(q^2 t_0 + q t_0^{-1}) T_1 + T_2 T_3}{q + q^{-1}}$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{(q^2 t_0 + q t_0^{-1}) T_3 + T_1 T_2}{q + q^{-1}}$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{(q^2 t_0 + q t_0^{-1}) T_2 + T_1 T_3}{q + q^{-1}} \quad \star$$

" Z_3 -symmetric Askey-Wilson relations"

pf: Get last equation \star

By Prop 32

$$\begin{aligned} qC + q^{-1}C' + AB &= R \\ q^{-1}C + qC' + BA &= R \\ R &= (q^2 t_0 + q t_0^{-1}) T_2 + T_1 T_3 \end{aligned}$$

Elim C' :

$$q(qC + AB - R) = q^{-1}(q^{-1}C + BA - R)$$

so

$$C(q^2 - q^{-2}) + qAB - q^{-1}BA = R(q - q^{-1})$$

so

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{R}{q + q^{-1}} \quad \checkmark$$

\star proved

Apply p twice to \star to get other two equations

COR 34 Assume $q^4 \neq 1$. Consider the subalgebra
of \hat{H}_q generated by A, B, C . In this subalgebra
each of

$$A + \frac{qBC - q^3CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^3AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^3BA}{q^2 - q^{-2}}$$

is central.

pf By th 33 and since q commutes with A, B, C \square

Motivated by Cor 34 we make a def.

DEF 35 Assume $q^4 \neq 1$. We define an F -algebra

Δ_q by generators and relations as follows.

The generators are A, B, C [view as abstract symbols,
not as el of \hat{H}_q]

Relations assert that each of

$$A + \frac{qBC - q^3CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^3AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^3BA}{q^2 - q^{-2}}$$

is central in Δ_q

Δ_q called the Universal Askey-Wilson algebra

We are going to show:

\exists injection of \mathbb{F} -algebra

$$\Delta_g \longrightarrow \hat{H}_g$$

that sends

$$A \longrightarrow t_0 t_1 + (t_0 t_1)^{-1}$$

$$B \longrightarrow t_0 t_3 + (t_0 t_3)^{-1}$$

$$C \longrightarrow t_0 t_2 + (t_0 t_2)^{-1}$$

In other words, the relations in Cor 34 are essentially the only ones relating A, B, C .

Next immediate goal: display a basis for \mathbb{F} -vector space \hat{H}_g

Going to show the following is a basis for \hat{H}_g :

$$Y^i X^j t_0^k T_1^r T_2^s T_3^t \quad i, j, k \in \mathbb{Z}, \quad r, s, t \in \mathbb{N}$$

The following is also a basis for \hat{H}_g :

$$Y^i X^j t_0^k T_0^l T_1^r T_2^s T_3^t \quad i, j \in \mathbb{Z}, \quad k \in \{0, 1\}, \quad l, r, s, t \in \mathbb{N}$$

We will also obtain reduction rules that show how to write any given element of \hat{H}_g as a linear comb of these basis vectors.

We mention some presentations for \hat{H}_g that illuminate various aspects

LEM 36 The \mathbb{F} -algebra \hat{H}_g is presented by generators and relations as follows.

The gens are $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$ The relations assert that each of $t_0 + t_0^{-1}$

$$t_0 X^{-1} + X t_0^{-1}$$
$$t_0 Y^{-1} + Y t_0^{-1} \quad q^{-1} t_0 X^{-1} Y^{-1} + q Y X t_0^{-1}$$

is central and

$$X X^{-1} = 1, \quad X^{-1} X = 1, \quad Y Y^{-1} = 1, \quad Y^{-1} Y = 1.$$
$$t_0 t_0^{-1} = 1, \quad t_0^{-1} t_0 = 1$$

pf let \check{H}_g denote the \mathbb{F} -algebra with above presentation show $\check{H}_g \cong \hat{H}_g$

claim: \exists \mathbb{F} -alg hom

$$\check{H}_g \longrightarrow \hat{H}_g$$
$$t_0 \longrightarrow t_0$$
$$X \longrightarrow t_3 t_0$$
$$Y \longrightarrow t_0 t_1$$



pf cl the def rels for \check{H}_g hold in \hat{H}_g by LEM 23

claim \exists \mathbb{F} -alg hom

$$\hat{H}_g \rightarrow \check{H}_g$$

$$t_0 \rightarrow t_0$$

$$t_1 \rightarrow t_0^{-1} Y$$

$$t_2 \rightarrow q^{-1} Y^{-1} t_0 X^{-1}$$

$$t_3 \rightarrow X t_0^{-1}$$



pf

show the def rels for \hat{H}_g hold in \check{H}_g :

D1 -

D2:

$$t_0 t_0^{-1} \text{ central in } \hat{H}_g? \checkmark$$

$$t_0^{-1} Y + Y^{-1} t_0 \text{ central in } \hat{H}_g?$$

"

$$t_0^{-1} (t_0 Y^{-1} + Y t_0^{-1}) t_0$$

"

$$t_0 Y^{-1} + Y t_0^{-1}$$

"central"

$$\underbrace{q^{-1} Y^{-1} t_0 X^{-1} + q X t_0^{-1} Y}_{\text{central in } \hat{H}_g?}$$

$$Y^{-1} (q^{-1} t_0 X^{-1} Y^{-1} + q Y X t_0^{-1}) Y$$

"

$$q^{-1} t_0 X^{-1} Y^{-1} + q Y X t_0^{-1}$$

"central"

$$X t_0^{-1} + t_0 X^{-1} \text{ central in } \hat{H}_g? \checkmark$$

D3:

$$t_0 (t_0^{-1} Y) (q^{-1} Y^{-1} t_0 X^{-1}) (X t_0^{-1}) \stackrel{?}{=} q^{-1}$$

ok.

claim \star, \star are inverses:

$$\begin{array}{l} \checkmark \\ H_9 \end{array} \rightarrow \begin{array}{l} \wedge \\ H_9 \end{array} \rightarrow \begin{array}{l} \checkmark \\ H_9 \end{array}$$

$$t_0 \rightarrow t_0 \rightarrow t_0 \checkmark$$

$$X \rightarrow t_3 t_0 \rightarrow X t_0^{-1} t_0 = X \checkmark$$

$$Y \rightarrow t_0 t_1 \rightarrow t_0 t_0^{-1} Y = Y \checkmark$$

$$\begin{array}{l} \wedge \\ H_9 \end{array} \rightarrow \begin{array}{l} \checkmark \\ H_9 \end{array} \rightarrow \begin{array}{l} \wedge \\ H_9 \end{array}$$

$$t_0 \rightarrow t_0 \rightarrow t_0$$

$$t_1 \rightarrow t_0^{-1} Y \rightarrow t_0^{-1} t_0 t_1 = t_1$$

$$t_2 \rightarrow q^{-1} Y^{-1} t_0 X^{-1} \rightarrow \underbrace{q^{-1} t_0^{-1} t_0^{-1} t_0^{-1} t_0^{-1} t_0^{-1} t_3^{-1}}_{\text{"}} = t_2 \checkmark$$

$$t_3 \rightarrow X t_0^{-1} \rightarrow t_3 t_0 t_0^{-1} = t_3 \checkmark$$

□

F arb $0 \neq q \in F$

Continue to discuss univ DAHA \hat{H}_q of type (C_1^V, C_1)

Gens $\{t_i^{\pm 1}\}_{i \in \mathbb{I}}$ $\mathbb{I} = \{0, 1, 2, 3\}$

Rebs $t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad i \in \mathbb{I} \quad (01)$

$t_i + t_i^{-1}$ is central $" \quad (02)$

$t_0 t_1 t_2 t_3 = q^{-1} \quad (03)$

Write $T_i = t_i + t_i^{-1} \quad i \in \mathbb{I}$

$X = t_3 t_0$

$Y = t_0 t_1$

Next goal: show the following is a basis for F -vector space \hat{H}_q :

$Y^i X^j t_0^k T_1^r T_2^s T_3^t$

$i, j, k \in \mathbb{Z},$
 $r, s, t \in \mathbb{N} = \{0, 1, 2, \dots\}$ *

We will first show the following is a basis for \hat{H}_q

$Y^i X^j t_0^k T_0^l T_1^r T_2^s T_3^t$

$i, j \in \mathbb{Z}$ **
 $k \in \{0, 1, 2\}, \quad l, r, s, t \in \mathbb{N}$

Last time

In Lem 36 We saw a presentation

of \hat{H}_g involving the gens $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}$

Here is a variation of that presentation

LEM 37 The \mathbb{F} -alg \hat{H}_g has a presentation

by gens $X^{\pm 1}, Y^{\pm 1}, t_0^{\pm 1}, \{T_i\}_{i \in \mathbb{I}}$

and rels

$$XX^{-1} = 1, \quad X^{-1}X = 1, \quad YY^{-1} = 1, \quad Y^{-1}Y = 1$$

$$t_0 t_0^{-1} = 1, \quad t_0^{-1} t_0 = 1$$

$$T_i \text{ central } \quad i \in \mathbb{I}$$

$$T_0 = t_0 + t_0^{-1}$$

$$T_1 = t_0 Y^{-1} + Y t_0^{-1}$$

$$T_2 = Y^{-1} t_0 X^{-1} Y^{-1} + Y Y X t_0^{-1}$$

$$T_3 = t_0 X^{-1} + X t_0^{-1}$$

Pf use L36

□

In Lem 37 let's remove t_0^{-1} using

$$t_0^{-1} = T_0 - t_0$$

this gives

LEM 38 The \mathbb{F} -alg \hat{H}_q has a presentation by generators

$$X^{\pm 1}, Y^{\pm 1}, t_0, \{T_i\}_{i \in \mathbb{I}}$$

and rels

$$X X^{-1} = 1, X^{-1} X = 1, Y Y^{-1} = 1, Y^{-1} Y = 1$$

$$T_i \text{ central } i \in \mathbb{I}$$

$$t_0^2 = t_0 T_0 - 1$$

$$T_1 = t_0 Y^{-1} + Y(T_0 - t_0)$$

$$T_2 = q^{-1} t_0 X^{-1} Y^{-1} + q Y X (T_0 - t_0)$$

$$T_3 = t_0 X^{-1} + X(T_0 - t_0)$$