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COUNTING 4–VERTEX CONFIGURATIONS IN P–AND Q–POLYNOMIAL ASSOCIATION SCHEMES

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Abstract

An open problem is whether certain symmetric association schemes arising from the finite projective, orthogonal, unitary, and symplectic geometries, all with the so-called P– and Q– polynomial property, are the unique ones with their own intersection numbers. The following result, which applies to all P– and Q– polynomial schemes, may shed light on this problem. If we say 4–tuples (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) of elements taken from the scheme $Y = (X, [R_i]\ 0 \leqslant i \leqslant d)$ have the same type if $(x_i, x_j) \in R_t$ implies $(y_i, y_j) \in R_t$ $(1 \leqslant i,j \leqslant 4)$, then we show the total number n_T of 4–tuples from Y of type T can be computed from the intersection numbers of Y and the numbers n_S for at most $[d/2]$ types S.

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For any positive integer d set $[d] = \{0,1,\dots,d\}$. A symmetric d-class association scheme (or simply, scheme) is a configuration $Y = (X, \{R_i\}_{i \in [d]})$ consisting of a finite set X and symmetric relations $R_0, R_1, \dots, R_d$ on X where i) $R_0 = \{(x,x) \mid x \in X\}$ is the identity relation, ii) for every $x,y \in X$, $(x,y) \in R_i$ for exactly one $i \in [d]$, and iii) for any $h,i,j \in [d]$ and any $x,y \in X$ with $(x,y) \in R_h$, the number of $z \in X$ where $(x,z) \in R_i$ and $(z,y) \in R_j$ depends only on h, i, and j. We denote this number by the intersection number p^h_{ij}.

The set X of all d-dimensional (maximal isotropic) subspaces in a projective (orthogonal, unitary, or symplectic) geometry forms such a scheme, if we set $(x,y) \in R_i$ for any $x,y \in X$ where $\dim(x \cap y) = d-i$, and in fact these examples are among the few known schemes with the so called P- and Q- polynomial property (defined below). Here we give new information on P- and Q- polynomial schemes that may help in their classification. See Bannai and Ito[1], Cohen[2], Egawa[3], Huang[4], Leonard[5], Neumaier[6], Sprague[7], and Terwilliger[8-12].

We fix a scheme $Y = (X, \{R_i\}_{i \in [d]})$ with $n = |X|$, set $k_i = p^0_{ii}$ $(i \in [d])$, and set $k = k_1$. Let EK_4 be the set of all 2-element subsets of a 4-element set K_4. The level $\lambda(T)$ of a function $T: EK_4 \to [d]$ (henceforth called a type function) is the minimal integer in its range, and any 4-tuple (x_1, x_2, x_3, x_4) of elements in X is said to have type T if $(x_i, x_j) \in R_{T(i,j)}$ for all $(i,j) \in$

EK_4. Denote by n_T the total number of 4-tuples from X of type T, and for any $i \in [d]$ set $n_i^* = n_C$, where $C = C(i)$ is the constant function of level i. We prove the following.

THEOREM 1. Let Y be a d-class P- and Q- polynomial scheme and let T be any type function. Then n_T can be computed from the intersection numbers of Y and $n_1^*, n_2^*, \ldots, n_p^*$, where p is the minimum of $\lambda(T)$ and the integer part of d/2.

We review some preliminaries found in Bannai and Ito[1] before proving the intermediate results Theorem 6 and Corollary 7, which may be of independent interest, and then prove Theorem 1.

Let A(Y) be the <u>Bose-Mesner Algebra</u> of Y (over $\mathbb{R}$), acting on a Euclidean space V, $\langle , \rangle$, that possesses an orthonormal basis which we identify with X. Let $V = \oplus V_i$ $(i \in [d])$ be the orthogonal decomposition of V into maximal A(Y)-invariant subspaces, let π_i denote the projection $V \to V_i$, and let the matrix E_i represent π_i relative to X $(i \in [d])$. The <u>Krein parameters</u> q^h_{ij} $(h, i, j \in [d])$ are defined by

$$E_i \circ E_j = n^{-1} \sum_{h \in [d]} q^h_{ij} E_h$$

where $\circ$ is Hadamard multiplication. Y is called P- <u>and</u> Q- <u>polynomial</u> (with respect to the given ordering of the relations and projections) if the <u>intersection matrix</u> B and its <u>dual</u> B^*, with ijth entries p^i_{1j} and q^i_{1j}, respectively $(i,j \in [d])$, are <u>tri-diagonal</u>, with non-zero entries directly above and below the main diagonal. In this paper we always assume Y is P- and Q- polynomial. For convenience set $F_i = \{\pi_0, \pi_1, \ldots, \pi_i\}$ $(i \in [d])$.

REMARK 2. Set $m_j = \dim V_j$ $(j \in [d])$. By [8], for $i,j \in [d]$ the <u>cosine</u> $c_i^{(j)}$ of the angle between $\pi_j(x)$ and $\pi_j(y)$ $((x,y) \in R_i)$ is

$$c_i^{(j)} = n m_j^{-1} \langle \pi_j(x), \pi_j(y) \rangle \qquad (1)$$

and can be computed from the intersection numbers of Y. We also have

$$m_r m_s c_i^{(r)} c_i^{(s)} = \sum_{h \in [d]} q^h_{rs} m_h c_i^{(h)} \qquad (i,r,s \in [d]). \qquad (2)$$

We write $c_i = c_i^{(1)}$, $c^{(i)} = c_1^{(i)}$ $(i \in [d])$, and by Bannai and Ito[1,p365] have

$$c_i \neq c_j \text{ and } c^{(i)} \neq c^{(j)} \text{ if } i \neq j \qquad (i,j \in [d]). \qquad (3)$$

Let the matrix Q have ijth entry $m_j c_i^{(j)}$, $(i,j \in [d])$. By Bannai and Ito[1] Q is essentially Vandermonde and hence nonsingular.

DEFINITION 3. Let G be the Cartesian product $[d] \times [d]$, and write $u = (u_x, u_y)$ for $u \in G$. Let $\partial(u,v) = |u_x - v_x| + |u_y - v_y|$ be the <u>distance</u> between $u,v \in G$, and for $u \in G$, $r \in [d]$, let $D(u,r) = \{v \mid v \in G, \partial(u,v) \leq r\}$ be the <u>diamond</u> of radius r centered at u. For $i \in \mathbb{Z}$ let $G_i = \{u \mid u \in G, u_x > u_y + i\}$. We will use the following constants in Theorem 6.

DEFINITION 4. A <u>path</u> P <u>of length</u> t <u>in</u> G_j is a sequence $(u_0, u_1, \ldots, u_t)$ with $u_i \in G_j$ $(i \in [t])$ and $\partial(u_i, u_{i+1}) \leq 1$ $(i \in [t-1])$. We say P goes from u_0 to u_t and write $|P| = t$. Abusing notation we write $P \in G_j$. If $|P| \geq 1$ set $P^* = (u_0, u_1, \ldots u_{t-1})$ and $P^{**} = (u_1, u_2, \ldots, u_{t-1})$, with $P^{**} = \phi$ if $t = 1$, and assign to P a sequence $(f_u \mid u \in P^*)$ of integers as follows. For each $i \in [t-1]$, let $u = u_i$, $u = (r,s)$ and set f_u equal to $p^r{}_{1,r+1}$, $p^r{}_{1,r-1}$, $-p^s{}_{1,s+1}$, $-p^s{}_{1,s-1}$, or $p^r{}_{1r} - p^s{}_{1s}$, depending on whether $u_{i+1} = (r+1,s)$, $(r-1,s)$, $(r,s+1)$, $(r,s-1)$, or (r,s), respectively. For all paths P in G with $|P| \geq 1$ and $P^{**} \in G_0$ define the <u>positive weight</u>

$$w^+(P) = f_{u_0} \prod_{u \in P^{**}} f_u \ (c_{u_x} - c_{u_y})^{-1}.$$

Define the <u>negative weight</u> of any path P in G for which $P^* \in G_0$ by

$$w^-(P) = \prod_{u \in P^*} f_u (c_{u_x} - c_{u_y})^{-1},$$

and set $w^-(P) = 1$ if $|P| = 0$. Finally for all $t \in [d]$, all $u \in G_t$ and all $v \in G_{-1}$ let $a_v^{\pm}(u,t) = \Sigma\ w^{\pm}(P)$, the sum being over all paths P from v to u having length t in case (−) and length t+1 in case (+).

DEFINITION 5. For all $x,y \in X$ and all $i,j \in [d]$, set $P_{ij}(x,y) = \Sigma z$, the sum (in V) being over all $z \in X$ where $(x,z) \in R_i$ and $(z,y) \in R_j$.

THEOREM 6. For $t \in [d]$, $u \in G_t$, and $x,y \in X$, we have

equation $(u,t)^-$: $$\sum_{v \in D(u,t)} a_v^-(u,t)\, \pi(P_v(x,y) - P_v(y,x)) = 0 \qquad (\pi \in F_t)$$

and

equation $(u,t)^+$: $$\sum_{v \in D(u,t+1)} a_v^+(u,t)\, \pi(P_v(x,y) + P_v(y,x)) = 0 \qquad (\pi \in F_t).$$

The constants $a_v^{\pm}(u,t)$ are from Definition 4.

<u>Proof.</u> Fix $x,y \in X$. By Bannai and Ito[1,p126] we have

$$\sum_{z \in X} \langle \pi_r x, \pi_r y\rangle \langle \pi_s y, \pi_s z\rangle\, \pi_t z = 0$$

for all $r,s,t \in [d]$ with $q^r_{st} = 0$. Summing over the possible inner products first, the Q-polynomial property implies

$$\sum_{i,j \in [d]} c_i^{(r)} c_j^{(s)}\, \pi P_{ij}(x,y) = 0 \qquad r,s \in [d],\ \pi \in F_t,\ t < |r-s|. \qquad (4)$$

Let $N = \{e_i \mid i \in [d]\}$ be the standard basis for $\mathbb{R}^{d+1}$, let $N^* = \{e_i^* \mid e_i^*$ the ith column of Q, $i \in [d]\}$ be another basis, and set $W = \mathbb{R}^{d+1} \otimes \mathbb{R}^{d+1}$. We abreviate $e_{ij} = e_i \otimes e_j$, $e_{ij}^* = e_i^* \otimes e_j^*$. For $t \in [d]$ define $H_t, W_t \in W$ by $H_t = \mathrm{span}\{ e_{ij} \mid |i-j| > t,\ i,j \in [d] \}$, $W_t = \mathrm{span}\{ e_{ij}^* \mid |i-j| > t,\ i,j \in [d] \}$, and decompose $H_t = H_t^- \oplus H_t^+$, setting $H_t^- = \mathrm{span}\{ e_{ij} - e_{ji} \mid (i,j) \in G_t\}$, and $H_t^+ = \mathrm{span}\{ e_{ij} + e_{ji} \mid (i,j) \in G_t\}$. We decompose $W_t = W_t^- \oplus W_t^+$ similarily, and note $\dim(W_t^{\pm}) = (d-t+1)(d-t)/2 \quad (t \in [d])$. Setting

$$e_u^-(t) = \sum_{(i,j) \in D(u,t)} a_{ij}^-(u,t)\,(e_{ij} - e_{ji}) \qquad t \in [d] \quad u \in G_t$$

and

$$e_u^+(t) = \sum_{(i,j) \in D(u,t+1)} a_{ij}^+(u,t)\,(e_{ij} + e_{ji}) \qquad t \in [d] \quad u \in G_t,$$

by (4) it suffices to prove $\{e_u^-(t) \mid u \in G_t\}$ and $\{e_u^+(t) \mid u \in G_t\}$ form bases for W_t^- and W_t^+, respectively. Define the linear transformations $M, M^* : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ by

$$M(e_i) = c_i(e_i) \qquad M^*(e_i^*) = c^{(i)}(e_i^*) \qquad (i \in [d]). \tag{5}$$

Let $M_1 : H_0 \to H_0$ be the restriction of $M \otimes \mathbb{I} - \mathbb{I} \otimes M$ to its invariant subspace H_0, and let $M_1^* : W_0 \to W_0$ be the restiction of $M^* \otimes \mathbb{I} - \mathbb{I} \otimes M^*$ to W_0. By (3),

M_1 and $M_1{}^*$ are invertible, and in fact $M_1(H_i{}^{\mp}) = H_i{}^{\pm}$, $M_1{}^*(W_i{}^{\mp}) = W_i{}^{\pm}$, for all $i \in [d]$. Since by (2) and [1,p72] the matrices representing M and M^* relative to N^* and N are the tri-diagonal matrices $m_1{}^{-1}B^*$ and $k^{-1}B$, repectively, we have $M_1(W_i{}^{\pm}) \subseteq W_{i-1}^{\mp}$ and $M_1{}^*(H_i{}^{\pm}) \subseteq H_{i-1}^{\mp}$ for all i $(1 \leq i \leq d)$. Since Definition 4 and a routine induction on t shows
$e_{rs}{}^-(t) = k^t(M_1{}^{-1}M_1{}^*)^t(e_{rs} - e_{sr})$ and $e_{rs}{}^+(t) = kM_1{}^*e_{rs}{}^-(t)$
$(t \in [d], (r,s) \in G_t)$, it suffices to show

$$W_t{}^- = (M_1{}^{-1}M_1{}^*)^t H_t{}^- \qquad (t \in [d]). \tag{6}$$

This equation follows from $H_0{}^- \cap W_t{}^- = W_t{}^-$ and $H_t{}^- \cap W_0 = H_t{}^-$ if we can show

$$M_1(W_{r+1}{}^- \cap H_s{}^-) = W_r{}^+ \cap H_s{}^+ \qquad (r \in [d-1],\ \ s \in [d]) \tag{7}$$

and

$$M_1{}^*(W_r{}^- \cap H_{s+1}{}^-) = W_r{}^+ \cap H_s{}^+ \qquad (r \in [d],\ \ s \in [d-1]). \tag{8}$$

To prove (7), it suffices to prove

$$M_1(W_{r+1}{}^-) = W_r{}^+ \cap H_0{}^+ \qquad r \in [d-1] \tag{9}$$

for we would then have $M_1(W_{r+1}{}^- \cap H_s{}^-) = M_1(W_{r+1}{}^-) \cap M_1(H_s{}^-) =$

$W_r^+ \cap H_0^+ \cap H_s^+ = W_r^+ \cap H_s^+$. Since $M_1(W_{r+1}^-) = M_1(W_{r+1}^- \cap H_0^-) = M_1(W_{r+1}^-) \cap M_1(H_0^-) \subseteq W_r^+ \cap H_0^+$, to prove (9) we need only check

$$\dim(W_r^+ \cap H_0^+) = (d-r)(d-r-1)/2 \qquad (10)$$

$$= \dim(W_{r+1}^-).$$

For this, we produce a dimension d-r subspace $S_r \subseteq W_r^+$ that intersects $W_r^+ \cap H_0^+$ trivially, where

$$W_r^+ = W_r^+ \cap H_0^+ + S_r \qquad (r \in [d]). \qquad (11)$$

We take $S_r = \text{span}\{ e_{i0}^* + e_{0i}^* \mid r+1 \leq i \leq d \}$. Upon writing these vectors in terms of $\{ e_{ij} \mid i,j \in [d]\}$ we find a linear combination

$$\sum_{i=r+1}^{d} \alpha_i(e_{i0}^* + e_{0i}^*) \in H_0^+$$

is equivalent to $Q[0,0,\ldots,0,\alpha_{r+1},\alpha_{r+2},\ldots,\alpha_d]^t = 0$, so $S_r \cap H_0^+$ is trivial. By writing the vectors

$$e_{xy}^* + e_{yx}^* - \sum_{i=r+1}^{d} q^i_{xy}(e_{i0}^* + e_{0i}^*) \qquad ((x,y) \in G_r)$$

in terms of $\{ e_{ij} \mid i,j \in [d] \}$ and applying (2), we find they are all in $W_r^+ \cap H_0^+$, yielding (11) and proving (10). Line (8) is proved by interchanging the roles of W_r, H_s, and M_1, M_1^* in the proof of (7). □

COROLLARY 7. let $t \in [d]$, set $L(t) = \{ (i,j) \mid 0 \leq i < t \text{ or } 0 \leq j \leq t \}$, and pick $u \in G$. From t, u, and the intersection numbers of Y we can compute

$\{ g_v \mid g_v \in \mathbb{R}, v \in L(t) \}$ where

$$\pi P_u(x,y) = \sum_{v \in L(t)} g_v \, \pi P_v(x,y) \tag{12}$$

for all $\pi \in F_t$ and all $x,y \in X$.

<u>Proof</u>. Set $u = (r,s)$ $(r,s \in [d])$. The Corollary is true if it is true under the assumption $u \in L(t+1) \setminus L(t)$ $(t \in [d-1])$, so we make this assumption and consider two cases.

<u>Case</u> 1. $t = r < s$. Here (12) follows from equation $(s,0,t)^-$ of Theorem 6.

<u>Case</u> 2. $t+1 = s \leq r$. We first apply the equation $a_u^+(r,0,t)(r,0,t+1)^- + a_u^-(r,0,t+1)(r,0,t)^+$ to obtain the vector $\pi P_u(x,y)$ in (12) as a linear combination of those $\pi P_{u'}(x,y)$ for which either i) $u' \in L(t)$ or ii) both $u' \in L(t+1) \setminus L(t)$ and $r' < s'$ $(u' = (r',s'))$, and then apply case 1 to those $\pi P_{u'}(x,y)$ of the second type. □

<u>Proof of Theorem</u> 1. Let $\lambda = \lambda(T)$. For each type function S let e(S) be the number of $u \in EK_4$ for which $S(u) = \lambda(S)$, except that $e(S) = 1$ if there are exactly two $u,v \in EK$ with $S(u), S(v) = \lambda(S)$, and these u,v are disjoint. Define a partial order $\ll$ on the set of all type functions, letting R, S satisfy $R \ll S$ if either i) $\lambda(R) < \lambda(S)$, ii) $\lambda(R) = \lambda(S)$ and $e(R) > e(S)$, or iii) $\lambda(R) = \lambda(S)$, $e(R) = e(S)$, and $R(u) \leq S(u)$ for all $u \in$

EK_4, with strict inequality for some u. It now suffices to assume T is either not constant or $\lambda > [d/2]$, and show n_T is computable from those $n_{T'}$ for which $T' \ll T$. There are 3 cases, the first being

1) $\lambda > [d/2]$.

If $\lambda \leq [d/2]$ then T is not constant, so we can label $K_4 = \{x,y,z,w\}$ so that

$T(x,z) > T(x,y) = \lambda$, and either

2) $T(y,z) = \lambda$

3a) $T(x,z) \geq T(u) > \lambda$ for all $u \in EK_4$ containing x or y, or

3b) $T(y,w)$ and $T(x,w)$ equal λ, and $T(x,z) \geq T(u) > \lambda$ for all $u \in EK_4$ containing z.

Let e, f, g, r, and s denote the integers $T(z,w)$, $T(x,w)$, $T(y,w)$, $T(y,z)$, and $T(x,z)$, respectively. In case 1 we label K_4 so $T(x,y) = \lambda$. For convenience set $(\sigma,\epsilon) = (d-[d/2]-1, [d/2]+1)$, $(\lambda-1,\lambda+1)$, or $(\min(\lambda,d-r), r)$, in case 1, 2, and 3, respectively, and let $J = [\sigma+\epsilon] \setminus [\epsilon-1]$. For each $i \in [d]$, let $S^{(i)}$ be the type function with $S^{(i)}(x,z) = i$ that agrees with T on all $p \in EK_4$ with $p \neq (x,z)$. Set $n_i = n_{S^{(i)}}$ $(i \in [d])$ and note $n_s = n_T$. By (1), for all $h \in [d]$ and in particular for all $h \in [\sigma]$, we have

$$\sum_{i \in [d]} n_i c_i^{(h)} = n\, m_h^{-1} \sum \langle \pi_h P_{er}(u,v), \pi_h P_{f\lambda}(u,v) \rangle, \qquad (13)$$

the second sum being over all $u,v \in X$ with $(u,v) \in R_g$. By Corollary 7 we replace each vector $\pi_h P_{er}(u,v)$ in (13) by a known linear combination of those $\pi_h P_{e'r'}(u,v)$ for which $e' < h$ or $r' \leq h$. In each case 1, 2, 3a, 3b and for each $h \in [d]$, evaluation of the inner product in (13) shows the right side of that equation is computable from the intersection numbers of Y and those $n_{T'}$ for which $T' \ll T$. Now the constants n_i ($i \in [\epsilon-1]$) each represent some $n_{T'}$ for which $T' \ll T$, and the P-polynomial property implies $n_j = 0$ for $j > \epsilon + \sigma$, so using (13) we can compute $\{ q_h \mid q_h \in \mathbb{R}, h \in [\sigma] \}$ from the intersection numbers and those $n_{T'}$ for which $T' \ll T$, such that

$$\sum_{i \in J} n_i c_i^{(h)} = q_h \qquad (h \in [\sigma]).$$

By remark 2 the coefficient matrix for the above system is essentially Vandermonde and hence nonsingular, allowing us to solve for each n_i ($i \in J$). □

REMARK. For each $i,j \in [d]$ let $D = D(i,j)$ be the square matrix of degree $(d+1)^2$, with rows and columns indexed by $G = [d] \times [d]$, where

$$D_{u,v} = \sum \langle \pi_j P_u(x,y), \pi_j P_v(x,y) \rangle \qquad u \in G, \quad v \in G,$$

the sum being over all $x,y \in X$ with $(x,y) \in R_i$. Equations like (13) show D is determined by the free parameters $n_1^*, \ldots, n_i^*$, and the intersection numbers of Y. The positive semi-definiteness of each D yields bounds on the free parameters and hence estimates for the n_T's.

REFERENCES

1. E. Bannai and T. Ito, "Algebraic Combinatorics I: Association Schemes," Benjamin-Cummings Lecture Note Series 58, Menlo Park, CA 1984.
2. A. Cohen, A synopsis of known distance-regular graphs with large diameters, Mathematisch Centrum Report ZW 168/81, Amsterdam, 1981.
3. Y. Egawa, Characterization of H(n,q) by the parameters, J. Combin. Theory Ser A, 31 (1981), 108-125.
4. T. Huang, Ph.D. Thesis, The Ohio State University, 1985.
5. D. Leonard, Parameters of association schemes that are both P- and Q- polynomial, J. Combin. Theory Ser. A 36 (1984), 355-363.
6. A. Neumaier, Characterization of a class of distance-regular graphs, J. Reine Angew. Math., to appear.
7. A. Sprague, Characterization of the projective graphs, J. Combin. Theory Ser. B 24 (1978), 294-300.
8. P. Terwilliger, A characterization of the P- and Q- polynomial schemes, submitted to J. Combin. Theory Ser. A.
9. P. Terwilliger, A class of distance-regular graphs that are Q-polynomial, submitted to J. Combin. Theory Ser. B.
10. P. Terwilliger, Root systems and the Johnson and Hamming graphs, submitted to Europ. J. Combin. Theory.
11. P. Terwilliger, The classification of distance-regular graphs of type IIB, submitted to Combinatorica.
12. P. Terwilliger, Towards a classification of distance-regular graphs with the Q-polynomial property, submitted to J. Combin. Theory Ser. B.