Classical t-Design. Yiwen Bai 846

Date

Definition 1.23 (t-(v,k,λ) Design)

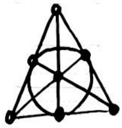
Let t, v, k, λ be positive integers and $t \le k \le v$.
V is a finite set consisting of v points.
$V^{(k)}$ is the set of all k-element subsets of V.
$\mathcal{B} \subset V^{(k)}$
A pair $(V, \mathcal{B})$ is called a t-(v,k,λ) design if there exists a positive integer λ such that for any $T \in V^{(t)}$ the following holds:

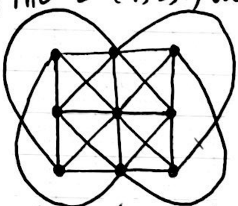
$$\left| \{ B \in \mathcal{B} \mid T \subset B \} \right| = \lambda$$

An element of V is called a point and an element of $\mathcal{B}$ is called a block.

$$b = |\mathcal{B}| = \frac{\binom{v}{t}}{\binom{k}{t}} \lambda$$

Example ① The 2-(7,3,1) design ② The 2-(9,3,1) design.





③ Trivial t-design. let $\mathcal{B} = V^{(k)}$ then $(V, \mathcal{B})$ is a t-design for any $t \le k$

$$\lambda = \binom{v-t}{k-t}$$

Since for any $T \in V^{(t)}$ there are $\binom{v-t}{k-t}$ k-element subsets of V containing T.

Remark: In some case, k-element subsets are allowed to appear repeatedly as elements of $\mathcal{B}$.
If repeated blocks are not allowed, a design is called a simple design.

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In what follows, we consider simple designs unless otherwise stated

Definition 1.27 (Isomorphism of block designs)
Two t-designs $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ are said to be isomorphic if there exists a bijection $\sigma: V \to V'$ which induces a bijection from $\mathcal{B}$ to $\mathcal{B}'$.
i.e. there exists a bijection $\sigma: V \to V'$
and a bijection $\rho: \mathcal{B} \to \mathcal{B}'$
such that $P \in B$ in $(V, \mathcal{B})$ implies
$\sigma(P) \in \rho(B)$ in $(V', \mathcal{B}')$

The set of isomorphisms of designs from $(V, \mathcal{B})$ to itself forms a group. This group is called the automorphism group of the design $(V, \mathcal{B})$ and denoted by $\mathrm{Aut}(V, \mathcal{B})$

Proposition 1.30

Let $(V, \mathcal{B})$ be a t-(v, k, λ) design. For any integer s with $0 \leq s \leq t$, $(V, \mathcal{B})$ is an s-(v, k, λ_s) design.

$$\lambda_s = \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} \lambda$$

(The concept of a 0-design has no special meaning but $\lambda_0 = b$ can be regarded as the number of blocks)

Proof: For $S \in V^{(s)}$, let $\lambda(S) = \left| \{ B \in \mathcal{B} \mid S \subset B \} \right|$

We prove that $\lambda(S)$ is independent of the choice of S by counting the number N of pairs (T, B) of $T \in V^{(t)}$, $B \in \mathcal{B}$ such that $S \subset T \subset B$

① $\left| \{ T \in V^{(t)} \mid S \subset T \} \right| = \binom{v-s}{t-s}$

For any $T \in V^{(t)}$, $\left| \{ B \in \mathcal{B} \mid T \subset B \} \right| = \lambda$

$\Rightarrow N = \binom{v-s}{t-s} \lambda$

② For any $B \in \mathcal{B}$, there are $\binom{k-s}{t-s}$ t-element subsets of B containing S.

$\Rightarrow N = \lambda(S) \binom{k-s}{t-s}$

Thus, $\lambda(S) = \dfrac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} \lambda$ for any $S \in V^{(s)}$

From the above proof, we obtain the necessary condition for the existence of t-design.

Proposition 1.31

A t-(v, k, λ) design exists only if λ_s is a integer for any integer $0 \leq s \leq t$.

Let $(V, \mathcal{B})$ be a 2-(v,k,λ) design and let r be the number of blocks containing a fixed point p.

$$r = \lambda_1 = \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}\lambda = \frac{v-1}{k-1}\lambda.$$

$$b = \lambda_0 = \frac{\binom{v}{t}}{\binom{k}{t}}\lambda = \frac{v(v-1)}{k(t-1)}\lambda.$$

we have

$$r(k-1) = (v-1)\lambda$$

and

$$bk = rv.$$

A non-trivial 2-(v,k,λ) design is called a balanced incomplete block design (BIBD), and a t-$(v,k,1)$ design with $\lambda = 1$ is called a Steiner system.

Definition 1.33 (Incidence matrix of a design $(V, \mathcal{B})$)
For a design $(V, \mathcal{B})$, we define a matrix M whose rows are indexed by $\mathcal{B}$ and whose columns are indexed by V.
For $P \in V$, $B \in \mathcal{B}$, define the (B,P)-entry of M as

$$M(B,P) = \begin{cases} 0 & \text{if } P \notin B \\ 1 & \text{if } P \in B \end{cases}$$

The matrix M is called the incidence matrix of a design $(V, \mathcal{B})$
The incidence matrix of the 2-$(7,3,1)$ design in Example ① is given as follows:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Definition 1.34 (Complementary design)

For a t-(v,k,λ) design $(V,\mathcal{B})$, define $\mathcal{B}'=\{V\backslash B \mid B\in\mathcal{B}\}$. Then $(V,\mathcal{B}')$ becomes a block design, let t-(v,k',λ') be its parameters. Then

$$k'=v-k, \quad \lambda'=\frac{\binom{v-k}{t}}{\binom{k}{t}}\lambda$$

The design $(V,\mathcal{B}')$ is called the complementary design of $(V,\mathcal{B})$.