ch 8 Special Counting Sequences

8.1 Catalan numbers

For $n \geq 0$ define

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

"nth Catalan number"

n	0	1	2	3	4	5	...
C_n	1	1	2	5	14	42	...

In Section 7.6 we considered triangular divisions of a convex $(n+1)$-gon

the answer was

$$\frac{1}{n}\binom{2n-2}{n-1} = C_{n-1}$$

LEM $\forall n \quad n \geq 1$

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}$$

pf

$$\frac{1}{n+1}\binom{2n}{n} \stackrel{?}{=} \binom{2n}{n} - \binom{2n}{n-1}$$

$$\binom{2n}{n-1} \stackrel{?}{=} \binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{n}{n+1}\binom{2n}{n}$$

$$\frac{(2n)!}{(n-1)!\,(n+1)!} \stackrel{?}{=} \frac{n}{n+1}\,\frac{(2n)!}{n!\,n!}$$

✓

$\square$

An interpretation of C_n

Fix integer $n \geq 0$

Consider a permutation

$$a_1 a_2 \cdots a_{2n}$$

of multiset

$$\{ n \cdot 1, \quad n \cdot -1 \}$$

*

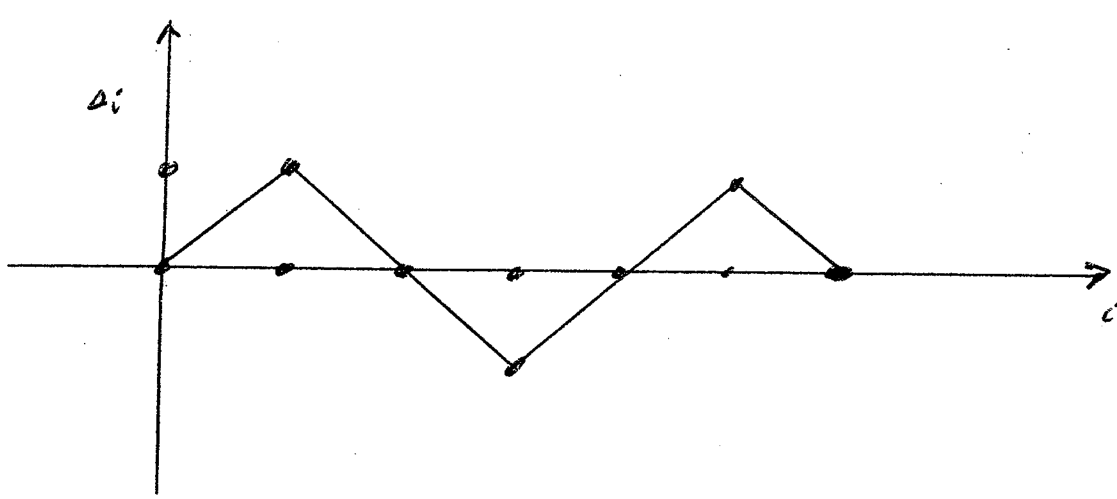
For $0 \leq i \leq 2n$ the ith partial sum is

$$s_i = a_1 + a_2 + \cdots + a_i$$

ex $n = 3$

$$a_1 a_2 a_3 a_4 a_5 a_6 = 1 \; -1 \; -1 \; 1 \; 1 \; -1$$

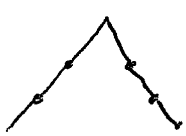
i	0	1	2	3	4	5	6
s_i	0	1	0	-1	0	1	0



<u>Problem</u> Let

h_n = # perms of $*$ that have all partial sums nonnegative.

Find h_n.

n	desc	h_n
0		1
1	 $1\ -1$	1
2	 $1\ 1\ -1\ -1$  $1\ -1\ 1\ -1$	2
3	 $1\ 1\ 1\ -1\ -1\ -1$  $1\ 1\ -1\ 1\ -1\ -1$  $1\ 1\ -1\ -1\ 1\ -1$  $1\ -1\ 1\ 1\ -1\ -1$  $1\ -1\ 1\ -1\ 1\ -1$	5
$\vdots$		$\vdots$

Thm For $n \geq 0$,

$$C_n = \# \text{ of perms } a_1 a_2 \cdots a_{2n} \text{ of}$$

$$\{ n \cdot 1, \; n \cdot -1 \} \qquad *$$

such that each partial sum is nonnegative:

$$a_1 + a_2 + \cdots + a_i \geq 0 \qquad 0 \leq i \leq 2n$$

pf: Assume $n \geq 1$ else triv.

Define

S_n = set of all perms of $*$

$$|S_n| = \binom{2n}{n}$$

Define

A_n = set of perms in S_n that have all partial sums nonneg.

show $|A_n| = C_n$

Define
$U_n = S_n \setminus A_n$

So

$$|A_n| + |U_n| = |S_n|$$

Suf to show $\quad |U_n| = \binom{2n}{n-1}$

Def

$T_n =$ set of perms of multiset

$$\{(n+1)\cdot 1,\ (n-1)\cdot -1\}$$

$$|T_n| = \binom{2n}{n-1}$$

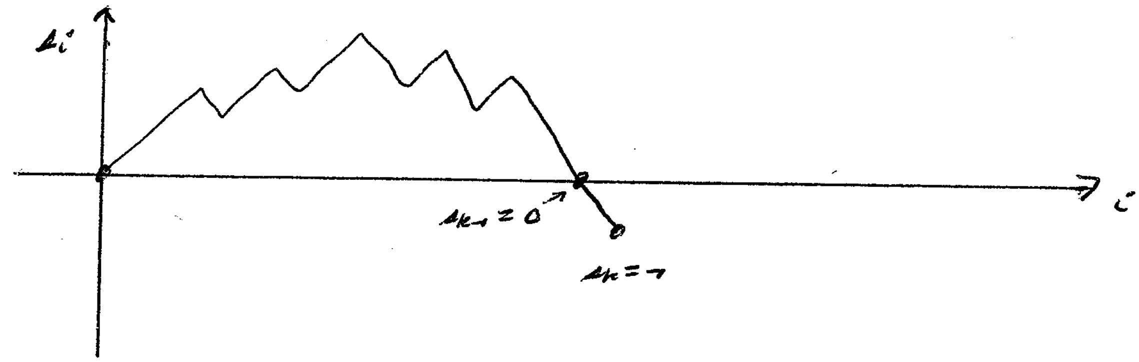
Display bijection

$$f: U_n \longrightarrow T_n$$

Given perm $a_1 a_2 \cdots a_{2n} \in U_n$

This perm has at least one negative partial sum

Pick minimal k such that kth partial sum s_k is neg



k is odd

$s_k = -1$

$s_{k-1} = 0$

$a_k = -1$

$a_1 a_2 \cdots a_{k-1}$ has $\frac{k-1}{2}$ 1's and $\frac{k-1}{2}$ -1's.

Replace each of $a_1, a_2, \ldots, a_k$ by its opposite and leave alone $a_{k+1}, \ldots, a_{2n}$:

$(-a_1)(-a_2) \cdots (-a_k)\, a_{k+1} \cdots a_{2n}$

This sequence has $n+1$ 1's and $n-1$ -1's

so it is in T_n

This gives function

$f: U_n \to T_n$

f Injective by constr

check f surjective:

Given perm

$$b_1 b_2 \cdots b_{2n} \in T_n$$

has $n+1$ 1's
$n-1$ -1's

Last partial sum is $n+1 - (n-1) = 2 > 0$

Pick minimal k such that kth partial sum is > 0

$$b_1 + b_2 + \cdots + b_k = 1$$

$$b_1 + \cdots + b_{k-1} = 0$$

$$b_k = 1$$

k odd
$b_1 b_2 \cdots b_{k-1}$ has $\frac{k-1}{2}$ 1's
 " $\frac{k-1}{2}$ -1's

For $1 \le i \le 2n$ define

$$a_i = \begin{cases} -b_i & \text{if } 1 \le i \le k \\ b_i & \text{if } k+1 \le i \le 2n \end{cases}$$

then

$$a_1 a_2 \cdots a_{2n} \in U_n$$

and

$$f(a_1 \cdots a_{2n}) = b_1 \cdots b_{2n}$$

So f is surj.
So f is bijection.

Now

$$|A_n| = \underset{\binom{2n}{n}}{\underset{\shortparallel}{|S_n|}} - \underset{\underset{\binom{2n}{n-1}}{\underset{\shortparallel}{|T_n|}}}{\underset{\shortparallel}{|U_n|}}$$

$$= \binom{2n}{n} - \binom{2n}{n-1}$$

$$= C_n$$

$\square$

A recurrence for C_n

Thm For $n \ge 1$

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$$

$$= \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

$\times$	$C_0=1$	$C_1=1$	$C_2=2$	$C_3=5$	$C_4=14$
$C_0=1$	1	1	2	5	14
$C_1=1$	1	1	2	5	
$C_2=2$	2	2	4		
$C_3=5$	5	5			
$C_4=14$	14				
$C_5=42$					

pf 1 We obtained this recurrence in Sec 7.6 in our discussion of triangular divisions of a convex polygon

pf 2 Use partial sum interp

Pf 2

Def $A = A_n =$ set of perms $a_1 a_2 \cdots a_{2n}$ of

$\{n \cdot 1, n \cdot -1\}$ with all partial sums $s_i \geq 0$ for $0 \leq i \leq 2n$.

Given $a_1 a_2 \cdots a_{2n} \in A$

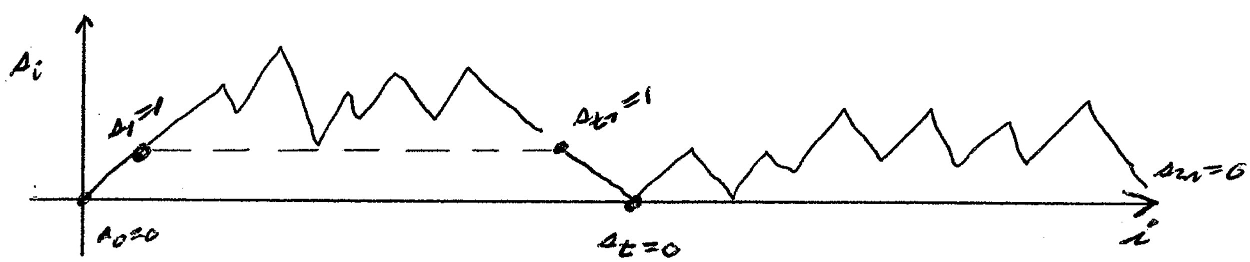
$a_1 = 1$,

$s_0 = 0$. $s_{2n} = 0$

$s_1 = 1$,

Def $t = \min\{ i \mid 2 \leq i \leq 2n, \ s_i = 0 \}$

"Landing number"



t even

$s_1 = a_1 = 1$

$s_t = 0$ $a_t = -1$

$s_{t-1} = 1$

$s_i \geq 1$ $1 \leq i \leq t-1$

Partition A according to landing number

For $0 \le k \le n-1$ define

$A(k) =$ set of perms in A that have landing number $2k+2$

$\{A(k)\}_{k=0}^{n-1}$ partition A

So $$|A| = \sum_{k=0}^{n-1} |A(k)|$$

For $0 \le k \le n-1$ find $|A(k)|$

We construct $a_1 a_2 \cdots a_{2n} \in A(k)$ in stages

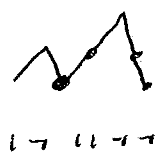
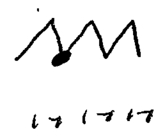
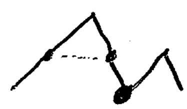
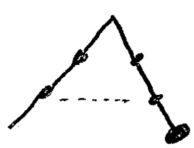
(write $t = 2k+2$)

stage	to do	# choices
1	pick a_1 $(=1)$	1
2	Pick $a_2 a_3 \cdots a_{t-1}$	C_k
3	pick a_t $(=-1)$	1
4	pick $a_{t+1} \cdots a_{2n}$	C_{n-1-k}

So $$|A(k)| = C_k C_{n-1-k}$$

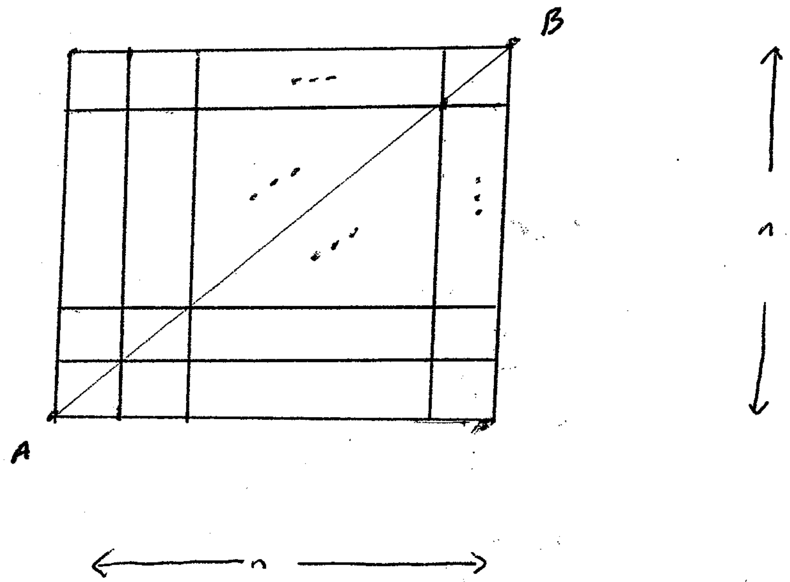
So $$C_n = |A| = \sum_{k=0}^{n-1} |A(k)| = \sum_{k=0}^{n-1} C_k C_{n-1-k} \qquad \square$$

ex $n=3$

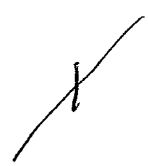
k	$A(k)$		$\lvert A(k)\rvert$	
0			2	$= C_0 C_2$
1			1	$= C_1 C_1$
2			2	$= C_2 C_0$
			$5 = C_3$	

Ex For $n \geq 1$

consider city blocks



How many paths of length $2n$ from A to B that do not cross the diagonal?



Let h_n = # paths from A to B that lie to NW of diagonal

Obs h_n = - - - SE

Ans = $2h_n$

Find h_n

Each path from $A \to B$ is sequence

$$a_1 a_2 \cdots a_{2n} \qquad a_i \in \{N, E\} \qquad 1 \le i \le 2n$$

View $N \leftrightarrow 1$

$E \leftrightarrow -1$

path stays NW of diag $\Leftrightarrow$ all partial sums ≥ 0

$$h_n = C_n$$

Ans $= 2C_n$

$$= \frac{2}{n+1}\binom{2n}{n}$$

$\square$

Lecture 28 Fri Nov 9

11/9/12

8.1 Cont

Recall nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad n=0,1,2..$$

C_n and Multiplication schemes

Given n variables, say $n=4$

a, b, c, d

We can compute the product $abcd$ in the following ways

$(((a\,b)c)\,d) \quad ((a\,b)(c\,d)) \quad ((a(b\;c))d) \quad (a((b\;c)\,d)) \quad (a(b(c\;d)))$

Call these *multiplication schemes*

Let

$h_n =$ # mult schemes involving n variables

Find h_n.

One checks

n	1	2	3	4	5	..
h_n	1	1	2	5	14	--

So it appears

$$h_n = C_{n-1} \qquad n = 1, 2, \dots$$

Thm For $n \geq 1$

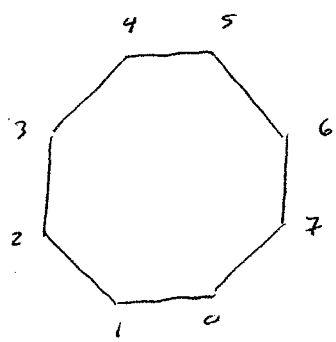
$$C_{n-1} = \#\text{ of multiplication schemes involving } n \text{ variables}$$

pf Call the variables $a_1, a_2, \dots, a_n$

Consider a convex $(n+1)$-gon P.

ex $n=7$

P:



Recall from Section 7.6

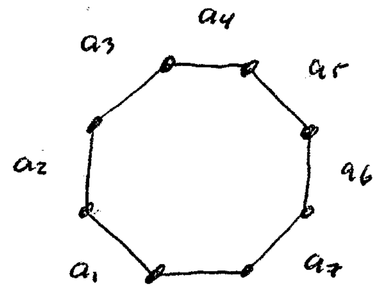
$$C_{n-1} = \#\text{ triangular divisions of } P$$

We display a bijection

$$\left\{ \begin{array}{c} \text{mult schemes} \\ \text{involving} \\ a_1, a_2, \dots, a_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{triangular} \\ \text{divisions of} \\ P \end{array} \right\} \qquad *$$

For $1 \le i \le n$ give the edge i-th of P the label a_i

ex $n=7$

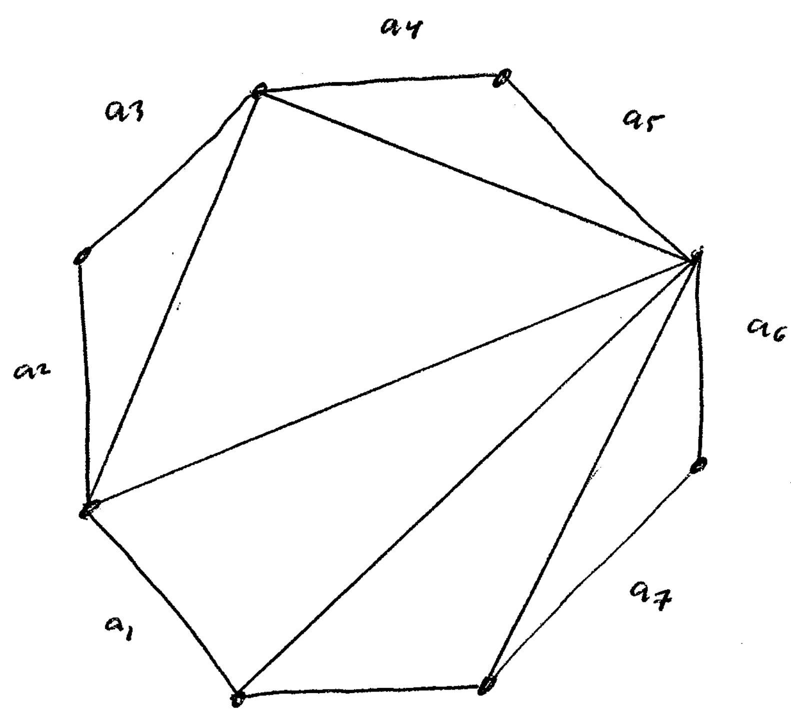


Given a mult scheme involving $a_1, a_2, \ldots a_n$

ex $n=7$.

$$\left(\left(a_1\left(\left(a_2\ a_3\right)\left(a_4\ a_5\right)\right)\right)\left(a_6\ a_7\right)\right)$$

Use this to divide P:

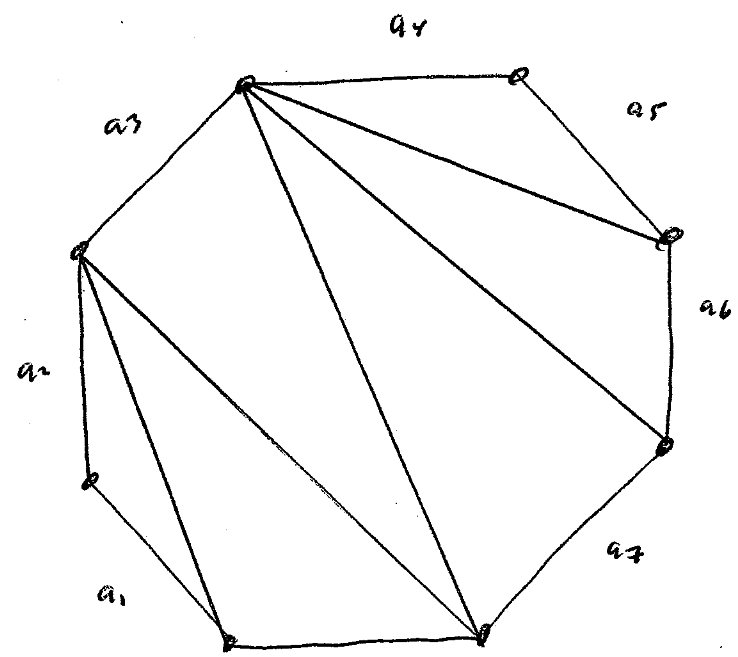


This gives triangular division of P.

Division process is reversible:

Given triangular division of P, find corresp mult scheme for $a_1, a_2, \ldots, a_n$

ex $n=7$



yields

$$((a_1 \quad a_2)(a_3(((a_4 \quad a_5) \quad a_6) \quad a_7)))$$

We have displayed a bijection $*$

Result follows. $\square$

Next goal Show directly that

$$C_{n-1} = \#\text{ mult schemes involving } n \text{ variables}$$

Def For $n \geq 1$ define

$$C_n^* = n!\, C_{n-1}$$

"nth pseudo Catalan number"

Obs

$$C_n^* = n! \frac{1}{n} \binom{2n-2}{n-1}$$

$$= (n-1)! \frac{(2n-2)!}{(n-1)!\,(n-1)!}$$

$$= \frac{(2n-2)!}{(n-1)!}$$

and

$$C_1^* = 1, \qquad C_2^* = 2, \qquad C_3^* = 12 \quad \cdots$$

For $n \geq 2$

$$\frac{C_n^*}{C_{n-1}^*} = \frac{(2n-2)!}{(n-1)!} \cdot \frac{(n-2)!}{(2n-4)!} = \frac{(2n-2)(2n-3)}{n-1} = 2(2n-3)$$

$$= 4n-6$$

So

$$C_n^* = (4n-6)\, C_{n-1}^* \qquad n = 2, 3, 4, \ldots$$

$$C_1^* = 1.$$

Direct pf that

C_{n-1} = # mult schemes involving n variables :

Call the variables $a_1, a_2, \ldots, a_n$

Let M_n = set of mult schemes involving any permutation of $a_1, a_2, \ldots, a_n$

So $|M_n| = n!\,($# of mult schemes involving $a_1, a_2, \ldots, a_n$ in order$)$

Suf to show

$$|M_n| = C_n^*$$

Suf to show

$$|M_1| = 1 \qquad *$$

and

$$|M_n| = (4n-6)\,|M_{n-1}| \qquad n \geq 2 \qquad **$$

$*$ is clear

show **

Given mult scheme in M_{n-1}

involves some perm of $a_1, a_2, \ldots, a_{n-1}$

Insert a_n in $4n-6$ poss ways to get mult scheme in M_n

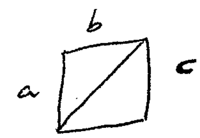
ex $n=4$

Given mult scheme for a, b, c

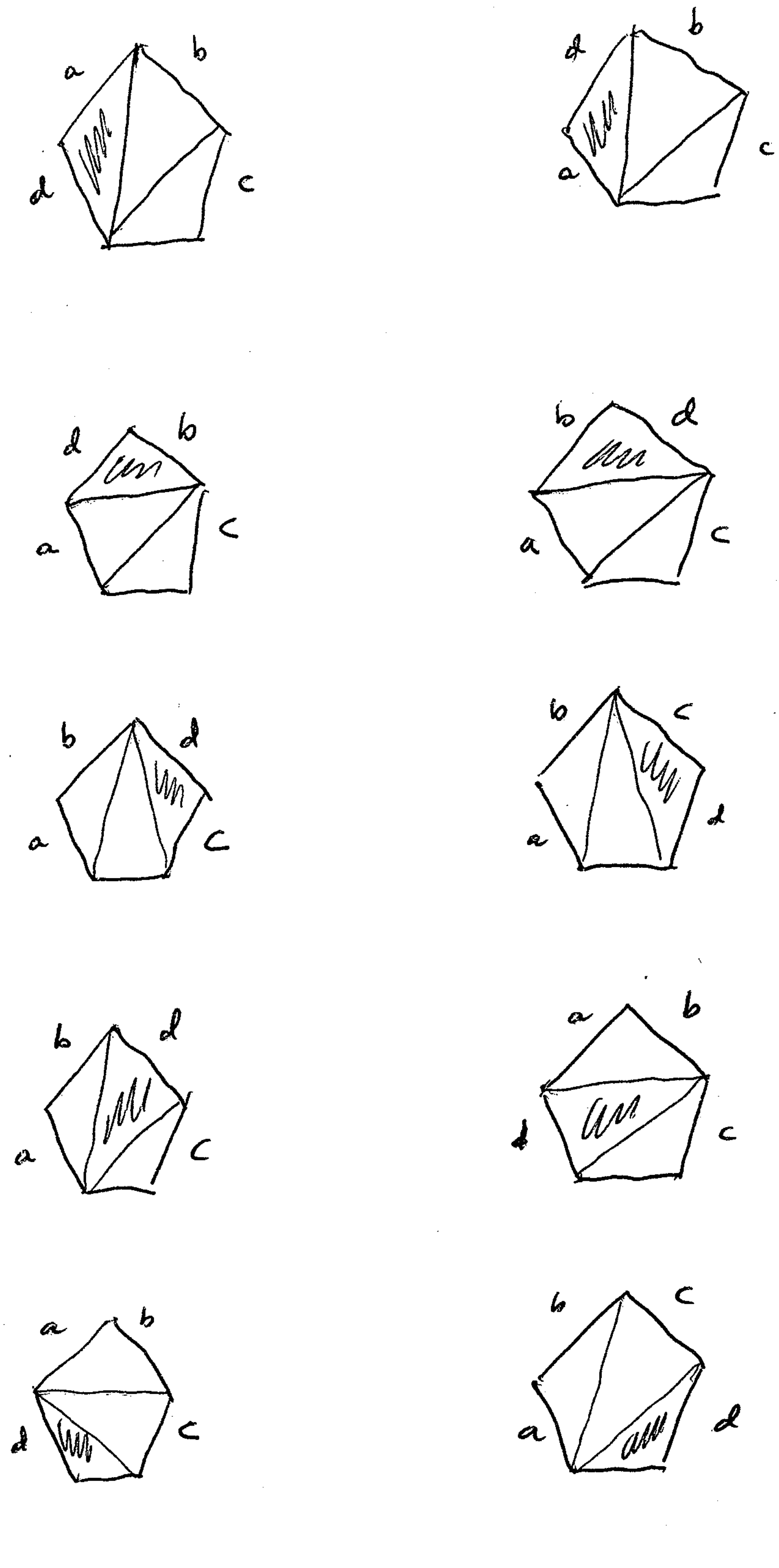
say $(ab)c$

insert d in $4n-6=10$ ways to get

mult scheme for a perm of a, b, c, d

View $((ab)c) \leftrightarrow$ 

added triangle shaded



□

8.2 Difference Sequences and Stirling numbers

Given a sequence of real numbers $\{h_n\}_{n=0}^{\infty}$

Define a new sequence $\{\Delta h_n\}_{n=0}^{\infty}$ where

$$\Delta h_n = h_{n+1} - h_n$$

View Δ as an operator that acts on all sequences $\{h_n\}_{n=0}^{\infty}$

Ex take $h_n = n^2$

$$\{n^2\}_{n=0}^{\infty}$$

$$\Delta \quad \downarrow \qquad (n+1)^2 - n^2 = 2n+1$$

$$\{2n+1\}_{n=0}^{\infty}$$

$$\Delta \quad \downarrow \qquad 2(n+1)+1 - (2n+1) = 2$$

$$\{2\}_{n=0}^{\infty}$$

$$2 - 2 = 0$$

$$\Delta \quad \downarrow$$

$$\{0\}_{n=0}^{\infty}$$

Define

$$\mathbb{O} = \{0\}_{n=0}^{\infty} = (0, 0, 0, \ldots \quad)$$

For $r \geq 0$ the operator Δ^r acts by applying Δ r times.

Δ^0 means "do nothing"

$$\Delta^0 h_n = h_n \qquad n = 0, 1, 2 \ldots$$

Δ^r sends $\{n^2\}_{n=0}^{\infty}$ to $\mathbb{O}$ for $r \geq 3$

View

$$\begin{array}{lcccccccccccc}
 & 0 & & 1 & & 4 & & 9 & & 16 & & 25 & \cdots \\
\Delta \downarrow & & 1 & & 3 & & 5 & & 7 & & 9 & \cdots & \\
\Delta \downarrow & & & 2 & & 2 & & 2 & & 2 & \cdots & & \\
\Delta \downarrow & & & & 0 & & 0 & & 0 & \cdots & & & \\
\Delta \downarrow & & & & & 0 & & 0 & \cdots & & & & \\
\vdots & & & & & & \vdots & & \vdots & & & &
\end{array}$$

"difference table"

Define

$\mathbb{V}$ = set of all sequences of real numbers $\{h_n\}_{n=0}^{\infty}$

Given $h, h' \in \mathbb{V}$

Write

$h = \{h_n\}_{n=0}^{\infty}$ $\qquad$ $h' = \{h_n'\}_{n=0}^{\infty}$

Define

$h + h' = \{h_n + h_n'\}_{n=0}^{\infty} \in \mathbb{V}$

Given a real number α

Define

$\alpha h = \{\alpha h_n\}_{n=0}^{\infty} \in \mathbb{V}$

Obs

$$\alpha(h + h') = \alpha h + \alpha h'$$

$$h + \mathbb{O} = \mathbb{O} + h = h$$

$\mathbb{V}$ is a "vector space"

11/9/12

12

the function

$$\Delta : \mathbb{V} \to \mathbb{V}$$

respects addition and scalar mult as follows:

$$\Delta(h+h') = \Delta h + \Delta h' \qquad \forall h, h' \in \mathbb{V}$$

$$\Delta(\alpha h) = \alpha \Delta h \qquad \forall \alpha \in \mathbb{R} \quad \forall h \in \mathbb{V}$$

" Δ is a linear transformation of $\mathbb{V}$ "

— o —

Question

what elements of $\mathbb{V}$ does Δ^r send to $\mathbb{O}$ for suf large r ?

thm Given any polynomial $f(x)$

let p = degree of f

Consider the sequence

$$\{f(n)\}_{n=0}^{\infty}$$

Then

Δ^{p+1} sends this sequence to $\mathbb{O}$

pf Use induction on p

Case $p=0$: triv ✓

Case $p \geq 1$:

Write

$$f(x) = a_0 + a_1 x + \cdots + a_p x^p$$

So
$$f(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_p n^p$$
$$= \sum_{k=0}^{p} a_p n^p$$

So
$$\Delta^{p+1}\{f(n)\}_{n=0}^{\infty} = \sum_{k=0}^{\infty} a_k \, \Delta^{p+1}\{n^k\}_{n=0}^{\infty}$$

Suf to show

$$\Delta^{p+1}\{n^k\}_{n=0}^{\infty} = \mathbb{O} \qquad 0 \leq k \leq p$$

Suf to show
$$\Delta^{p+1}\{n^p\}_{n=0}^{\infty} = \mathbb{O}$$

Write

$$\Delta \{n^p\}_{n=0}^{\infty} = \{h_n\}_{n=0}^{\infty}$$

$$h_n = (n+1)^p - n^p$$

$$= \sum_{k=0}^{p} \binom{p}{k} n^k \qquad - n^p$$

$$= \sum_{k=0}^{p-1} \binom{p}{k} n^k$$

$$= g(n)$$

$$g(x) = \sum_{k=0}^{p-1} \binom{p}{k} x^k$$

$g(x)$ has degree $p-1$

So by induction

$$\Delta^p \{h_n\}_{n=0}^{\infty} = \emptyset$$

Now

$$\Delta^{p+1} \{n^p\}_{n=0}^{\infty} = \Delta^p \Delta \{n^p\}_{n=0}^{\infty}$$

$$= \Delta^p \{h_n\}_{n=0}^{\infty}$$

$$= \emptyset$$

Result follows. $\square$

Lecture 29 Monday Nov 12

11/12/18

Next general goal:

Given $h = \{h_n\}_{n=0}^{\infty} \in \mathbb{V}$

Suppose $\exists\ p \geq 0$ s.t.

$$\Delta^{p+1} h = 0$$

show there exists a polynomial $f(x)$ of degree $\leq p$

such that $h_n = f(n)$ for $n \geq 0$

Given a sequence $\{h_n\}_{n=0}^{\infty}$

Construct difference table

$$
\begin{array}{ccccccccccc}
h_0 & & h_1 & & h_2 & & h_3 & & h_4 & & \cdots \\
& \Delta h_0 & & \Delta h_1 & & \Delta h_2 & & \Delta h_3 & & \cdots & \\
& & \Delta^2 h_0 & & \Delta^2 h_1 & & \Delta^2 h_2 & & \cdots & & \\
& & & \Delta^3 h_0 & & \Delta^3 h_1 & & \cdots & & & \\
& & & & \Delta^4 h_0 & & \cdots & & & & \\
& & & & & \cdot\cdot & & & & &
\end{array}
$$

diagonal sequence

the corresp <u>diagonal sequence</u> is

$$\{\Delta^n h_0\}_{n=0}^{\infty}$$

Ex Define

$$h_n = \binom{n}{3} \qquad n = 0, 1, 2, \ldots$$

Find the corresp diagonal sequence

Sol Create difference table

$$\begin{array}{ccccccccccccc}
\binom{0}{3} & & \binom{1}{3} & & \binom{2}{3} & & \binom{3}{3} & & \binom{4}{3} & & \binom{5}{3} & & \binom{6}{3} & \cdots \\
\| & & \| & & \| & & \| & & \| & & \| & & \| \\
0 & & 0 & & 0 & & 1 & & 4 & & 10 & & 20 \\
& 0 & & 0 & & 1 & & 3 & & 6 & & 10 & & \cdots \\
& & 0 & & 1 & & 2 & & 3 & & 4 & & \cdots \\
& & & 1 & & 1 & & 1 & & 1 & & \cdots \\
& & & & 0 & & 0 & & 0 & & \cdots \\
& & & & & & \cdots
\end{array}$$

Diagonal sequence is

$$0, 0, 0, 1, 0, 0, \cdots$$

LEM Fix an integer $p \geq 0$

For the sequence

$$\left\{ \binom{n}{p} \right\}_{n=0}^{\infty}$$

The corresp diagonal sequence is

$$0, 0, \ldots, 0, 1, 0, 0, \ldots$$
$$\uparrow$$
Location p

pf Denote the diagonal sequence by $\{c_n\}_{n=0}^{\infty}$

Show

$$c_n = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{if } n \neq p \end{cases} \qquad n \geq 0$$

Create difference table

$$\binom{0}{p} \quad \binom{1}{p} \quad \binom{2}{p} \quad \cdots \quad \binom{p-1}{p} \quad \binom{p}{p} \quad \binom{p+1}{p} \quad \cdots$$

$$\begin{matrix} \| & \| & \| & & \| & \| & \| \\ 0 & 0 & 0 & & 0 & 1 & p+1 \end{matrix}$$

$0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad p \quad \cdots$

$0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad p-1 \quad \cdots$

$0 \quad 0 \quad \cdots \quad 0 \quad 1 \quad p-2 \quad \cdots$

$\cdots$

$0 \quad 1 \quad 2 \quad \cdots$

$1 \quad 1 \quad \cdots$

row p $\quad 0 \quad \cdots$

So $\quad c_n = 0 \qquad 0 \le n \le p-1$

and $\quad c_p = 1$

Recall $\quad \binom{n}{p} = \dfrac{n(n-1)(n-2)\cdots(n-p+1)}{p!} =$ poly in n with degree p

So by prev thm

Δ^{p+1} sends $\{\binom{n}{p}\}_{n=0}^{\infty}$ to $\mathbf{0}$

Therefore

$c_n = 0 \quad$ if $\quad n \ge p+1$

□

Ex Given a sequence $\{h_n\}_{n=0}^{\infty}$

Suppose its diagonal sequence is all zeros

$0, 0, 0, \ldots$

Find h_n

Sol Difference table is

$$
\begin{array}{ccccccccc}
\underset{0}{\overset{h_0}{\|}} & & \underset{0}{\overset{h_1}{\|}} & & \underset{0}{\overset{h_2}{\|}} & & \underset{0}{\overset{h_3}{\|}} & & \cdots \\
& 0 & & \underset{0}{\overset{*}{\|}} & & \underset{0}{\overset{*}{\|}} & & & \\
& & 0 & & \underset{0}{\overset{*}{\|}} & & \cdots & & \\
& & & 0 & & & & & \\
& & & & \ddots & & & &
\end{array}
$$

$h_n = 0 \qquad n = 0, 1, 2, \ldots$

So $\{h_n\}_{n=0}^{\infty} = 0$ $\square$

Cor. A sequence $\{h_n\}_{n=0}^{\infty}$ is uniquely determined by its diagonal sequence

pf. Suppose $\{h_n\}_{n=0}^{\infty}$ and $\{h_n'\}_{n=0}^{\infty}$ have the same diagonal sequence. Show $h_n = h_n'$ $n = 0, 1, 2, \ldots$

Then the sequence $\{h_n - h_n'\}_{n=0}^{\infty}$ has diagonal sequence $0, 0, 0, \ldots$

Now $\{h_n - h_n'\}_{n=0}^{\infty} = 0$

so $h_n = h_n'$ $n = 0, 1, 2, \ldots$ $\square$

Ex Find the sequence $\{h_n\}_{n\geq 0}^{\infty}$ whose diagonal sequence is

$$1, 3, 2, 5, 0, 0, \ldots \quad *$$

Sol. View $*$ as

$$1 \text{ times } 1, 0, 0, 0, \ldots$$
$$+ \ 3 \text{ times } 0, 1, 0, 0, \ldots$$
$$+ \ 2 \text{ times } 0, 0, 1, 0, \ldots$$
$$+ \ 5 \text{ times } 0, 0, 0, 1, \ldots$$

So

$$h_n = 1\binom{n}{0} + 3\binom{n}{1} + 2\binom{n}{2} + 5\binom{n}{3} \qquad n \geq 0$$

Thm Fix integer $p \geq 0$

Given real numbers

$$c_0, c_1, \ldots, c_p$$

Let $\{h_n\}_{n=0}^{\infty}$ denote the sequence with diagonal sequence

$$c_0, c_1, c_2, \ldots, c_p, 0, 0, 0, \ldots$$

Then

$$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + c_2 \binom{n}{2} + \cdots + c_p \binom{n}{p}$$

$\square$

pf clear from prev Lem + Cor

thm Given sequence $h = \{h_n\}_{n=0}^{\infty}$

Assume $\exists$ integer $p \ge 0$ such that

$$\Delta^{p+1} h = 0$$

then $\exists$ polynomial $f(x)$ of degree $\le p$ such that

$$h_n = f(n) \qquad n = 0, 1, 2, \ldots$$

pf. Let $\{c_n\}_{n=0}^{\infty}$ denote the diagonal sequence for h.

Obs $\quad c_n = 0 \quad$ if $\quad n \ge p+1$

By prev thm

$$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \cdots + c_p \binom{n}{p} \qquad n = 0, 1, 2, \ldots$$

Define

$$f(x) = c_0 \binom{x}{0} + c_1 \binom{x}{1} + \cdots + c_p \binom{x}{p}$$

$f(x)$ has degree $\begin{cases} p & \text{if } c_p \ne 0 \\ < p & \text{if } c_p = 0 \end{cases}$

$$h_n = f(n) \qquad n = 0, 1, 2, \ldots$$

$\square$

Ex Find a formula for

$1^4 + 2^4 + 3^4 + \cdots + n^4 \qquad n \geq 0$

Sol. Consider sequence $\{n^4\}_{n=0}^{\infty}$

Find the diagonal sequence

$$
\begin{array}{ccccccccccc}
0^4 & & 1^4 & & 2^4 & & 3^4 & & 4^4 & & 5^4 \\
\| & & \| & & \| & & \| & & \| & & \| \\
0 & & 1 & & 16 & & 81 & & 256 & & 625 \\
& 1 & & 15 & & 65 & & 175 & & 369 & \\
& & 14 & & 50 & & 110 & & 194 & & \\
& & & 36 & & 60 & & 84 & & & \\
& & & & 24 & & 24 & & & & \\
& & & & & 0 & & & & &
\end{array}
$$

Diag sequence is

$0, 1, 14, 36, 24, 0, 0, \ldots$

So

$$n^4 = \binom{n}{1} + 14\binom{n}{2} + 36\binom{n}{3} + 24\binom{n}{4}$$

11/12/12
12

Recall

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} \qquad n, k \geq 0$$

So

$1^4 + 2^4 + 3^4 + \cdots + n^4 =$

$$\binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5}$$

Ex Given a polynomial $f(x)$ deg p

Find formula for

$$f(0)+f(1)+\cdots+f(n) \qquad n=0,1,2,\ldots$$

Sol. Consider sequence $\{f(n)\}_{n=0}^{\infty}$

Let $\{c_n\}_{n=0}^{\infty}$ denote corresp diagonal sequence.

Obs

$$c_n = 0 \qquad \text{if } n>p$$

$$f(n) = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_p\binom{n}{p} \qquad n \geq 0$$

Now

$f(0)+f(1)+\cdots+f(n) =$

$$c_0\binom{n+1}{1} + c_1\binom{n+1}{2} + \cdots + c_p\binom{n+1}{p+1} \qquad n=0,1,2,\ldots$$

$\square$

Lecture 30 Wednesday Nov 14 11/14/12

1

Stirling numbers

Consider the sequence of polynomials

$$1, \quad x, \quad x^2, \quad x^3, \ldots \qquad *$$

Another sequence of polynomials

$$1, \quad x, \quad x(x-1), \quad x(x-1)(x-2), \ldots \qquad **$$

$$\begin{array}{cccc} 1 & x & x(x-1) & x(x-1)(x-2) \\ \| & \| & \| & \| \\ [x]_0 & [x]_1 & [x]_2 & [x]_3 \end{array}$$

Two problems:

(i) Write $*$ in terms of $**$

(coefs give Stirling numbers of 2nd kind)

(ii) Write $**$ in terms of $*$

(coefs give Stirling numbers of 1st kind)

We start with (i)

11/14/12

Ex Write X^3 in terms of **

Sol Consider sequence $\{n^3\}_{n=0}^{\infty}$

Create difference table

$$\begin{array}{ccccccccc}
0 & & 1 & & 8 & & 27 & & 64 \quad \cdots \\
& 1 & & 7 & & 19 & & 37 & \\
& & 6 & & 12 & & 18 & & \\
& & & 6 & & 6 & & & \\
& & & & 0 & & & &
\end{array}$$

So

$$n^3 = 0\binom{n}{0} + 1\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3}$$

$$n^3 = n + 6\,\frac{n(n-1)}{2} + 6\,\frac{n(n-1)(n-2)}{3\cdot 2}$$

$$= n + 3n(n-1) + n(n-1)(n-2) \qquad n = 0, 1, 2 \ldots$$

So

$$x^3 = x + 3x(x-1) + x(x-1)(x-2)$$

$$= [x]_1 + 3[x]_2 + [x]_3$$

$\square$

More generally we have

4/11/12

	$[x]_0$	$[x]_1$	$[x]_2$	$[x]_3$	$[x]_4$	$[x]_5$	$\cdots$
1	1						
x	0	1					
x^2	0	1	1				
x^3	0	1	3	1			
x^4	0	1	7	6	1		
x^5	0	1	15	25	10	1	
$\vdots$			$\cdots$				

11/14/12

View as analog to Pascal triangle :

$$\underset{S(0,0)}{\underset{\shortparallel}{1}}$$

$$\underset{S(1,0)}{\underset{\shortparallel}{0}} \qquad \underset{S(1,1)}{\underset{\shortparallel}{1}}$$

$$\underset{S(2,0)}{\underset{\shortparallel}{0}} \qquad \underset{S(2,1)}{\underset{\shortparallel}{1}} \qquad \underset{S(2,2)}{\underset{\shortparallel}{1}}$$

$$0 \qquad 1 \qquad 3 \qquad 1$$

$$0 \qquad 1 \qquad 7 \qquad 6 \qquad 1$$

$$0 \qquad 1 \qquad 15 \qquad 25 \qquad 10 \qquad 1$$

$$\cdots$$

"Stirling numbers of 2nd kind"

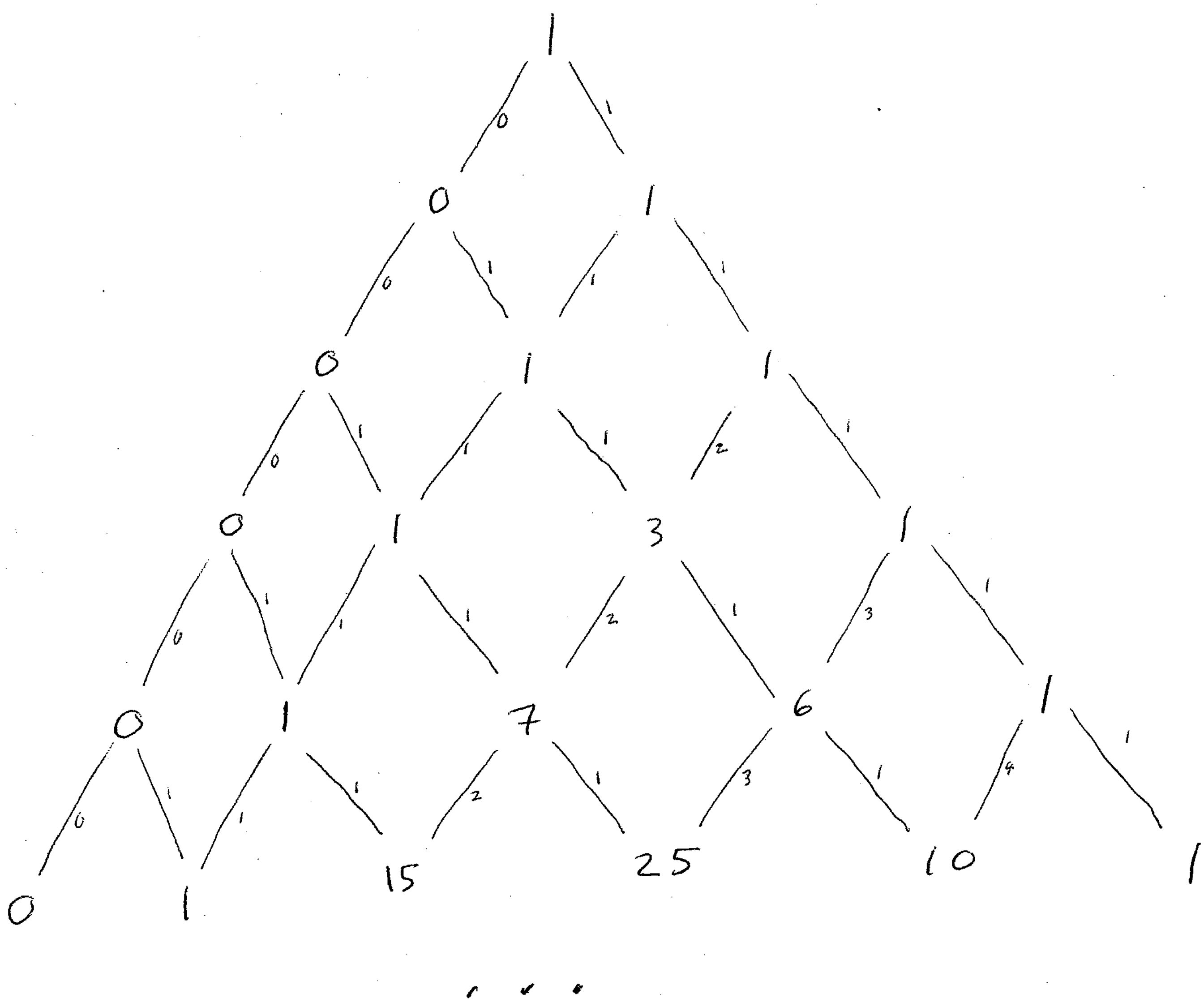
By construction for $n \geq 0$

$$x^n = \sum_{k=0}^{n} S(n,k)\,[x]_k$$

Obs

- $S(n,0) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$
- $S(n,1) = 1 \qquad n \geq 1$
- $S(n,n) = 1 \qquad n \geq 0$

Next goal: show $S(n,k)$ satisfy a recurrence similar to Pascal's formula



For notational convenience define

$$S(n,k) = 0 \quad \text{if } k<0 \text{ or } k>n$$

Thm For $0 \le k \le n$

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

pf Assume $n \ge 1$ else triv

Recall

$$X^n = \sum_{k=0}^{n} S(n,k)\,[x]_k$$

So

$$X^{n-1} = \sum_{k=0}^{n-1} S(n-1,k)\,[x]_k$$

Also

$$X^n = X^{n-1}\,X$$

$$= \sum_{k=0}^{n-1} S(n-1,k)\,\underbrace{[x]_k}_{x(x-1)(x-2)\cdots(x-k+1)}\;\underbrace{x}_{x-k+k}$$

$$\Big[\quad [x]_k\, x = [x]_{k+1} + k\,[x]_k \quad\Big]$$

$$= \sum_{k=0}^{n-1} S(n-1,k)\,[x]_{k+1} \quad + \sum_{k=0}^{n-1 \,\nearrow\, n} k\,S(n-1,k)\,[x]_k$$

$$\Big\downarrow \; k \to k-1$$

$$\sum_{\substack{k=1 \\ \downarrow \\ k=0}}^{n} S(n-1,k-1)\,[x]_k$$

$$= \sum_{k=0}^{n} \Big(S(n-1,k-1) + k\,S(n-1,k) \Big)\,[x]_k$$

Compare with ☆ to get

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k) \qquad 0 \le k \le n$$

$\square$

11/14/12 8

Thm Fn $0 \le k \le n$

$$S(n,k) = \frac{1}{k!} \sum_{t=0}^{k} \binom{k}{t} (-1)^t (k-t)^n$$

where $0^0 = 1$

pf Ind on n

$n=0$ ✓

$n \ge 1$:

Case $k=0$

$$\begin{array}{ccccc} S(n,0) & \overset{?}{=} & \frac{1}{0!} & \binom{0}{0} & (-1)^0 & 0^n \\ \| & & \| & \| & \| & \| \\ 0 & & 1 & 1 & 1 & 0 \end{array}$$

✓

Case $1 \le k \le n$

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

$$= \frac{1}{(k-1)!} \sum_{t=0}^{k-1} \binom{k-1}{t} (-1)^t (k-1-t)^{n-1}$$

$$+ \frac{k}{k!} \sum_{t=0}^{k} \binom{k}{t} (-1)^t (k-t)^{n-1}$$

$$= \frac{1}{(k-1)!} \sum_{t=1}^{k} \binom{k-1}{t-1} (-1)^{t-1} (k-t)^{n-1}$$

$\to 0$

$\to t-1$

$$+ \frac{k}{k!} \sum_{t=0}^{k} \binom{k}{t} (-1)^{t} (k-t)^{n-1}$$

$$= \frac{k}{k!} \sum_{t=0}^{k} \Bigg(\underbrace{\binom{k}{t} - \binom{k-1}{t-1}}_{= \binom{k-1}{t}} \Bigg) (-1)^t (k-t)^{n-1}$$

$$\stackrel{?}{=} \frac{1}{k!} \sum_{t=0}^{k} \binom{k}{t} (-1)^t (k-t)^n$$

need

$$k \binom{k-1}{t} = \binom{k}{t} (k-t) \qquad 0 \le t \le k$$

$t=k$ ✓ $\qquad 0=0$

$0 \le t \le k-1$:

$$k \frac{(k-1)!}{t!\,(k-1-t)!} \stackrel{?}{=} \frac{k!\,(k-t)}{t!\,(k-t)!} \quad ✓$$

$\square$

Next goal:

A combinatorial interp of $S(n,k)$

Given sets X, Y with

$$|X| = n, \qquad |Y| = k$$

LEM the number of surjective functions $X \to Y$ is

$$\sum_{t=0}^{k} \binom{k}{t} (-1)^t (k-t)^n$$

pf. Use incl/excl

Define

S = set of all functions $X \to Y$

$$|S| = k^n$$

For $1 \le i \le k$ define

A_i = set of functions $X \to Y \setminus i$

We seek

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_k|$$

For $s \subseteq \{1, 2, \dots, k\}$

define

$$A_s = \bigcap_{i \in s} A_i$$

$$= \text{set of functions} \quad X \longrightarrow (Y \setminus s)$$

Obs

$$|A_s| = |Y \setminus s|^n$$

$$= (k - |s|)^n$$

By incl/excl

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_k| = \sum_{s \subseteq \{1, 2, \dots, k\}} |A_s| (-1)^{|s|}$$

$$= \sum_{t=0}^{k} \sum_{\substack{s \subseteq \{1, 2, \dots, k\} \\ |s| = t}} |A_s| (-1)^{|s|}$$

$$= \sum_{t=0}^{k} \binom{k}{t} (k-t)^n (-1)^t$$

$\square$

COR For $0 \le k \le n$

(i) $k!\, S(n,k) =$ # surjective functions $\{1,2,\dots,n\} \to \{1,2,\dots,k\}$

(ii) $k!\, S(n,k) =$ # ways to put n distinguishable objects into k distinguishable boxes, leaving no box empty

(iii) $S(n,k) =$ # ways to put n distinguishable objects into k indistinguishable boxes, leaving no box empty

pf By Prev Lem $\square$

11/14/12

Ex $n = 5$

$S(n,k)$ = # partitions of n distinguishable objects into k indistinguishable nonempty boxes

k	desc		$S(5,k)$
0	ϕ		0
1	abc de		1
2	ab cd, e	a bc, de	$5 + \binom{5}{2} = 15$
3	a bc, d, e	ab, cd, e	$\binom{5}{3} + 5 \times 3 = 25$
4	ab, c, d, e		$\binom{5}{2} = 10$
5	a, b, c, d, e		1

Lecture 21 Friday Nov 16 11/16/12 1

Def

For $n \geq 0$ define

$$B_n = \sum_{k=0}^{n} S(n,k)$$

"nth Bell number"

n	$S(n,k)$	B_n
0	1	1
1	0 1	1
2	0 1 1	2
3	0 1 3 1	5
4	0 1 7 6 1	15
5	0 1 15 25 10 1	52
$\vdots$	$\dots$	$\vdots$

By the prev Cor

B_n = # ways to place n distinguishable objects into nonempty indistinguishable boxes

Pattern of Bell numbers B_n

B_n ↘ weights

$\binom{n}{k}$

$$
\begin{array}{ccccccccc|l}
 & & & & 1 & & & & & \\
 & & & 1 & & & & & & 1 = 1\cdot 1 \\
 & & 1 & & 1 & & & & & 2 = 1\cdot 1 + 1\cdot 1 \\
 & 1 & & 2 & & 1 & & & & 5 = 1\cdot 1 + 2\cdot 1 + 1\cdot 2 \\
1 & & 3 & & 3 & & 1 & & & 15 = 1\cdot 1 + 3\cdot 1 + 3\cdot 2 + 1\cdot 5 \\
1 & 4 & & 6 & & 4 & & 1 & & 52 = 1\cdot 1 + 4\cdot 1 + 6\cdot 2 + 4\cdot 5 + 1\cdot 15 \\
 & & & \ldots & & & & & & \ddots
\end{array}
$$

Thm For $n \geq 1$

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

Pf

Let S = set of all partitions of $\{1, 2, \ldots, n\}$ into nonempty indistinguishable boxes

$$|S| = B_n$$

Given a partition p in S

Let

$$\text{index of } p = \left| \{ i \mid 1 \le i \le n-1, \ i \text{ not in same box as } n \} \right|$$

$0 \le \text{index} \le n-1$

For $0 \le k \le n-1$ define

S_k = set of partitions in S with index k

$$S = \bigcup_{k=0}^{n-1} S_k \qquad \text{(disjoint union)}$$

So

$$|S| = \sum_{k=0}^{n-1} |S_k|$$

For $0 \le k \le n-1$ find $|S_k|$

To construct an element of S_k we proceed in stages

stage	to do	# choices
1	pick a k-subset X of $\{1,2,\ldots,n-1\}$	$\binom{n-1}{k}$
2	partition X into nonempty indist boxes	B_k
3	add another box containing n and $\{1,2,\ldots,n-1\}\setminus X$	1

So $$|S_k| = \binom{n-1}{k} B_k$$

So $$B_n = |S| = \sum_{k=0}^{n-1} |S_k|$$
$$= \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

$\square$

Stirling numbers of 1st kind

Write $\{[x]_n\}_{n=0}^{\infty}$ in terms of $\{x^n\}_{n=0}^{\infty}$

Ex $[x]_3$ in terms of $\{x^n\}_{n=0}^{\infty}$

$$[x]_3 = x(x-1)(x-2)$$

$$= x^3 - 3x^2 + 2x$$

$$= 0 \cdot 1 + 2 \cdot x - 3 \cdot x^2 + 1 x^3$$

More generally we have

	x^0	x^1	x^2	x^3	x^4	x^5
$[x]_0$	1					
$[x]_1$	0	1				
$[x]_2$	0	-1	1			
$[x]_3$	0	2	-3	1		
$[x]_4$	0	-6	11	-6	1	
$[x]_5$	0	24	-50	35	-10	1
$\vdots$			$\cdots$			

the signs alternate – we ignore them

11/16/12 7

View as analog of Pascal triangle

$$
\begin{array}{ccccccccccc}
 & & & & & \underset{s(0,0)}{\overset{1}{\shortparallel}} & & & & & \\
 & & & & \underset{s(1,0)}{\overset{0}{\shortparallel}} & & \underset{s(1,1)}{\overset{1}{\shortparallel}} & & & & \\
 & & & \underset{s(2,0)}{\overset{0}{\shortparallel}} & & \underset{s(2,1)}{\overset{1}{\shortparallel}} & & \underset{s(2,2)}{\overset{1}{\shortparallel}} & & & \\
 & & 0 & & 2 & & 3 & & 1 & & \\
 & 0 & & 6 & & 11 & & 6 & & 1 & \\
0 & & 24 & & 50 & & 35 & & 10 & & 1
\end{array}
$$

...

" Stirling numbers of 1st Kind "

Stirling numbers	
$s(n,k)$	1st kind
$S(n,k)$	2nd kind

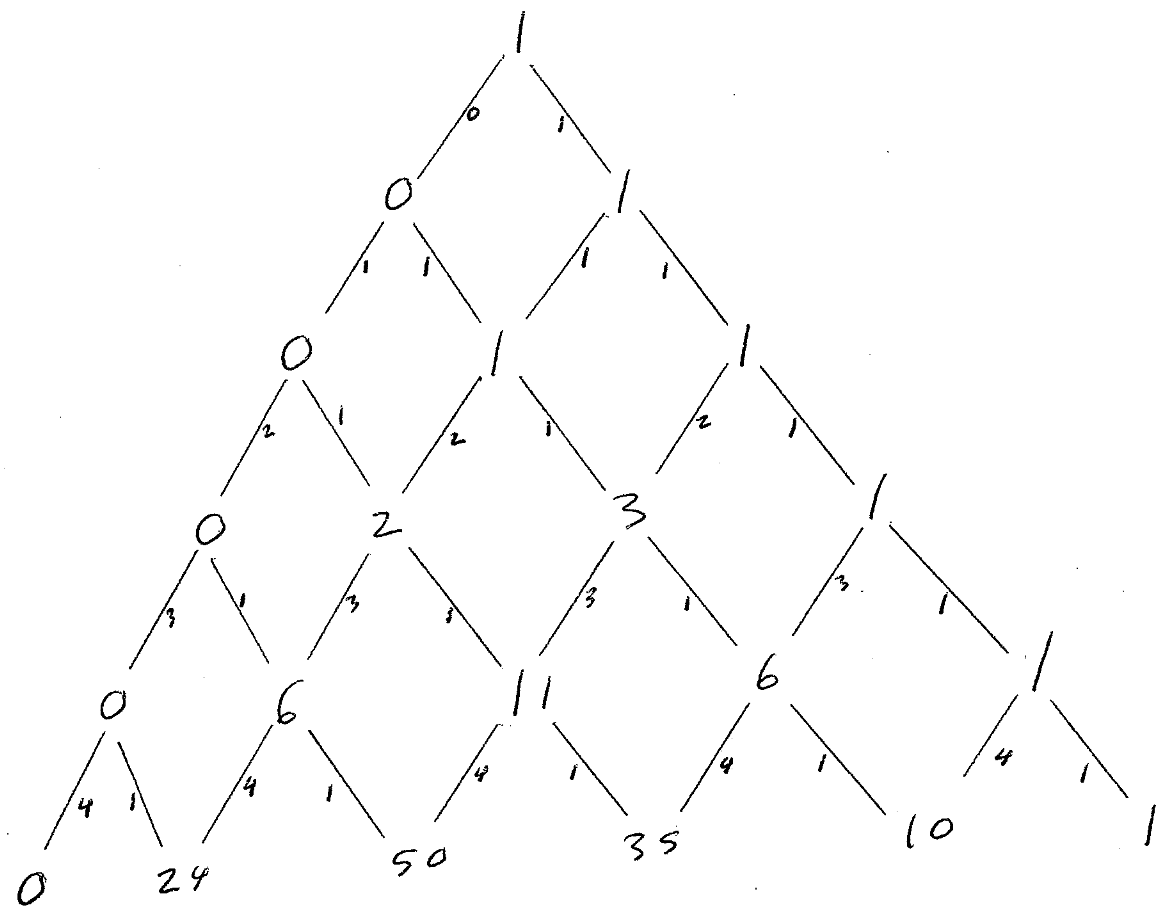
By construction

$$[X]_n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k)\, x^k \qquad n = 0, 1, 2, \dots$$

Obs

- $s(n,0) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$
- $s(n,1) = (n-1)! \qquad n = 1, 2, \dots$
- $s(n,n) = 1 \qquad n = 0, 1, 2, \dots$

Next goal: show $s(n,k)$ satisfy a recurrence similar to Pascal's formula



. . .

For notational convenience define

$$s(n,k) = 0 \quad \text{if} \quad k < 0 \text{ or } k > n$$

Thm for $0 \le k \le n$

$$s(n,k) = s(n-1,k-1) + (n-1)\, s(n-1,k)$$

pf Assume $n \ge 1$ else trivial

Recall

$$[x]_n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k)\, x^k \qquad \star$$

So

$$[x]_{n-1} = \sum_{k=0}^{n-1} (-1)^{n-1-k} s(n-1,k)\, x^k$$

Obs

$$\begin{aligned} [x]_n &= x(x-1)(x-2)\cdots(x-n+2)(x-n+1) \\ &= [x]_{n-1}(x-n+1) \\ &= \sum_{k=0}^{n-1} (-1)^{n-1-k} s(n-1,k)\, x^k (x-n+1) \end{aligned}$$

11/16/12
10

$$= \sum_{k=0}^{n-1} (-1)^{n-1-k} s(n-1,k)\, x^{k+1}$$

$k \to k+1$

$$- (n-1) \sum_{k=0}^{n-1 \nearrow n} (-1)^{n-1-k} s(n-1,k)\, x^{k}$$

$$= \sum_{k=1 \downarrow 0}^{n} (-1)^{n-k} s(n-1,k-1)\, x^{k} - (n-1) \sum_{k=0}^{n} (-1)^{n-1-k} s(n-1,k)\, x^{k}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \Big(s(n-1,k-1) + (n-1) s(n-1,k) \Big) x^{k}$$

Compare this to ☆ to get

$$s(n,k) = s(n-1,k-1) + (n-1) s(n-1,k) \qquad 0 \le k \le n$$

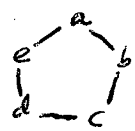
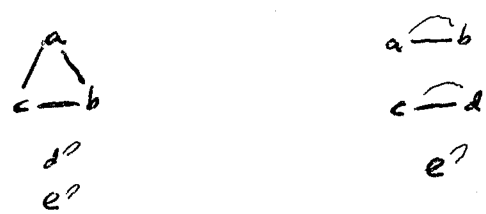
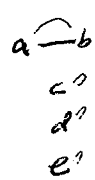
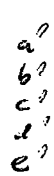
□

Next goal: Show that for $0 \le k \le n$

$s(n,k)$ = # arrangements of n distinguishable objects into k nonempty circular permutations.

11/16/12

ex $n = 5$ $\quad s(n,k) = \#$ arrangements of n dist objects into k nonempty circular permutations

k	desc	$s(5,k)$
0	ϕ	0
1		$4! = 24$
2		$5 \times 3! + \binom{5}{2} 2! = 50$
3		$\binom{5}{2} \times 2! + 5 \times 3 = 35$
4		$\binom{5}{2} = 10$
5		1

Thm for $0 \le k \le n$

$s(n,k)$ = # arrangements of n distinguishable objects into k nonempty circular permutations

pf Ind on n

$n = 0$ ✓

$n \geq 1$:

Call the objects $1, 2, \ldots, n$

Define

S = set of all arrangements of $1, 2, \ldots, n$ into k nonempty circular permutations

Show $|S| = s(n,k)$

Define S' = set of elements in S for which the circular perm containing n has length 1

S'' = complement of S' in S

$S = S' \cup S''$ (disj union)

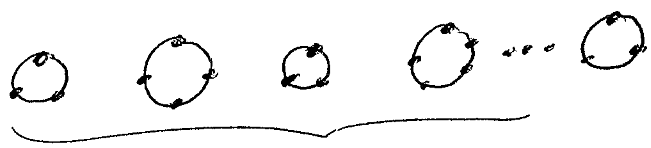
$|S| = |S'| + |S''|$

$|S'| =$ # of arrangements of $1, 2, \ldots, n-1$ into $k-1$ nonempty circular perms

$= s(n-1, k-1)$

Find $|S''|$

We construct an element of S'' in stages

stage	to do	# choices
1	pick an arrangement of $1, 2, \ldots, n-1$ into k nonempty circular perms  $n-1$ dots, k circles	$s(n-1, k)$
2	insert "n"	$n-1$

$$|S''| = (n-1)\, s(n-1, k)$$

So $|S| = |S'| + |S''|$

$= s(n-1, k-1) + (n-1) s(n-1, k)$

$= s(n, k)$ $\square$

This Wed Nov 21 I understand most of you will be travelling. I expect no one will show up

475 Exam II

Monday Nov 26

Covers Ch 6, 7, $8(\underbrace{8.1-8.3 \text{ only}}_{\text{HW } 1-30})$

Exam problems from these sections with numbers changed

Ch 8 HW due at exam

I will post HW sols before thanksgiving.

Lecture 32 Monday Nov 19 11/19/12

8.3 Partition numbers

Given integer $n \geq 1$

A <u>partition of n</u> is a representation of n as a sum of pos integers

— summation order unimportant. WLOG list them in non inc order

Partitions of $n = 5$:

	Notation
$1+1+1+1+1$	1^4
$2+1+1+1$	$2\,1^3$
$2+2+1$	$2^2 1$
$3+1+1$	$3\,1^2$
$3+2$	32
$4+1$	41
5	5

Def For $n \geq 1$

p_n = number of partitions of n

Take $p_0 = 1$.

Given partition of n:

$$1^{a_1} 2^{a_2} \cdots n^{a_n}$$

then

$$1 \cdot a_1 + 2a_2 + \cdots + n a_n = n \qquad *$$

View p_n = # of nonnegative integer solutions $a_1, a_2, \dots, a_n$ to $*$.

We now consider the generating function

$$\sum_{n=0}^{\infty} p_n x^n$$

thm We have

$$\sum_{n=0}^{\infty} p_n x^n = \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3}\cdots$$

$$= \prod_{k=1}^{\infty} (1-x^k)^{-1}$$

pf RHS =

$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots$$

$$= \left(\sum_{a_1=0}^{\infty} x^{a_1}\right)\left(\sum_{a_2=0}^{\infty} x^{2\cdot a_2}\right)\left(\sum_{a_3=0}^{\infty} x^{3\cdot a_3}\right)\cdots$$

$$= \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\sum_{a_3=0}^{\infty}\cdots \; x^{a_1+2a_2+3a_3+\cdots}$$

$$= \sum_{n=0}^{\infty}\left(\begin{array}{c}\#\text{ non neg integral sols to}\\ a_1+2a_2+\cdots+n a_n=n\end{array}\right) x^n$$

$$= \sum_{n=0}^{\infty} p_n x^n$$

□

The Ferrers diagram

Given a partition λ of n, say $n=12$

λ: $\quad 12 = 4+3+2+2+1$

Rep by diagram:

$$\lambda: \quad \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & & & \end{array} \qquad \begin{array}{c} 4 \\ 3 \\ 2 \\ 2 \\ 1 \end{array}$$

The conjugate partition λ^* of n has the transposed diagram:

$$\lambda^*: \quad \begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & & & \\ \bullet & & & & \end{array} \qquad \begin{array}{c} 5 \\ 4 \\ 2 \\ 1 \end{array}$$

$\lambda^* = \quad 12 = 5+4+2+1$

Def A partition λ of n is <u>self-conjugate</u> whenever

$$\lambda^* = \lambda$$

ex The self-conjugate partitions of $n=8$ are

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \\ \bullet & & & \end{array} \qquad\qquad \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \end{array}$$

Thm For $n \geq 1$ the following are equal:

(i) # of self-conjugate partitions of n

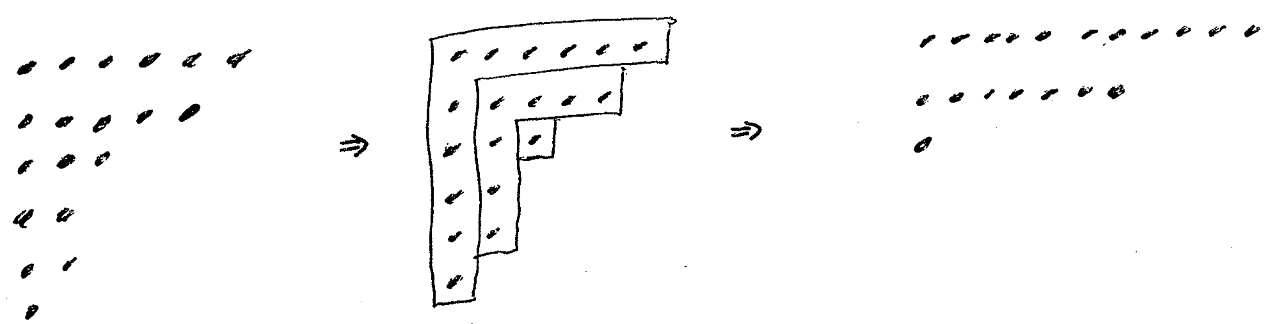
(ii) # of partitions of n into distinct odd parts pf define

S_n = set of self-conjugate partitions of n

D_n = set of partitions of n into dist odd parts

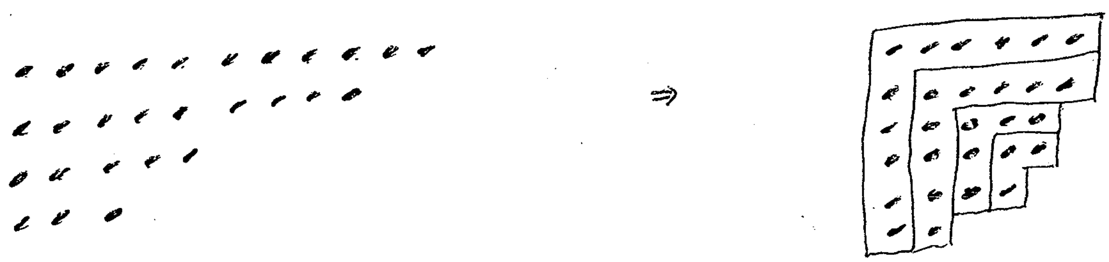
Display bijection

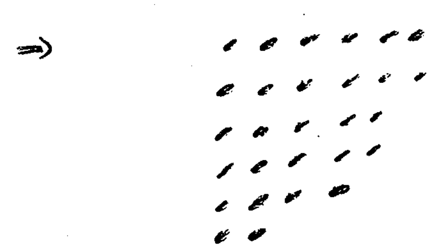
$$S_n \longrightarrow D_n$$



inverse bijection:

11
9
5
3





$\square$

Problem For $n \geq 1$ compare

$\#$ partitions of n into odd parts $\quad *$

$\#$ ... distinct parts $\quad **$

Compare gen functions:

Gen function for $*$ is

$$\frac{1}{1-x}\ \frac{1}{1-x^3}\ \frac{1}{1-x^5} \cdots \qquad (1)$$

Gen function for $**$ is

$$(1+x)(1+x^2)(1+x^3) \cdots \qquad (2)$$

Obs $(2) =$

$$\frac{\cancel{1-x^2}}{1-x}\ \frac{\cancel{1-x^4}}{\cancel{1-x^2}}\ \frac{\cancel{1-x^6}}{1-x^3}\ \frac{1-x^8}{\cancel{1-x^4}}$$

$$= \frac{1}{1-x}\ \frac{1}{1-x^3} \cdots$$

$$= (1)$$

This shows:

Thm (Euler) For $n \geq 1$ the following are equal:

(i) # partitions of n into odd parts

(ii) # partitions of n into dist parts

ex $n = 5$

partitions into odd parts	partitions into dist parts
$1+1+1+1+1$	$3+2$
$3+1+1$	$4+1$
5	5

Given $n \geq 1$

let P_n = set of partitions of n

We give a partial order $\leq$ on P_n called majorization (also called dominance order)

Given partitions λ, μ of n:

λ: $n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots$ $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$

μ: $n = \mu_1 + \mu_2 + \mu_3 + \cdots$ $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$

view as ∞ sequence with finitely many nonzero terms

Write

$$\lambda \geq \mu$$

whenever

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \quad \text{for } i = 1, 2, 3, \ldots$$

Ex $n = 6$

λ:

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet \\ \bullet \end{array}$$

λ_i	$\lambda_1 + \cdots + \lambda_i$
4	4
1	5
1	6

μ

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet \\ \bullet \end{array}$$

μ_i	$\mu_1 + \cdots + \mu_i$
3	3
2	5
1	6

$\lambda \geq \mu$ ✓

Ex $n=6$ cnt

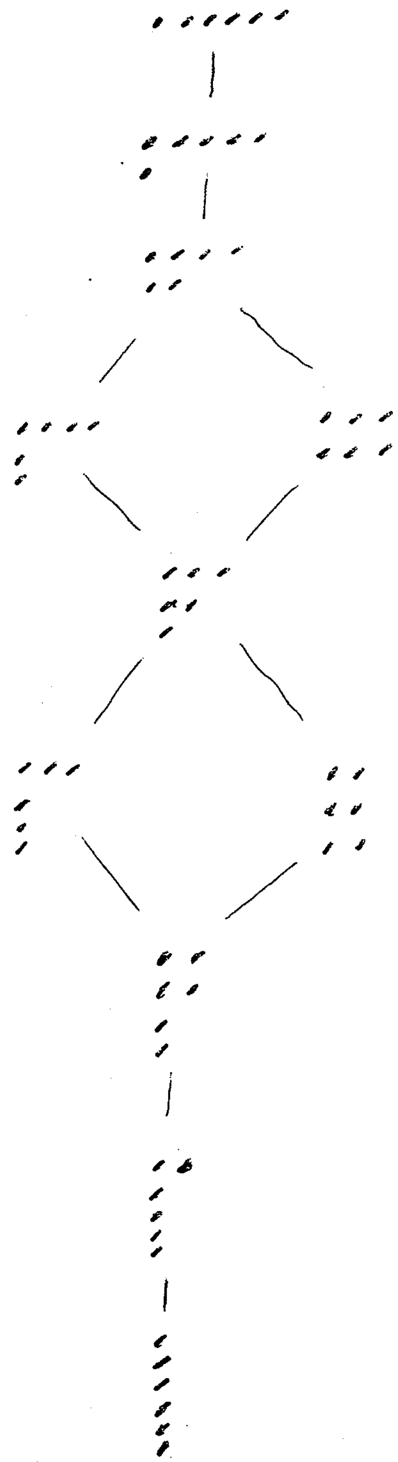
		λ_i	$\lambda_1+\cdots+\lambda_i$
λ	• • •	3	3
	•	1	4
	•	1	5
	•	1	6

		μ_i	$\mu_1+\cdots+\mu_i$
μ	• •	2	2
	• •	2	4
	• •	2	6
		0	6

$\lambda \not\leq \mu$ and $\mu \not\leq \lambda$ incomparable

For $n=6$ We show the covering relation for majorization

max



min

$p_6 = 11$

11/19/12

12

We have discussed the Majorization partial order on P_n

There is also a linear order on P_n called lexicographic order

LEX order on P_n:

Given partition $\lambda \vdash n$

λ: $n = \lambda_1 + \lambda_2 + \cdots$ $\lambda_1 \geq \lambda_2 \geq \cdots$

View as 'word'

$\lambda_1 \lambda_2 \cdots$

where 'letters'

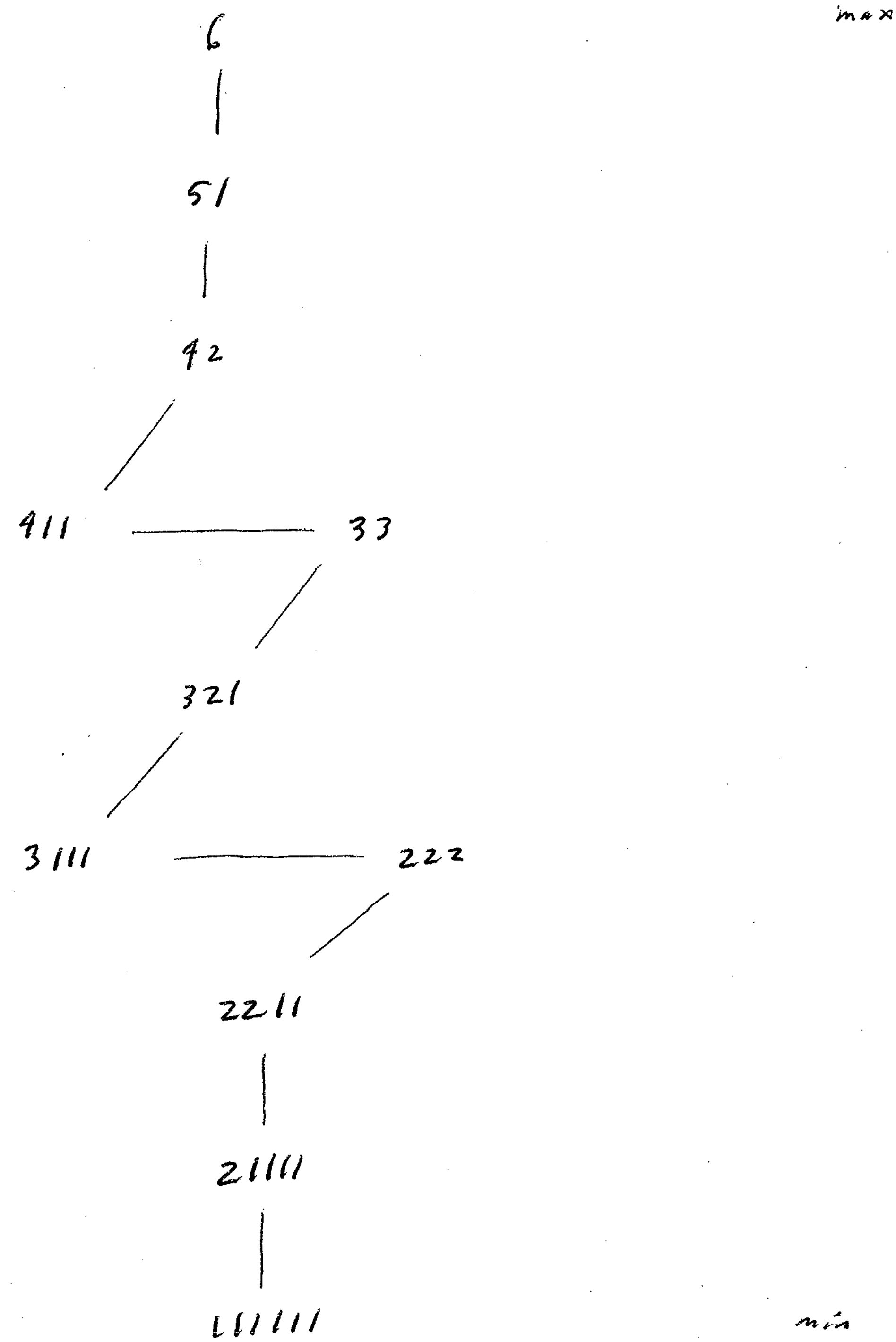
$\lambda_1, \lambda_2, \ldots$

satisfy

$0 < 1 < 2 < \cdots$

LEX order is usual dictionary ordering of these words.

ex $n=6$ covering rel for LEX order:



obs: LEX order is a linear extension of Majorization

Thm For $n \geq 1$ the LEX order on P_n

linearly extends the MAJ partial order on P_n

In other words, given partitions λ, μ of n

such that

$$\lambda \underset{MAJ}{\leq} \mu$$

then

$$\lambda \underset{LEX}{\leq} \mu$$

pf Assume $\lambda \neq \mu$ else triv.

Write

$\lambda :$ $\quad n = \lambda_1 + \lambda_2 + \cdots \quad \lambda_1 \geq \lambda_2 \geq \cdots$

$\mu :$ $\quad n = \mu_1 + \mu_2 + \cdots \quad \mu_1 \geq \mu_2 \geq \cdots$

Since $\lambda \neq \mu$ $\exists\, i$

$$\lambda_i \neq \mu_i$$

Let

$$t = \min\{ i \mid \lambda_i \neq \mu_i \}$$

By constr

$$\lambda_i = \mu_i \qquad 1 \leq i \leq t-$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_t \leq \mu_1 + \mu_2 + \cdots + \mu_t$$

So

$$\lambda_t \leq \mu_t$$

But $\lambda_t \neq \mu_t$ so

$$\lambda_t < \mu_t$$

Now

$$\lambda \underset{LEX}{<} \mu \qquad \square$$