

ch 8 Special Counting Sequences

8.1 Catalan numbers

For $n \geq 0$ define

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

"nth Catalan number"

n	0	1	2	3	4	5	...
C_n	1	1	2	5	14	42	...

In Section 7.6 we considered triangular divisions
of a convex $(n+1)$ -gon

The answer was

$$\frac{1}{n} \binom{2n-2}{n-1} = C_{n-1}$$

LEM $F_n = n2^n$

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}$$

pf

$$\frac{1}{n+1} \binom{2n}{n} \stackrel{?}{=} \binom{2n}{n} - \binom{2n}{n+1}$$

$$\binom{2n}{n+1} \stackrel{?}{=} \binom{2n}{n} - \frac{1}{n+1} \binom{2n}{n}$$

$$= \frac{n}{n+1} \binom{2n}{n}$$

$$\frac{(2n)!}{(n+1)!(n)!} \stackrel{?}{=} \frac{n}{n+1} \frac{(2n)!}{n!n!}$$

✓

□

An interpretation of C_n

Fix integer $n \geq 0$

Consider a permutation

$$a_1 a_2 \dots a_n$$

of multi set

$$\{ n \cdot 1, n \cdot -1 \}$$

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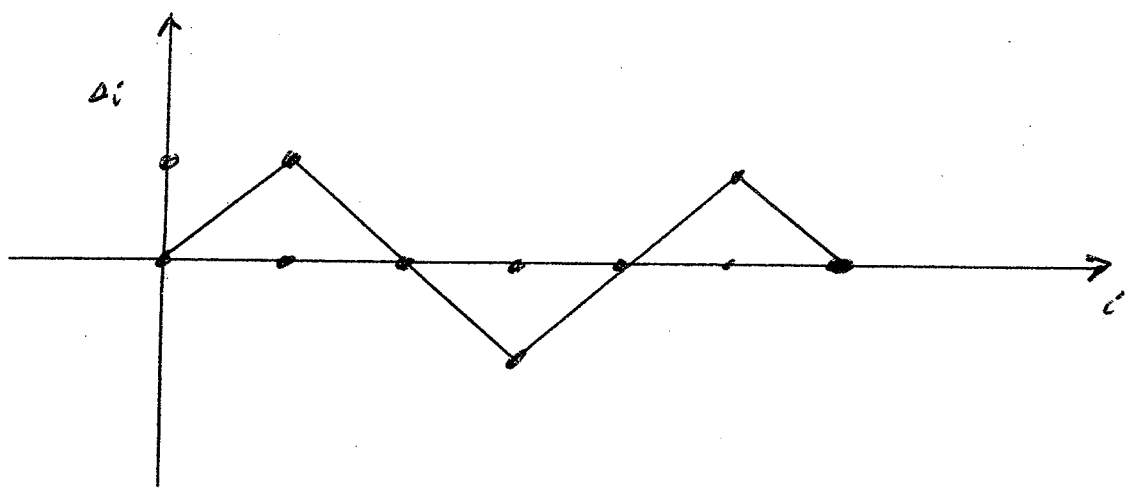
For $0 \leq i \leq n$ the i th partial sum is

$$A_i = a_1 + a_2 + \dots + a_i$$

ex $n = 3$

$$a_1 a_2 a_3 a_4 a_5 a_6 = 1 - 1 - 1 \quad 1 \quad 1 - 1$$

i	0	1	2	3	4	5	6
A_i	0	1	0	-1	0	1	0







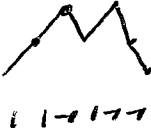



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Problem

Let

$h_n = \#$ perms of $\{1, \dots, n\}$ that have all partial sums nonnegative.

Find h_n

n	desc	h_n
0		1
1		1
2		2
		
3		5
		
		
		
		
⋮		⋮

Thm For $n \geq 0$,

$$C_n = \# \text{ of perms } a_1, a_2, \dots, a_n \text{ of}$$

$$\{1, 1, \dots, 1\}$$

*

such that each partial sum is nonnegative:

$$a_1 + a_2 + \dots + a_i \geq 0 \quad 0 \leq i \leq n$$

pf: Assume $n \geq 1$ else triv.

define

$S_n =$ set of all perms of *

$$|S_n| = \binom{2n}{n}$$

define

$A_n =$ set of perms in S_n that have all partial sums nonneg.

show $|A_n| = C_n$

define

$$U_n = S_n \setminus A_n$$

So

$$|A_n| + |U_n| = |S_n|$$

Suf to show

$$|U_n| = \binom{2n}{n}$$

Def

$T_n =$ set of perms of multiset

$$\{ (n+1) \cdot 1, (n-1) \cdot 1 \}$$

$$|T_n| = \binom{2n}{n}$$

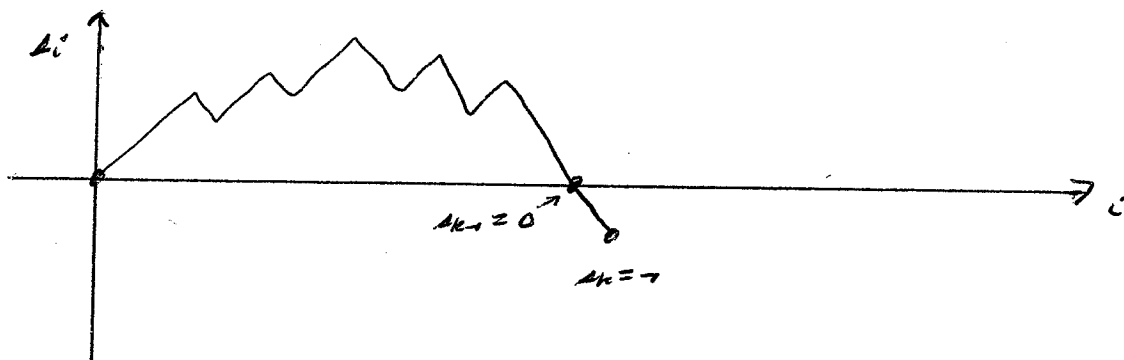
Display bijection

$$f: U_n \rightarrow T_n$$

Given perm
 $a_1 a_2 \dots a_n \in U_n$

This perm has at least one negative partial sum

Pick minimal k such that k th partial sum s_k is neg.



k is odd

$$A_k = -1$$

$$A_{k-1} = 0$$

$$a_k = -1$$

a_1, a_2, \dots, a_{k-1} has $\frac{k-1}{2}$ 1's and $\frac{k-1}{2}$ -1's.

Replace each of a_1, a_2, \dots, a_k by its opposite
and leave alone a_{k+1}, \dots, a_n :

$$(-a_1, (-a_2, \dots, (-a_k), a_{k+1}, \dots, a_n$$

This sequence has $n-k$ 1's and $n-k$ -1's
so it is in T_n

This gives function

$$f: U_n \rightarrow T_n$$

f Injective by const

check f surjective:

Given perm

$$b_1 b_2 \dots b_{2n} \in T_n$$

has $n+1$ 1's
 $n-1$ -1's

Last partial sum is $n+1 - (n-1) = 2 > 0$

Pick minimal k such that left partial sum is > 0

$$b_1 + b_2 + \dots + b_k = 1$$

$$b_k = 1$$

$$b_1 + \dots + b_{2n} = 0$$

k odd
 b_1, b_2, \dots, b_{k-1} has $\frac{k+1}{2}$ 1's
-- $\frac{k-1}{2}$ -1's

For $1 \leq i \leq 2n$ define

$$a_i = \begin{cases} -b_i & \text{if } 1 \leq i \leq k \\ b_i & \text{if } k+1 \leq i \leq 2n \end{cases}$$

then $a_1, a_2, \dots, a_{2n} \in U_n$

and $f(a_1, \dots, a_{2n}) = b_1, \dots, b_{2n}$

- Σ_0 f is surj.
- Σ_0 f is bijecton.

Now

$$|A_n| = \overset{u}{|S_n|} - \overset{u}{|U_n|}$$
$$= \overset{u}{\binom{2n}{n}} - \overset{u}{\binom{2n}{n+1}}$$

$$= \binom{2n}{n} - \binom{2n}{n+1}$$

$$= C_n$$

□

A recurrence for C_n

Thm $F_n \quad n \geq 1$

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \dots + C_{n-1} C_0$$

$$= \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

n	$C_0=1$	$C_1=1$	$C_2=2$	$C_3=5$	$C_4=14$
$C_0=1$	1	1	2	5	14
$C_1=1$	1	1	2	5	
$C_2=2$	2	2	4		
$C_3=5$	5	5			
$C_4=14$	14				
$C_5=42$					

of which is

pf1 We obtained this recurrence in Sec 7.6
in our discussion of triangular divisions of a convex polygon

pf2 Use partial sum interp

Pf2

Def $A = A_n =$ set of perms a_1, a_2, \dots, a_{2n} of $\{1, -1, 1, -1, \dots\}$ with all partial sums $A_i \geq 0$ for $0 \leq i \leq 2n$.

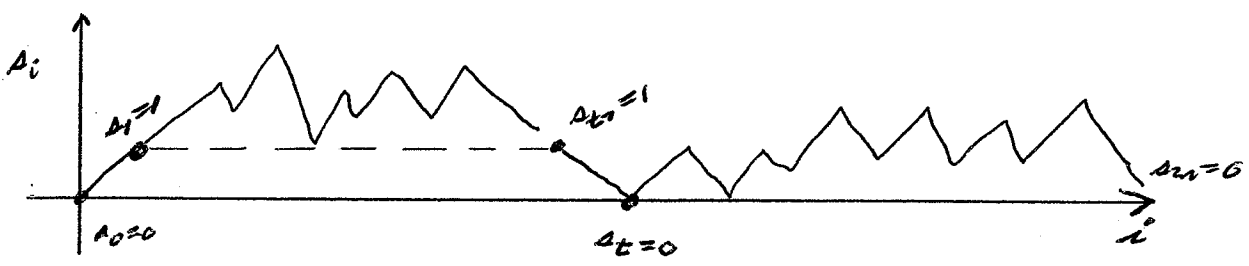
Given $a_1, a_2, \dots, a_{2n} \in A$

$a_1 = 1,$ $a_{2n} = -1$

$A_0 = 0,$ $A_{2n} = 0$

$A_1 = 1,$

Def $t = \min \{ i \mid 2 \leq i \leq 2n, A_i = 0 \}$
"Landing number"



t even

$a_1 = a_t = 1$

$A_t = 0$ $a_t = -1$

$A_{t-1} = 1$

$A_i \geq 1$ $1 \leq i \leq t-1$

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Partition A according to landing number

For $0 \leq k \leq n-1$ define

$A(k) =$ set of perms in A that have landing number $2k+2$

$\{A(k)\}_{k=0}^{n-1}$ partition A

$$\text{So } |A| = \sum_{k=0}^{n-1} |A(k)|$$

For $0 \leq k \leq n-1$ find $|A(k)|$




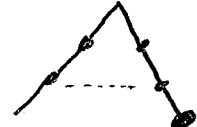

We construct $a_1, a_2, \dots, a_{2n} \in A(k)$ in stages
(write $t = 2k+2$)

stage	to do	# choices
1	pick a_1 ($= 1$)	1
2	pick a_2, a_3, \dots, a_t	C_k
3	pick a_t ($= -1$)	1
4	pick a_{t+1}, \dots, a_{2n}	C_{n-1-k}

$$\text{So } |A(k)| = C_k C_{n-1-k}$$

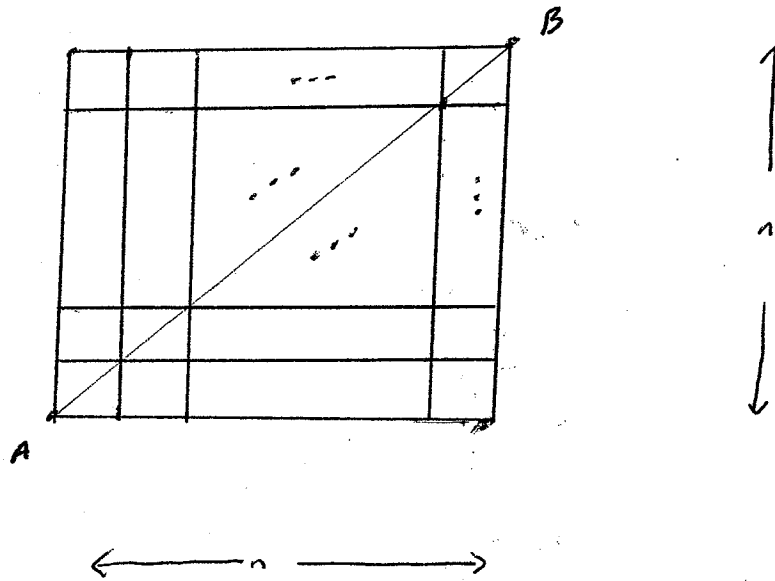
$$\text{So } C_n = |A| = \sum_{k=0}^{n-1} |A(k)| = \sum_{k=0}^{n-1} C_k C_{n-1-k} \quad \square$$

ex n=3

k	$A(k)$		$ A(k) $
0	 1 2 3 4	 1 2 3 4	2 = $C_0 C_2$
1			1 = $C_1 C_4$
2			2 = $C_2 C_4$
			5 = C_3

Ex For $n \geq 1$

Consider city blocks



How many paths of length $2n$ from A to B that do not cross the diagonal?



Let $h_n = \#$ paths from A to B that lie to NW of diagonal

Obs $h_n = \dots$ SE

Ans = $2h_n$

Find h_n

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Each path from A to B is sequence

$$a_1, a_2, \dots, a_{2n}$$

$$a_i \in \{N, E\} \quad 1 \leq i \leq 2n$$

$$\text{View } N \leftrightarrow 1$$

$$E \leftrightarrow -1$$

path stays NW of diag \Leftrightarrow all partial sums ≥ 0

$$h_n = C_n$$

$$\text{Ans} = 2C_n$$

$$= \frac{2}{n+1} \binom{2n}{n}$$

□

8.1 Cont

Recall n th Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

 $n = 0, 1, 2, \dots$ C_n and Multiplication schemesGiven n variables, say $n=4$ a, b, c, d We can compute the product $abcd$ in the following ways

$$(((a b) c) d) \quad ((a b)(c d)) \quad ((a(b c))d) \quad (a(b(c d))) \quad (a(b(c d)))$$

Call these multiplication schemes

Let

 $h_n = \#$ mult schemes involving n variablesFind h_n
One checks

n	1	2	3	4	5	...
h_n	1	1	2	5	14	...

So it appears

$$h_n = C_{n-2} \quad n=1, 2, \dots$$

Thm For $n \geq 1$

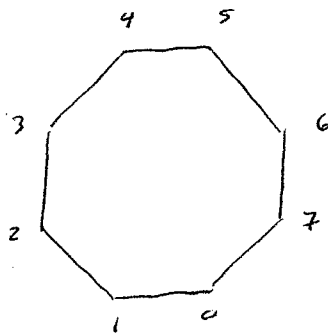
$$C_{n-2} = \# \text{ of multiplication schemes involving } n \text{ variables}$$

pf Call the variables a_1, a_2, \dots, a_n

Consider a convex $(n+1)$ -gon $P_0 \subset \mathbb{R}^2$

ex $n=7$

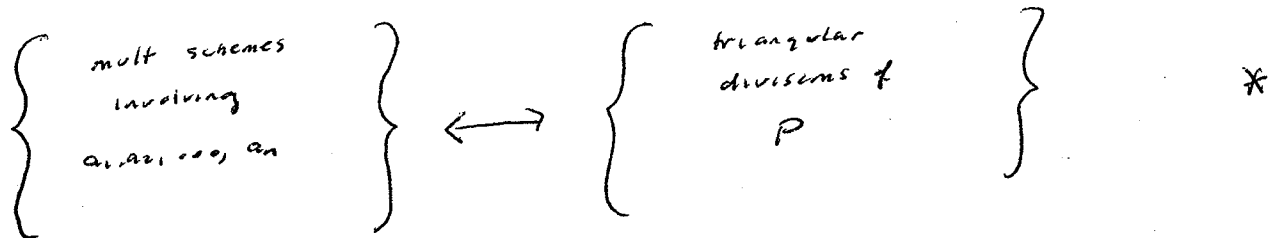
P :



Recall from Section 7.6

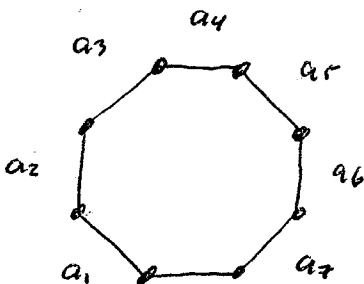
$$C_{n-2} = \# \text{ triangular divisions of } P$$

We display a bijection



For $1 \leq i \leq n$ give the edge e_i in P the label a_i

ex $n=7$

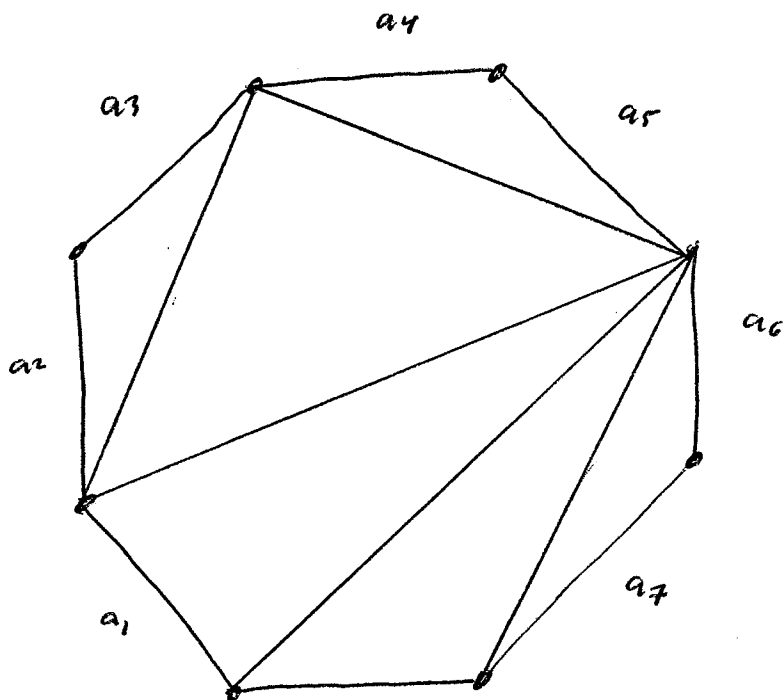


Given a mult scheme involving a_1, a_2, \dots, a_n

ex $n=7$

$$\left(\left(a_1 \left(a_2 a_3 \right) \left(a_4 a_5 \right) \right) \left(a_6 a_7 \right) \right)$$

Use this to divide P :

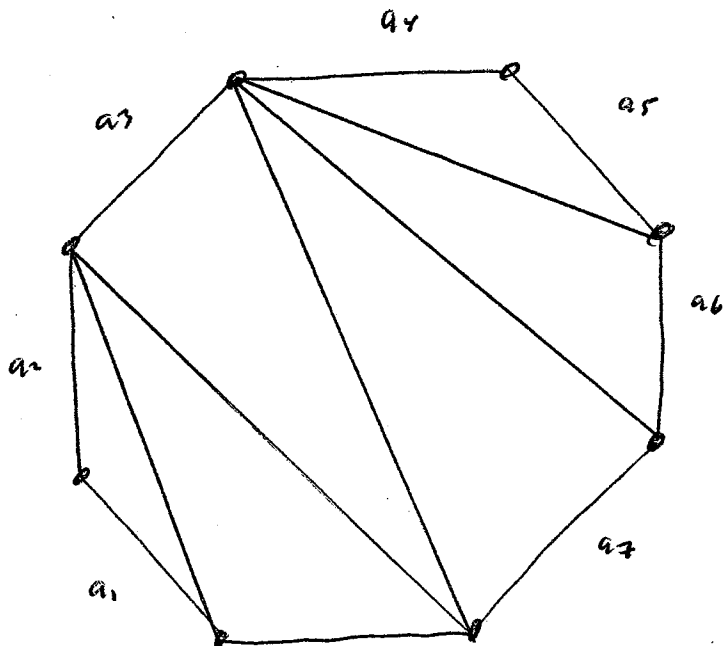


this gives triangular division of P .

Division process is reversible:

Given triangular division of P , find corresp mult scheme
for a_1, a_2, \dots, a_n

ex $n=7$



yields

$$\left((a_1 \ a_2) (a_3 \left((a_4 \ a_5) a_6 \right) a_7) \right)$$

We have displayed a bijection *

Result follows.

□

Next goal Show directly that

$$C_{n-1} = \# \text{ mult schemes involving } n \text{ variables}$$

Def For $n \geq 1$ define

$$C_n^* = n! C_{n-1}$$

"nth pseudo Catalan number"

Obs

$$\begin{aligned}
C_n^* &= n! \frac{1}{n} \binom{2n-2}{n-1} \\
&= (n-1)! \frac{(2n-2)!}{(n-1)! (n-1)!} \\
&= \frac{(2n-2)!}{(n-1)!}
\end{aligned}$$

and

$$C_1^* = 1, \quad C_2^* = 2, \quad C_3^* = 12 \quad \dots$$

For $n \geq 2$

$$\begin{aligned}
\frac{C_n^*}{C_{n-1}^*} &= \frac{(2n-2)!}{(n-1)!} \cdot \frac{(n-2)!}{(2n-4)!} = \frac{(2n-2)(2n-3)}{n-1} = 2(2n-3) \\
&= 4n-6
\end{aligned}$$

So

$$C_n^* = (4n-6) C_{n-1}^*$$

$$n = 2, 3, 4, \dots$$

$$C_1^* = 1$$

Direct pf that

$$C_{n-1}^* = \# \text{ mult schemes involving } n \text{ variables}^*$$

Call the variables a_1, a_2, \dots, a_n

Let $M_n =$ set of mult schemes involving any permutation
of a_1, a_2, \dots, a_n

$$\text{So } |M_n| = n! \left(\# \text{ of mult schemes involving } a_1, a_2, \dots, a_n \text{ in order} \right)$$

Suf to show

$$|M_n| = C_n^*$$

Suf to show

$$|M_1| = 1$$

and

$$|M_n| = (4n-6) |M_{n-1}|$$

$$n \geq 2$$

* is clear

*

**

show **

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Given mult scheme in M_{n-1}
involves some perm of a_1, a_2, \dots, a_{n-1}

Insert a_n in $4n-6$ poss ways to get mult scheme in M_n

ex $n=4$

Given mult scheme for a, b, c

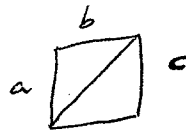
say $(ab)c$

insert d in $4n-6=10$ ways to get
mult scheme for a perm of a, b, c, d

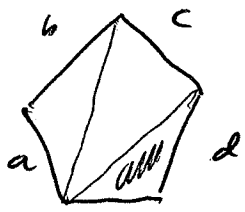
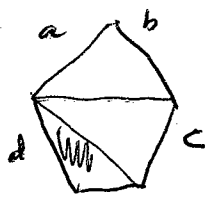
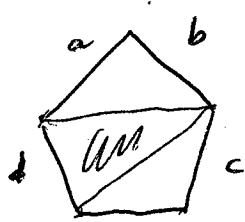
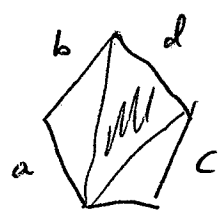
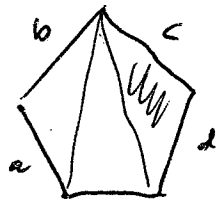
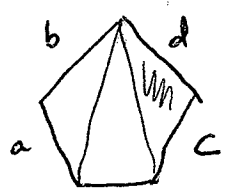
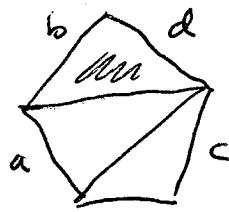
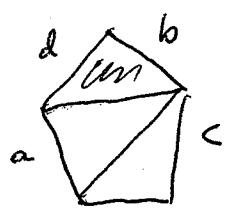
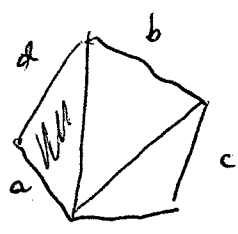
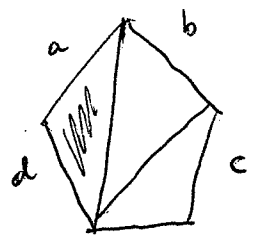
View

$((ab)c)$

\leftrightarrow



shaded triangle shaded



8.2 Difference Sequences and Stirling numbers

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Given a sequence of real numbers $\{h_n\}_{n=0}^{\infty}$

Define a new sequence $\{\Delta h_n\}_{n=0}^{\infty}$ where

$$\Delta h_n = h_{n+1} - h_n$$

View Δ as an operator that acts on all sequences $\{h_n\}_{n=0}^{\infty}$

Ex take $h_n = n^2$

$$\Delta \quad \downarrow$$

$$\{n^2\}_{n=0}^{\infty}$$

$$(n+1)^2 - n^2 = 2n+1$$

$$\Delta \quad \downarrow$$

$$\{2n+1\}_{n=0}^{\infty}$$

$$2(n+1)+1 - (2n+1) = 2$$

$$\Delta \quad \downarrow$$

$$\{2\}_{n=0}^{\infty}$$

$$2 - 2 = 0$$

$$\Delta \quad \downarrow$$

$$\{0\}_{n=0}^{\infty}$$

Define

$$\mathbb{0} = \{0\}_{n=0}^{\infty} \\ = (0, 0, 0, \dots)$$

For $r \geq 0$ the operator Δ^r acts by applying Δ r times.

Δ^0 means "do nothing"

$$\Delta^0 h_n = h_n$$

$$n = 0, 1, 2, \dots$$

Δ^r sends $\{n^2\}_{n=0}^{\infty}$ to $\mathbb{0}$ for $r \geq 3$

View

	0	1	4	9	16	25	---
$\Delta \downarrow$		1	3	5	7	9	---
$\Delta \downarrow$			2	2	2	2	---
$\Delta \downarrow$				0	0	0	---
$\Delta \downarrow$					0	0	---
\vdots							

"difference table"

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Define

$V =$ set of all sequences of real numbers $\{h_n\}_{n=0}^{\infty}$

Given $h, h' \in V$

Write

$$h = \{h_n\}_{n=0}^{\infty}$$

$$h' = \{h'_n\}_{n=0}^{\infty}$$

Define

$$h+h' = \{h_n+h'_n\}_{n=0}^{\infty}$$

$\in V$

Given a real number α

Define

$$\alpha h = \{\alpha h_n\}_{n=0}^{\infty}$$

$\in V$

Obs

$$\alpha(h+h') = \alpha h + \alpha h'$$

$$h + 0 = 0 + h = h$$

V is a "vector space"

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the function

$$\Delta: V \rightarrow V$$

respects addition and scalar mult as follows:

$$\Delta(h+h') = \Delta h + \Delta h'$$

$\forall h, h' \in V$

$$\Delta(\alpha h) = \alpha \Delta h$$

$\forall \alpha \in \mathbb{R} \quad \forall h \in V$

" Δ is a linear transformation of V "

— o —

Question

what elements of V does Δ^r
send to $\mathbb{0}$ for suf large r ?

Thm Given any polynomial $f(x)$

let $p = \text{degree of } f$

Consider the sequence

$$\{f(n)\}_{n=0}^{\infty}$$

Then

Δ^{p+1} sends this sequence to $\textcircled{0}$

pf Use induction on p

Case $p=0$: true ✓

Case $p \geq 1$:

Write

$$f(x) = a_0 + a_1 x + \dots + a_p x^p$$

$$\begin{aligned} \text{So } f(n) &= a_0 + a_1 n + a_2 n^2 + \dots + a_p n^p \\ &= \sum_{k=0}^p a_k n^k \end{aligned}$$

$$\text{So } \Delta^{p+1} \{f(n)\}_{n=0}^{\infty} = \sum_{k=0}^p a_k \Delta^{p+1} \{n^k\}_{n=0}^{\infty}$$

So to show

$$\Delta^{p+1} \{n^k\}_{n=0}^{\infty} = \textcircled{0}$$

$0 \leq k \leq p$

So to show

$$\Delta^{p+1} \{n^p\}_{n=0}^{\infty} = \textcircled{0}$$

Write

$$\Delta \{n^p\}_{n=0}^{\infty} = \{h_n\}_{n=0}^{\infty}$$

$$h_n = (n+1)^p - n^p$$

$$= \sum_{k=0}^p \binom{p}{k} n^k - n^p$$

$$= \sum_{k=0}^{p-1} \binom{p}{k} n^k$$

$$= g(n)$$

$$g(x) = \sum_{k=0}^{p-1} \binom{p}{k} x^k$$

$g(x)$ has degree $p-1$

So by induction

$$\Delta^p \{h_n\}_{n=0}^{\infty} = \textcircled{0}$$

Now

$$\Delta^{p+1} \{n^p\}_{n=0}^{\infty} = \Delta^p \Delta \{n^p\}_{n=0}^{\infty}$$

$$= \Delta^p \{h_n\}_{n=0}^{\infty}$$

$$= \textcircled{0}$$

Result follows.

□

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Next general goal:

Given $h = \{h_n\}_{n=0}^{\infty} \in \mathbb{V}$

Suppose $\exists p \geq 0$ s.t.

$$\Delta^{p+1} h = 0$$

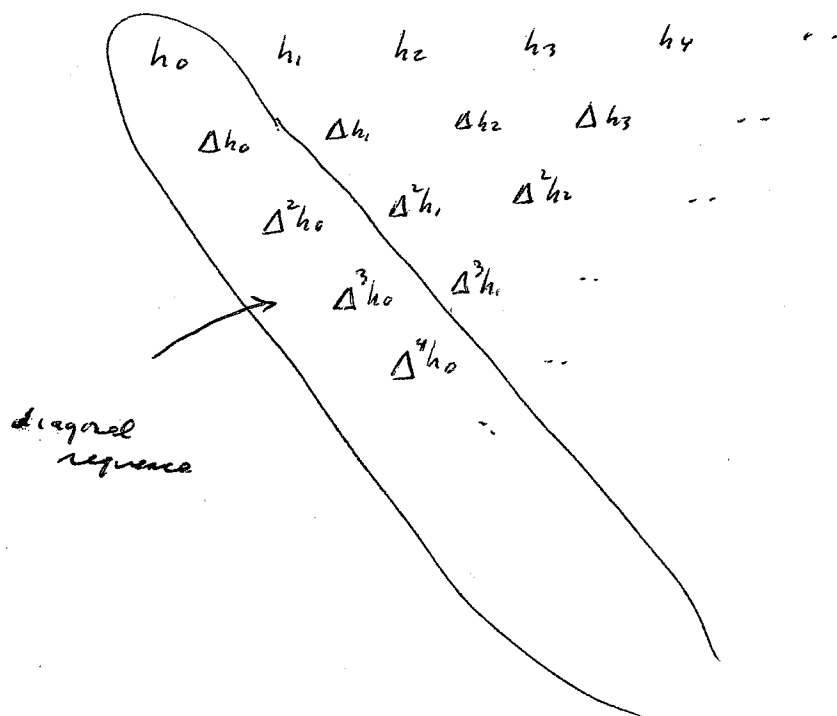
show there exists a polynomial $f(x)$ of degree $\leq p$

such that

$$h_n = f(n) \quad \forall n \geq 0$$

Given a sequence $\{h_n\}_{n=0}^{\infty}$

Construct difference table



the corresp diagonal sequence is

$$\{\Delta^n h_0\}_{n=0}^{\infty}$$

Ex Define

$$h_n = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad n = 0, 1, 2, \dots$$

Find the corresp diagonal sequence

Sol. Create difference table

$\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 3 \\ 10 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 3 \\ 20 \end{pmatrix}$...
	0	0	1	3	6	10	...
		0	1	2	3	4	...
			1	1	1	1	...
				0	0	0	...
							...

Diagonal sequence is

0, 0, 0, 1, 0, 0, ...

LEM Fix an integer $p \geq 0$

For the sequence

$$\left\{ \binom{n}{p} \right\}_{n=0}^{\infty}$$

The corresp diagonal sequence is

$$0, 0, \dots, 0, 1, 0, 0, \dots$$

↑
Location p

pf

Denote the diagonal sequence by $\{c_n\}_{n=0}^{\infty}$

show

$$c_n = \begin{cases} 1 & \text{if } n=p \\ 0 & \text{if } n \neq p \end{cases} \quad n \geq 0$$

Create difference table

$\binom{0}{p}$	$\binom{1}{p}$	$\binom{2}{p}$...	$\binom{p-1}{p}$	$\binom{p}{p}$	$\binom{p+1}{p}$...
0	0	0	...	0	1	p	...
	0	0	0	...	0	1	p ²
		0	0	...	0	1	p ²
			...	0	1	p ²	...
				...			
				0	1	2	...
				1	1	...	
				0	...		

row p

So $c_n = 0$ $0 \leq n \leq p-1$
 and $c_p = 1$

Recall $\binom{n}{p} = \frac{n(n-1)(n-2) \dots (n-p+1)}{p!} = \text{poly in } n \text{ with degree } p$

So by prev thm

Δ^{p+1} sends $\{\binom{n}{p}\}_{n=0}^{\infty}$ to $\mathbb{0}$

Therefore $c_n = 0$ if $n \geq p+1$



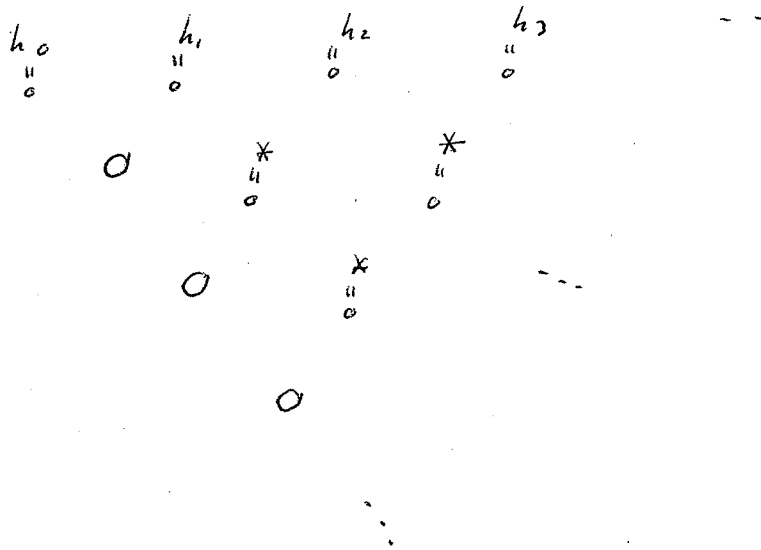
Ex Given a sequence $\{h_n\}_{n=0}^{\infty}$

Suppose its diagonal sequence is all zeros

$$0, 0, 0, \dots$$

Find h_n

Sol Difference table is



$$h_n = 0 \quad n = 0, 1, 2, \dots$$

So $\{h_n\}_{n=0}^{\infty} = \textcircled{0}$

□

Cor. A sequence $\{h_n\}_{n=0}^{\infty}$ is uniquely determined
by its diagonal sequence

pf. Suppose $\{h_n\}_{n=0}^{\infty}$ and $\{h'_n\}_{n=0}^{\infty}$ have the
same diagonal sequence. Show $h_n = h'_n \quad n = 0, 1, 2, \dots$

Then the sequence $\{h_n - h'_n\}_{n=0}^{\infty}$ has diagonal
sequence $0, 0, 0, \dots$

Now $\{h_n - h'_n\}_{n=0}^{\infty} = \textcircled{0}$

so $h_n = h'_n \quad n = 0, 1, 2, \dots$

□

Ex Find the sequence $\{h_n\}_{n=0}^{\infty}$ whose diagonal sequence is

$$1, 3, 2, 5, 0, 0, \dots$$

*

Sol. View * as

$$1 \text{ times } 1, 0, 0, 0, \dots$$

$$+ 3 \text{ times } 0, 1, 0, 0, \dots$$

$$+ 2 \text{ times } 0, 0, 1, 0, \dots$$

$$+ 5 \text{ times } 0, 0, 0, 1, \dots$$

So

$$h_n = 1 \binom{n}{0} + 3 \binom{n}{1} + 2 \binom{n}{2} + 5 \binom{n}{3}$$

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Thm Fix integer $p \geq 0$

Given real numbers

$$c_0, c_1, \dots, c_p$$

Let $\{h_n\}_{n=0}^{\infty}$ denote the sequence with diagonal sequence

$$c_0, c_1, c_2, \dots, c_p, 0, 0, 0, \dots$$

then

$$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + c_2 \binom{n}{2} + \dots + c_p \binom{n}{p}$$

pf clear from prev Lem + Cor

□

Thm Given sequence $h = \{h_n\}_{n=0}^{\infty}$

Assume \exists integer $p \geq 0$ such that

$$\Delta^{p+1} h = 0$$

then \exists polynomial $f(x)$ of degree $\leq p$ such that

$$h_n = f(n) \quad n = 0, 1, 2, \dots$$

pf. Let $\{c_n\}_{n=0}^{\infty}$ denote the diagonal sequence for h .

Obs $c_n = 0$ if $n \geq p+1$

By prev thm

$$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \dots + c_p \binom{n}{p} \quad n = 0, 1, 2, \dots$$

Define

$$f(x) = c_0 \binom{x}{0} + c_1 \binom{x}{1} + \dots + c_p \binom{x}{p}$$

$f(x)$ has degree $\begin{cases} p & \text{if } c_p \neq 0 \\ < p & \text{if } c_p = 0 \end{cases}$

$$h_n = f(n) \quad n = 0, 1, 2, \dots$$

□

Ex Find a formula for

$$1^4 + 2^4 + 3^4 + \dots + n^4 \quad n \geq 0$$

Sol. Consider sequence $\{n^4\}_{n=0}^{\infty}$

Find the diagonal sequence

0^4	1^4	2^4	3^4	4^4	5^4
0	1	16	81	256	625
	1	15	65	175	369
		14	50	110	194
			36	60	84
				24	24
					0

Diag sequence is

$$0, 1, 14, 36, 24, 0, 0, \dots$$

So

$$n^4 = \binom{n}{1} + 14 \binom{n}{2} + 36 \binom{n}{3} + 24 \binom{n}{4}$$

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Recall

$$\binom{0}{k} + \binom{1}{k} + \binom{2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$$

$n, k \geq 0$

So

$$1^4 + 2^4 + 3^4 + \dots + n^4 =$$

$$\binom{n+1}{2} + 14 \binom{n+1}{3} + 36 \binom{n+1}{4} + 24 \binom{n+1}{5}$$

Ex Given a polynomial $f(x)$ deg p

Find formula for

$$f(0) + f(1) + \dots + f(n) \quad n = 0, 1, 2, \dots$$

Sol. Consider sequence $\{f(n)\}_{n=0}^{\infty}$

Let $\{c_n\}_{n=0}^{\infty}$ denote corresp diagonal sequence.

Obs

$$c_n = 0 \quad \text{if } n > p$$

$$f(n) = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \dots + c_p \binom{n}{p} \quad n \geq 0$$

Now

$$f(0) + f(1) + \dots + f(n) =$$

$$c_0 \binom{n+1}{0} + c_1 \binom{n+1}{1} + \dots + c_p \binom{n+1}{p} \quad n = 0, 1, 2, \dots$$

□

Stirling numbers

Consider the sequence of polynomials

$$1, x, x^2, x^3, \dots$$

*

Another sequence of polynomials

$$1, x, x(x-1), x(x-1)(x-2), \dots$$

$$\begin{matrix} \text{"} & \text{"} & \text{"} & \text{"} \\ [x]_0 & [x]_1 & [x]_2 & [x]_3 \end{matrix}$$

**

Two problems:

(i) Write * in terms of **

(coeffs give Stirling numbers of 2nd kind)

(ii) Write ** in terms of *

(coeffs give Stirling numbers of 1st kind)

We start with (i)

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Ex Write x^3 in terms of x^k

Sol Consider sequence $\{n^3\}_{n=0}^{\infty}$

Create difference table

0	1	8	27	64	...
	1	7	19	37	
		6	12	18	
			6	6	
				0	

So

$$n^3 = 0 \binom{n}{0} + 1 \binom{n}{1} + 6 \binom{n}{2} + 6 \binom{n}{3}$$

$$n^3 = n + 6 \frac{n(n-1)}{2} + 6 \frac{n(n-1)(n-2)}{3 \cdot 2}$$

$$= n + 3n(n-1) + n(n-1)(n-2)$$

$n = 0, 1, 2, \dots$

So

$$\begin{aligned} x^3 &= x + 3x(x-1) + x(x-1)(x-2) \\ &= [x]_1 + 3[x]_2 + [x]_3 \end{aligned}$$

□

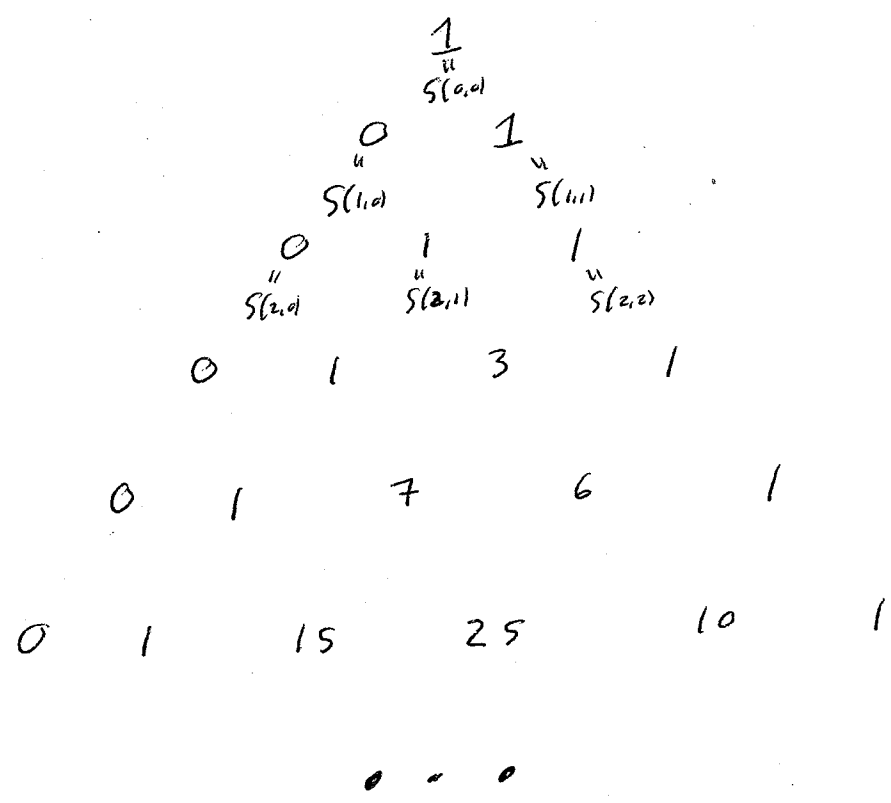
More generally we have

$(x^2 - 1)^2 = x^4 - 2x^2 + 1$

$\frac{1}{4} \sqrt{1/2}$

	$[x]_0$	$[x]_1$	$[x]_2$	$[x]_3$	$[x]_4$	$[x]_5$...
1	1						
x	0	1					
x^2	0	1	1				
x^3	0	1	3	1			
x^4	0	1	7	6	1		
x^5	0	1	15	25	10	1	
\vdots			...				

View as analog to Pascal triangle:



"Stirling numbers of 2nd kind"

By construction for $n \geq 0$

$$x^n = \sum_{k=0}^n S(n,k) [x]_k$$

Obs

- $S(n,0) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$

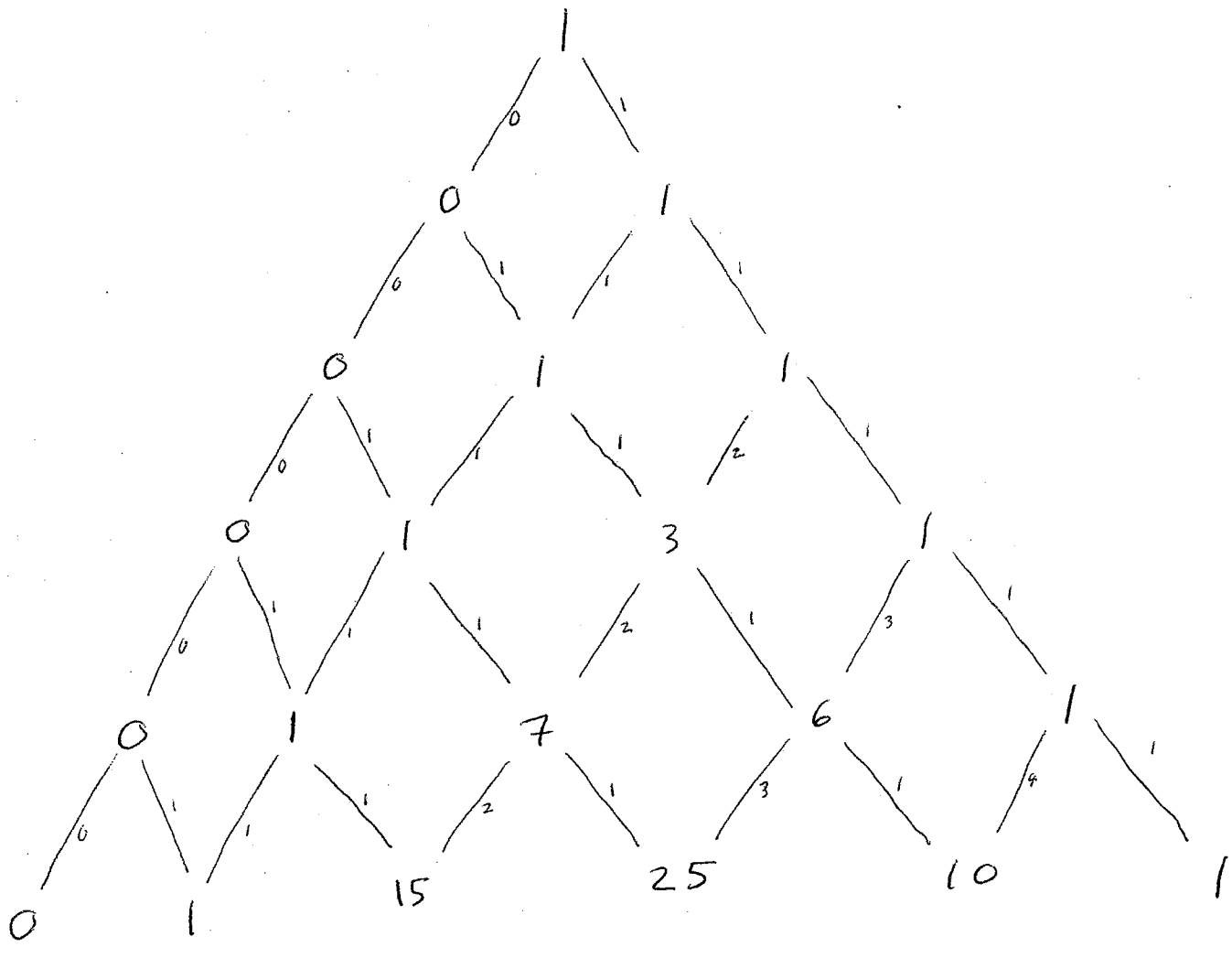
- $S(n,1) = 1 \quad n \geq 1$

- $S(n,n) = 1 \quad n \geq 0$

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Next goal: show $S(n,k)$ satisfy a recurrence

Similar to Pascal's formula



...

For notational convenience define

$$S(n, k) = 0 \quad \text{if } k < 0 \text{ or } k > n$$

thm For $0 \leq k \leq n$

$$S(n, k) = S(n-1, k) + k S(n-1, k-1)$$

pf Assume $n \geq 1$ else triv

Recall

$$X^n = \sum_{k=0}^n S(n, k) [x]_k$$



So

$$X^{n+1} = \sum_{k=0}^{n+1} S(n+1, k) [x]_k$$

Also

$$\begin{aligned}
 X^n &= X^{n-1} X \\
 &= \sum_{k=0}^{n-1} S(n-1, k) [x]_k \underset{\substack{\parallel \\ X(x-1)(x-2)\dots(x-k+1)}}{X} \underset{X-k+k}{x}
 \end{aligned}$$

$$\left[[x]_k x = [x]_{k+1} + k [x]_k \right]$$

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$$\begin{aligned}
 &= \sum_{k=0}^{n-1} S(n-1, k) [x]_{kn} + \sum_{k=0}^{n-1} k S(n-1, k) [x]_k \\
 &\quad \downarrow k \rightarrow kn \\
 &\sum_{k=1}^n S(n-1, k-1) [x]_k \\
 &\quad \downarrow \\
 &\quad k=0 \\
 &= \sum_{k=0}^n \left(S(n-1, k-1) + k S(n-1, k) \right) [x]_k
 \end{aligned}$$

Compare with \star to get

$$S(n, k) = S(n-1, k-1) + k S(n-1, k)$$

OSK EN

□

Thm F_n $0 \leq k \leq n$

$$S(n, k) = \frac{1}{k!} \sum_{t=0}^k \binom{k}{t} (-1)^t (k-t)^n$$

where $0^0 = 1$

pf Ind m n

$n=0$ ✓

$n \geq 1$:

Case $k=0$

$$\begin{array}{rcl}
 S(n, 0) & = & \frac{1}{0!} \binom{0}{0} (-1)^0 0^n \\
 \text{"} & & \text{"} \quad \text{"} \quad \text{"} \\
 0 & & 1 \quad 1 \quad 1
 \end{array}$$

✓

Case $1 \leq k \leq n$

$$\begin{aligned}
 S(n, k) &= S(n-1, k+1) + k S(n-1, k) \\
 &= \frac{1}{(k+1)!} \sum_{t=0}^{k+1} \binom{k+1}{t} (-1)^t (k+1-t)^{n-1} \\
 &\quad + \frac{k}{k!} \sum_{t=0}^k \binom{k}{t} (-1)^t (k-t)^{n-1}
 \end{aligned}$$