7. Recurrence Relations and Generating functions

7.1 Some number sequences

Consider a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$:

$$a_0, a_1, a_2, \ldots$$

Gives a sequence of partial sums $\{s_n\}_{n=0}^{\infty}$

$$s_n = a_0 + a_1 + \cdots + a_n \qquad n = 0, 1, 2 \ldots$$

Examples

- Arithmetic sequences

$$\underbrace{a_n - a_{n-1}}_{= k} \text{ is indep of } n \text{ for } n \geq 1$$

Here $$a_n = a_0 + nk \qquad n = 0, 1, 2, \ldots$$

$$a_0 + a_1 + \cdots + a_n = (n+1)a_0 + k\binom{n+1}{2}$$

- Geometric sequences

$$\underbrace{a_n/a_{n-1}}_{=q} \text{ indep of } n \text{ for } n \geq 1$$

Here

$$a_n = a_0 q^n \qquad n = 0, 1, 2 \ldots$$

$$a_0 + a_1 + \cdots + a_n = a_0(1 + q + q^2 + \cdots + q^n)$$

$$= \begin{cases} a_0 \dfrac{q^{n+1} - 1}{q - 1} & \text{if } q \neq 1 \\ a_0(n+1) & \text{if } q = 1 \end{cases}$$

The Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$

[saw this in ch 1]

Satisfies

$$f_n = f_{n-1} + f_{n-2} \qquad n \geq 2$$

$$f_0 = 0, \qquad f_1 = 1$$

n	0	1	2	3	4	5	6	7	8	9	10	..
f_n	0	1	1	2	3	5	8	13	21	34	55	..

Find partial sums

$f_0+f_1+\cdots+f_n$

n	0	1	2	3	4	5	6	7	8	$\cdots$	n
$f_0+f_1+\cdots+f_n$	0	1	2	4	7	12	20	33	54	$\cdots$	$f_{n+2}-1$?

LEM For $n \geq 0$

$$f_0+f_1+\cdots+f_n = f_{n+2}-1$$

pf By induction on n

$n=0$: $\underset{\overset{\|}{0}}{f_0} \overset{?}{=} \underset{\overset{\|}{1}}{f_2} - 1$ ✓

$n \geq 1$:

$$\underbrace{f_0+f_1+\cdots+f_{n-1}}_{\overset{\text{ind}}{=}\ f_{n+1}-1} + f_n \overset{?}{=} \underset{\overset{\|}{f_{n-1}+f_n}}{f_{n+2}} - 1 \quad ✓$$

Next goal: Find f_n in closed form

First ignore initial conds $f_0 = 0,\ f_1 = 1$

Consider any sequence that satisfies

$$f_n = f_{n-1} + f_{n-2} \qquad n \geq 2 \qquad *$$

Hunt for solutions of form

$$f_n = x^n \qquad x \in \mathbb{R}$$

Require

$$x^n = x^{n-1} + x^{n-2} \qquad n \geq 2$$

so

$$x^2 = x + 1$$

Find x:

$$x^2 - x - 1 = 0$$

By quad formula

$$x = \frac{1 \mp \sqrt{1^2 - 4(1)(-1)}}{2}$$

$$= \frac{1 \mp \sqrt{5}}{2}$$

Both sequences

$$f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

satisfy $*$.

Now for any $a, b \in \mathbb{R}$ the sequence

$$f_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

satisfies $*$

Now bring in init conds

$$0 = f_0 = a + b \quad \rightarrow b = -a$$

$$1 = f_1 = a\,\frac{1+\sqrt{5}}{2} + b\,\frac{1-\sqrt{5}}{2}$$

$$= a\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

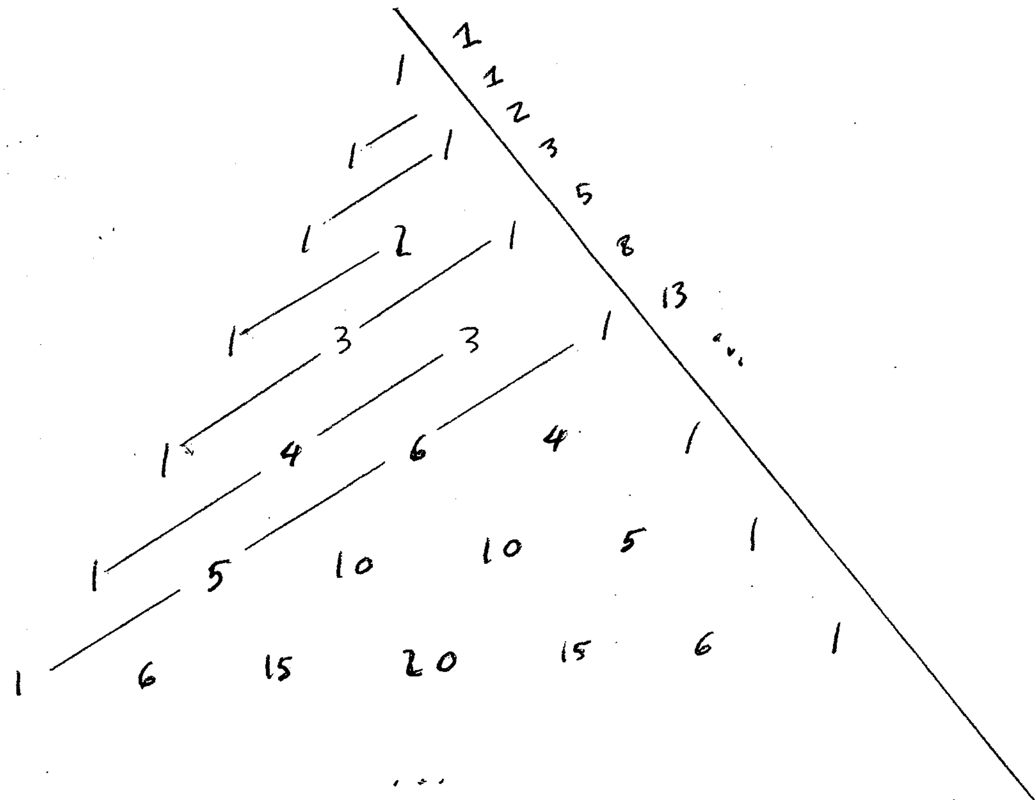
$$= a\sqrt{5}$$

$$a = \frac{1}{\sqrt{5}} \qquad b = \frac{-1}{\sqrt{5}}$$

So

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad n = 0, 1, 2, \ldots$$

Recall Pascal's triangle



LEM F_n $n \geq 0$

$$\underbrace{\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots}_{F_n} = f_{n+1}$$

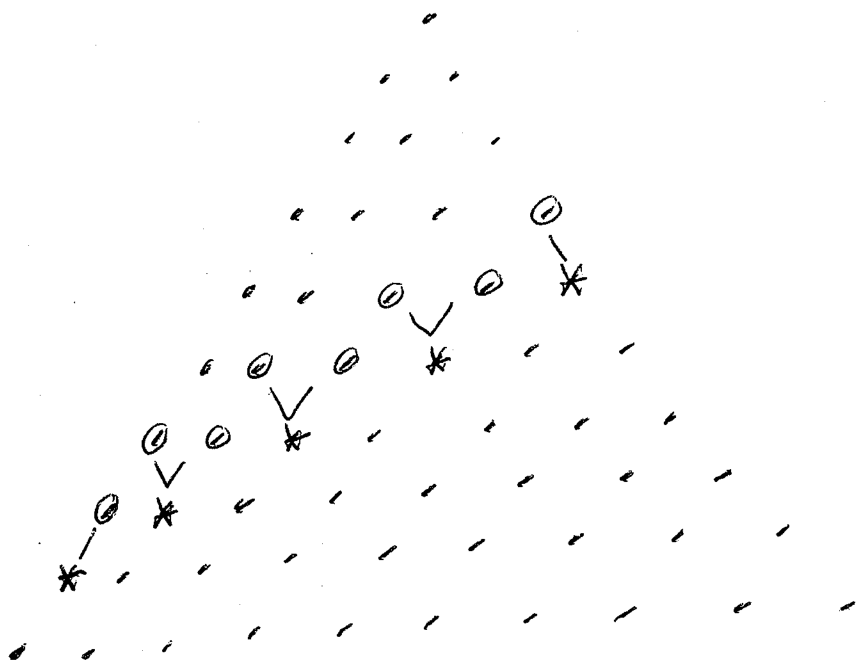
pf Suf to show

$F_0 = f_1$ ✓

$F_1 = f_2$ ✓

$F_n = F_{n-1} + F_{n-2}$ $n \geq 2$

show $F_n = F_{n-1} + F_{n-2}$



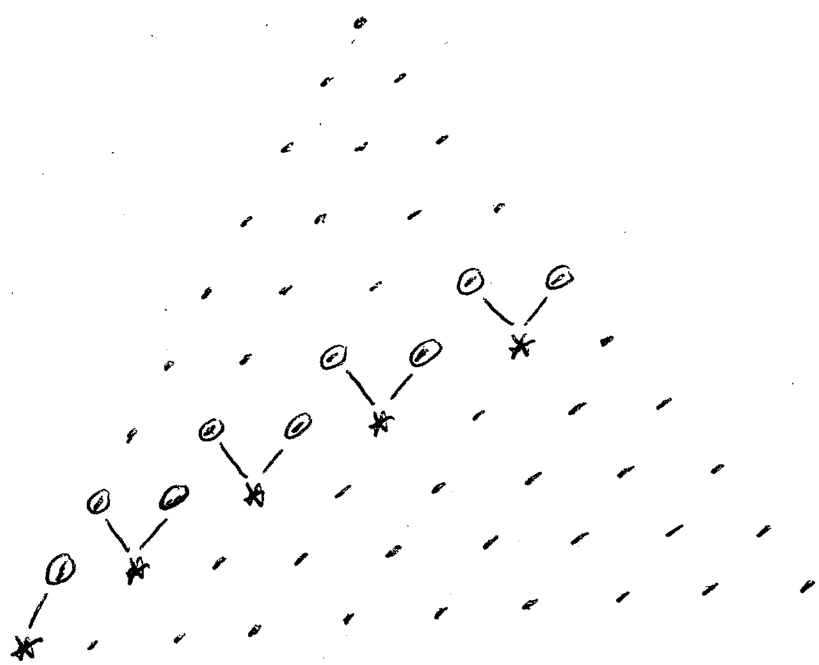
show

sum of ⊙

=

sum of *

follows from Pascal's formula



We mention some basic facts about Fibonacci numbers

LEM For $n \geq 1$

f_n, f_{n+1} are relatively prime

pf Ind on n

$n=1$ $f_1=1$ $f_2=1$ ✓

$n \geq 2$: Given pos integer x such that

$x \mid f_n$ $x \mid f_{n+1}$

show $x=1$

$$f_{n+1} = f_n + f_{n-1}$$

$x \mid f_{n+1}$ and $x \mid f_n$ so $x \mid f_{n-1}$

x divides $\underbrace{f_n, f_{n-1}}_{\text{rel pr}}$

$x=1$

□

LEM For integers $r, s \geq 0$

$$\underbrace{f_r f_s + f_{r+1} f_{s+1}}_{F(r,s)}$$

depends only on $r+s$

pf Show

$$F(r,s) = F(r-1, s+1) \qquad r \geq 1,\ s \geq 0$$

$$f_r f_s + f_{r+1} f_{s+1} \overset{?}{=} \underbrace{f_{r-1} f_{s+1} + f_r \underset{f_{s+1}+f_s}{\underset{\shortparallel}{f_{s+2}}}}_{\left(\underbrace{f_{r-1}+f_r}_{\substack{\shortparallel \\ f_{r+1}}}\right) f_{s+1} \; + \; f_r f_s}$$

✓

COR For integers $r, s \geq 0$

$$f_r f_s + f_{r+1} f_{s+1} = f_{r+s+1}$$

pf By prev Lem

$$\begin{aligned} F(r,s) &= F(r+s, 0) \\ &= f_{r+s} \underset{\substack{\shortparallel \\ 0}}{f_0} + f_{r+s+1} \underset{\substack{\shortparallel \\ 1}}{f_1} \\ &= f_{r+s+1} \end{aligned}$$

COR (alt) For $r, s \geq 1$

$$f_{r+s} = f_r f_{s-1} + f_s f_{r+1}$$
$$= f_s f_{r-1} + f_r f_{s+1}$$

LEM For $r, s \geq 1$

Given an integer x that divides at least two of

f_r, f_s, f_{r+s}

Then x divides all three

pf assume x / f_r and x / f_s

Then x / f_{r+s} by Prev Cor

Assume x / f_r and x / f_{r+s}

By Prev Cor
$x / f_s f_{r+1}$

But x, f_{r+1} are rel prime since

x / f_r and f_r, f_{r+1} rel prime

So x / f_s

$\square$

LEM For $n \geq 1$

f_n divides $f_{2n}, f_{3n}, f_{4n}, \ldots$

pf Show

$$f_n \mid f_{kn} \quad \text{for } k = 1, 2, \ldots$$

Ind on k

$k = 1$ ✓

$k \geq 2$ By ind

$$f_n \mid f_{(k-1)n}$$

Consider

$$f_n \qquad f_{(k-1)n} \qquad f_{kn}$$

Apply Prev Lem with

$r = n$

$s = (k-1)n$

$x = f_n$

Get $f_n \mid f_{kn}$

□

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LEM Given rel prime integers $m, n \geq 1$

Then

f_m, f_n rel prime

pf Induction on $\min(m,n)$

Case $\min(m,n) = 1$ ✓

Case $\min(m,n) > 1$: WLOG $m > n$

Divide m by n and consider remainder:

$$m = qn + r \qquad 1 \leq r < n$$

Obs n, r rel prime since m, n rel prime.

Now f_n, f_r are rel prime by ind and since $r < n$.

Given pos integer x s.t.

$$x / f_m \qquad x / f_n$$

show $x = 1$.

Consider

$$f_m, \quad f_{qn}, \quad f_r \qquad *$$

x / f_n so x / f_{qn}

x divides at least 2 of $*$ $\to$ x divides all 3

x / f_r

Now x / f_n and x / f_r so $x = 1$ $\square$

thm Given integers $m, n \geq 1$

let
$$d = GCD(m, n)$$
$\uparrow$ greatest common divisor

then
$$f_d = GCD(f_m, f_n)$$

Pf Induction on $\min(m, n)$

Case $\min(m,n) = 1$ ✓

Case $\min(m,n) > 1$: WLOG $m > n$

Divide m by n and consider remainder:
$$m = qn + r \qquad 1 \leq r \leq n-1$$

Obs
$$GCD(n, r) = GCD(m, n) = d$$

By ind and since $r \leq n-1$
$$f_d = GCD(f_n, f_r)$$

$d | m$ and $d | n$ so $f_d | f_m$ and $f_d | f_n$.

Conversely, given pos integer x s.t.
$$x | f_m \quad \text{and} \quad x | f_n$$
show $x | f_d$.

Consider
$$f_m, \; f_{qn}, \; f_r \qquad *$$

$x | f_m$.
$x | f_n$ so $x | f_{qn}$.
x divides at least two of $*$ $\to$ x divides all three of $*$.
$x | f_r$

So $x | f_n$ and $x | f_r$ so $x \,|\, \underset{f_d}{\underset{\shortparallel}{GCD(f_n, f_r)}}$ so $x | f_d$ $\square$

Lecture 23 Monday Oct 29

7.2 Generating Functions

Given a sequence of real numbers

$$\{h_n\}_{n=0}^{\infty}$$

Corresp <u>generating function</u> is

$$g(x) = \sum_{n=0}^{\infty} h_n x^n \qquad x = \text{variable}$$

Formal sum — dont consider convergence

To understand h_n sometimes easier to work with $g(x)$

ex

$$h_n = 1 \qquad \forall n$$

$$g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

ex

$$h_n = \frac{1}{n!} \qquad n = 0, 1, 2 \ldots$$

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

ex Given integer $m \geq 0$

Consider sequence

$$\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$$

Find gen function.

Sol. View sequence as ∞ by adding 0 terms

$$\binom{m}{0}, \ldots, \binom{m}{m}, 0, 0, \ldots$$

$$g(x) = \sum_{n=0}^{m} \binom{m}{n} x^n$$

$$= (1+x)^m$$

Ex Fix integer $k \geq 1$

For $n \geq 0$ let

$h_n =$ # integral solutions to

$e_1 + e_2 + \cdots + e_k = n$

$e_1, e_2, \ldots, e_k \geq 0$

Find gen function for $\{h_n\}_{n=0}^{\infty}$

Sol I We saw earlier

$$h_n = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

So

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

$$= \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

$$= (1-x)^{-k} \qquad \text{by Ch 5.5}$$

Sol II Consider

$$(1-x)^{-k}$$

Recall

$$\frac{1}{1-x} = 1+x+x^2+\cdots$$

$$= \sum_{e=0}^{\infty} x^e$$

So

$$(1-x)^{-k} = \frac{1}{1-x}\,\frac{1}{1-x}\cdots\frac{1}{1-x}$$

$$= \left(\sum_{e_1=0}^{\infty} x^{e_1}\right)\left(\sum_{e_2=0}^{\infty} x^{e_2}\right)\cdots\left(\sum_{e_k=0}^{\infty} x^{e_k}\right)$$

$$= \sum_{e_1=0}^{\infty}\sum_{e_2=0}^{\infty}\cdots\sum_{e_k=0}^{\infty} x^{e_1+e_2+\cdots+e_k}$$

$$= \sum_{n=0}^{\infty} \left(\begin{array}{c} \#\text{ of integral solutions to} \\ e_1+e_2+\cdots+e_k=n \\ e_1\ge 0,\ e_2\ge 0,\ \ldots,\ e_k\ge 0 \end{array} \right) x^n$$

$$= \sum_{n=0}^{\infty} h_n x^n$$

$$= g(x)$$

$\square$

Ex For $n \geq 0$ define

$h_n =$ # integral solutions to

$$e_1 + e_2 + e_3 + e_4 = n$$

$$e_1, e_2, e_3, e_4 \geq 0$$

e_1 even, e_2 odd, $e_3 \leq 4$, $e_4 \geq 1$

Find gen function for $\{h_n\}_{n=0}^{\infty}$

Sol Use variation on Sol II in prev example

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

$$= \Bigg(\sum_{\substack{e_1=0 \\ e_1 \text{ even}}}^{\infty} x^{e_1}\Bigg)\Bigg(\sum_{\substack{e_2=0 \\ e_2 \text{ odd}}}^{\infty} x^{e_2}\Bigg)\Bigg(\sum_{e_3=0}^{4} x^{e_3}\Bigg)\Bigg(\sum_{e_4=1}^{\infty} x^{e_4}\Bigg)$$

$$= \underbrace{(1+x^2+x^4+\cdots)}_{= \frac{1}{1-x^2}}\underbrace{(x+x^3+x^5+\cdots)}_{= \frac{x}{1-x^2}}\underbrace{(1+x+x^2+x^3+x^4)}_{= \frac{1-x^5}{1-x}}\underbrace{(x+x^2+\cdots)}_{= \frac{x}{1-x}}$$

$$= \frac{x^2(1-x^5)}{(1-x)^2(1-x^2)^2}$$

$\square$

Ex Given $n \geq 0$

How many ways to pick n fruits from an unlimited supply of

apples, oranges, bananas, pears

such that

- $\underbrace{\# \text{ apples}}_{e_1}$ is even
- $\underbrace{\# \text{ oranges}}_{e_2} \leq 2$
- $\underbrace{\# \text{ bananas}}_{e_3}$ divisible by 3
- $\underbrace{\# \text{ pears}}_{e_4} \leq 1$

Sol Call this h_n

$h_n =$ # non neg integral solutions to

$$e_1 + e_2 + e_3 + e_4 = n$$

e_1 even, $e_2 \leq 2$, $3 / e_3$, $e_4 \leq 1$

Gen function

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

$$= (1 + x^2 + x^4 + \cdots)(1 + x + x^2)(1 + x^3 + x^6 + \cdots)(1 + x)$$

$$= \frac{1}{1-x^2} \quad \frac{1-x^3}{1-x} \quad \frac{1}{1-x^3} \quad \frac{1-x^2}{1-x}$$

$$= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n$$

So $h_n = n+1$ $\square$

Ex Given $n \ge 0$

Find the number of integral solutions to

$$3e_1 + 4e_2 + 2e_3 + 5e_4 = n \qquad (*)$$

$$e_1 \ge 0, \quad e_2 \ge 0, \quad e_3 \ge 0, \quad e_4 \ge 0$$

Sol Change variables:

$$E_1 = 3e_1$$

$$E_2 = 4e_2$$

$$E_3 = 2e_3$$

$$E_4 = 5e_4$$

$(*)$ becomes

$$E_1 + E_2 + E_3 + E_4 = n$$

$$E_1, E_2, E_3, E_4 \ge 0$$

$$3 \mid E_1, \quad 4 \mid E_2, \quad 2 \mid E_3, \quad 5 \mid E_4$$

Gen function is

$$g(x) = (1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)$$

$$= \frac{1}{1-x^3} \quad \frac{1}{1-x^4} \quad \frac{1}{1-x^2} \quad \frac{1}{1-x^5}$$

ans = coeff of x^n in $g(x)$ $\square$

Ex Given integer $n \geq 1$

For a permutation $a_1 a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$

Recall the inversion sequence

$$(b_1, b_2, \ldots, b_n)$$

b_i = # of elements among $a_1, a_2, \ldots, a_n$ that are greater than i and to left of i

Recall

$$0 \leq b_1 \leq n-1$$

$$0 \leq b_2 \leq n-2$$

$$\vdots$$

$$0 \leq b_{n-1} \leq 1$$

$$b_n = 0$$

inversions of $a_1 a_2 \cdots a_n$ $= b_1 + b_2 + \cdots + b_n$

$$\leq n-1 + n-2 + \cdots + 2 + 1 + 0$$

$$= \binom{n}{2}$$

For $0 \leq k \leq \binom{n}{2}$ let

h_k = # perms of $\{1, 2, \ldots, n\}$ that have exactly k inversions

Find gen function for the h_k.

Sol For $0 \le k \le \binom{n}{2}$

$h_k = \#$ integral sols to

$b_1 + b_2 + \cdots + b_n = k$

$0 \le b_1 \le n-1$

$0 \le b_2 \le n-2$

$\vdots$

$0 \le b_{n-1} \le 1$

$b_n = 0$

Gen function is

$$g(x) = \sum_{k=0}^{\binom{n}{2}} h_k x^k$$

$$= (1+x+x^2+\cdots+x^{n-1})(1+x+x^2+\cdots+x^{n-2}) \cdots (1+x+x^2)(1+x)(1)$$

$$= \frac{1-x^n}{1-x} \frac{1-x^{n-1}}{1-x} \cdots \frac{1-x^3}{1-x} \frac{1-x^2}{1-x} \frac{1-x}{1-x}$$

$$= \prod_{j=1}^{n} \left(\frac{1-x^j}{1-x} \right)$$

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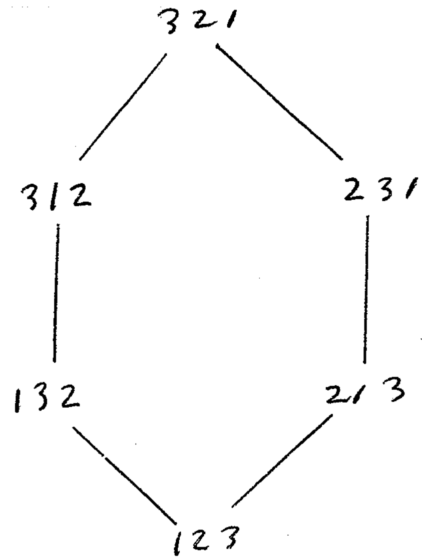
To illustrate take $n=3$

perm # inv



3

2

1

0

$$g(x) = 1 + 2x + 2x^2 + x^3$$

$$= (1 + x + x^2)(1+x)$$

Lecture 24 Wednesday Oct 31

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7.3 Exponential Generating Functions

Given a sequence of real numbers

$$\{h_n\}_{n=0}^{\infty}$$

then exponential generating function is

$$g^e(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$$

ex

$$h_n = 1 \qquad n = 0, 1, 2.$$

$$g^e(x) = \sum_{n=0}^{\infty} 1 \frac{x^n}{n!}$$

$$= e^x$$

Ex Fix integer $m \geq 0$

For $0 \leq n \leq m$ let

$h_n =$ # n-perms of $\{1, 2, \ldots, m\}$

Find exp gen function

Sol Recall

$$h_n = P(m,n)$$
$$= m(m-1) \cdots (m-n+1)$$

$$g^e(x) = \sum_{n=0}^{m} P(m,n) \frac{x^n}{n!}$$

$$= \sum_{n=0}^{m} \binom{m}{n} x^n$$

$$= (1+x)^m$$

□

Ex Fix integer $k \geq 1$

Consider multiset

$$\{ n_1 \cdot a_1, \; n_2 \cdot a_2, \; \dots, \; n_k \cdot a_k \} \qquad *$$

each n_i = pos integer or ∞

For $n \geq 0$ let

h_n = # n-perms of $*$

Find exp gen function for $\{h_n\}_{n=0}^{\infty}$

Sol We get an n-perm of $*$ in two stages

Stage 1 Pick an n-combination of $*$:

$$\{ b_1 \cdot a_1, \; b_2 \cdot a_2, \; \dots \; b_k \cdot a_k \}$$

$$0 \leq b_1 \leq n_1$$
$$0 \leq b_2 \leq n_2$$
$$\vdots$$
$$0 \leq b_k \leq n_k$$

$$b_1 + b_2 + \dots + b_k = n$$

<u>Stage 2</u> order the elements of the n-comb

ways is

$$\frac{n!}{b_1!\, b_2! \cdots b_k!}$$

So

h_n = # n-perms of $*$

$$= \sum \frac{n!}{b_1!\, b_2! \cdots b_k!}$$

↑

sum over all sequences $b_1, b_2, \ldots, b_k$ such that

$0 \le b_1 \le n_1$

$0 \le b_2 \le n_2$

$\vdots$

$0 \le b_k \le n_k$

$b_1 + b_2 + \cdots + b_k = n$

Now

$$g^e(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\substack{0 \le b_i \le n_i \\ b_1 + \cdots + b_k = n}} \frac{\cancel{n!}}{b_1!\, b_2! \cdots b_k!} \right) \frac{x^n}{\cancel{n!}}$$

$$x^n = x^{b_1 + b_2 + \cdots + b_k}$$

$$= \sum_{b_1=0}^{n_1} \sum_{b_2=0}^{n_2} \cdots \sum_{b_k=0}^{n_k} \frac{x^{b_1+b_2+\cdots+b_k}}{b_1!\, b_2! \cdots b_k!}$$

$$= \left(\sum_{b_1=0}^{n_1} \frac{x^{b_1}}{b_1!} \right) \left(\sum_{b_2=0}^{n_2} \frac{x^{b_2}}{b_2!} \right) \cdots \left(\sum_{b_k=0}^{n_k} \frac{x^{b_k}}{b_k!} \right)$$

[define
$$f_m(x) = \begin{cases} 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^m}{m!} & \text{if } m < \infty \\ e^x & \text{if } m = \infty \end{cases}$$
]

$$= f_{n_1}(x)\, f_{n_2}(x) \cdots f_{n_k}(x)$$

$\square$

Ex For $n \geq 0$ define

h_n = # n-digit numbers using digits 1, 2, 3 with 1 used an even number of times

Find exp gen function

Sol Obs

h_n = # n-perms of

$$\{\infty \cdot 1, \infty \cdot 2, \infty \cdot 3\}$$

such that 1 is used an even number of times

As in prev example,

$$g^e(x) =$$

$$\underbrace{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)}_{=\frac{e^x+e^{-x}}{2}}\underbrace{\left(1+x+\frac{x^2}{2!}+\cdots\right)}_{=e^x}\underbrace{\left(1+x+\frac{x^2}{2!}+\cdots\right)}_{=e^x}$$

$$= \frac{e^{3x}+e^x}{2}$$

$$= \sum_{n=0}^{\infty}\left(\frac{3^n+1}{2}\right)\frac{x^n}{n!}$$

So $h_n = \dfrac{3^n+1}{2}$ $\quad n = 0, 1, 2, \ldots$

$\square$

Ex for $n \geq 0$ let

$h_n =$ # ways to color a $1 \times n$ chessboard with colors

Red, White, Blue

such that

- number of red squares is even
- at least one blue square

Find h_n

Sol, Compute exp gen function

Obs

$h_n =$ # n-perms of

$\{\infty \cdot R, \ \infty \cdot W, \ \infty \cdot B\}$

such that

- R is used an even number of times
- B used at least once

$g^e(x) =$

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+x+\frac{x^2}{2!}+\cdots\right)\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)$$

$$\left\{\frac{e^x+e^{-x}}{2}\right. \qquad e^x \qquad e^x-1$$

$$= \frac{e^{3x}-e^{2x}+e^x-1}{2}$$

$$= -\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{3^n-2^n+1}{2}\right)\frac{x^n}{n!}$$

for $n=0$ this is $\frac{1-1+1}{2} = \frac{1}{2}$

$$= \sum_{n=1}^{\infty} \frac{3^n-2^n+1}{2} \frac{x^n}{n!}$$

so

$$h_n = \begin{cases} \frac{3^n-2^n+1}{2} & \text{if } n \geq 1 \\ 0 & \text{if } n=0 \end{cases}$$

$\square$

7.4 Linear homogeneous Recurrence Relations

Given a sequence $\{h_n\}_{n=0}^{\infty}$ of real numbers

suppose it satisfies a recurrence such as

$$h_n = a h_{n-1} + b h_{n-2} + c \qquad n \geq 2 \qquad *$$

Possibly a, b, c depend on n.

Ex Fibonacci numbers

$$f_n = f_{n-1} + f_{n-2} \qquad n \geq 2$$

$$f_0 = 0, \qquad f_1 = 1$$

Ex. Derangement numbers

View I:

$$D_n = (n-1) D_{n-1} + (n-1) D_{n-2} \qquad n \geq 2$$

$$D_0 = 1, \qquad D_1 = 0$$

View II:

$$D_n = n D_{n-1} + (-1)^n \qquad n \geq 1$$

$$D_0 = 1$$

Special cases of *

- a, b, c indep of n "constant coefficients"
- $c = 0$ "homogeneous"

Until further notice assume both constant coef + homog

Ex Solve the recurrence

$$h_n = 3h_{n-1} + 4h_{n-2} - 12h_{n-3} \qquad n \ge 3 \qquad *$$

subject to

$$h_0 = 5, \quad h_1 = -11, \quad h_2 = 15 \qquad **$$

Sol First ignore ** and find gen sol to *

Hunt for sol of form

$$h_n = q^n \qquad n \ge 0$$

Require

$$q^n = 3q^{n-1} + 4q^{n-2} - 12q^{n-3} \qquad n \geq 3$$

Take $n \to 3$

$$q^3 = 3q^2 + 4q - 12$$

or $$q^3 - 3q^2 - 4q + 12 = 0$$

So q is a root of polynomial

$$x^3 - 3x^2 - 4x + 12$$ "characteristic poly"

Factor

$$x^3 - 3x^2 - 4x + 12 = (x^2 - 4)(x-3)$$
$$= (x-2)(x+2)(x-3)$$

$$q \in \{2, -2, 3\}$$

So far we found 3 sols

Sol 1 $\quad h_n = 2^n$

Sol 2 $\quad h_n = (-2)^n$

Sol 3 $\quad h_n = 3^n$

We combine above 3 sols to get general sol:

For arbitrary real numbers a, b, c

$$h_n = a\,2^n + b(-2)^n + c\,3^n \qquad n = 0, 1, 2 \ldots \qquad ☆$$

is a sol to $*$

<u>claim</u> Any sol to $*$ has form ☆

<u>pf claim</u> Call the sol $\{H_n\}_{n=0}^{\infty}$

Consider H_0, H_1, H_2

Solve the equations

$$\begin{aligned} a + b + c &= H_0 \\ 2a - 2b + 3c &= H_1 \\ 2^2 a + (-2)^2 b + 3^2 c &= H_2 \end{aligned}$$

for a, b, c. Using these a, b, c define

$$h_n = a\,2^n + b(-2)^n + c\,3^n \qquad n = 0, 1, 2 \ldots$$

We have

- $\{h_n\}_{n=0}^{\infty}$ is sol to $*$
- $\{H_n\}_{n=0}^{\infty}$ is sol to $*$
- $H_n = h_n$ for $n = 0, 1, 2$

So $H_n = h_n$ for $n \geq 0$ claim proved.

So far

$$h_n = a2^n + b(-2)^n + c3^n \qquad n=0,1,2,\ldots$$

Now consider init conds **

Require

$n=0$: $a + b + c = 5$

$n=1$: $2a - 2b + 3c = -11$

$n=2$: $4a + 4b + 9c = 15$

solving this system (ex) we find

$a=1 \qquad b=5 \qquad c=-1$

So –

$$h_n = 2^n + 5(-2)^n - 3^n \qquad n \geq 0$$

$\square$

— o —

Above method works in general as long as char poly has all roots distinct

We now consider when char poly has repeated roots

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7.4 cont

Summary Solve the rec

$h_n = 3h_{n-1} + 4h_{n-2} - 12h_{n-3}$ $n \geq 3$ *

subject to **

$h_0 = 5$ $h_1 = -11$ $h_2 = 15$

Sol steps

- Factor char poly

  $x^3 - 3x^2 - 4x + 12 = (x-2)(x+2)(x-3)$

- gen sol to * is

  $h_n = a2^n + b(-2)^n + c3^n$ $n = 0, 1, 2.$

- Use ** to find

  $a = 1,$ $b = 5,$ $c = -1$

Sol is

$h_n = 2^n + 5(-2)^n - 3^n$ $n = 0, 1, 2, \dots$

— o —

Above method works provided the char poly has no repeated roots.

We now consider case of repeated roots.

ex Solve the recurrence

$$h_n = 6h_{n-1} - 12h_{n-2} + 8h_{n-3} \qquad n \geq 3 \qquad *$$

subject to

$$h_0 = 5, \qquad h_1 = 0, \qquad h_2 = -12 \qquad **$$

Sol Factor char poly

$$x^3 - 6x^2 + 12x - 8$$

$$= x^3 - 3 \cdot x^2 \cdot 2 + 3 \cdot x \cdot 2^2 - 2^3$$

$$= (x-2)^3$$

Repeated root 2, 2, 2.

<u>claim</u> Each of the following is a sol to $*$:

(i) $h_n = 2^n$

(ii) $h_n = n2^n$

(iii) $h_n = n^2 2^n$

<u>pf cl</u> (i) as before

(ii)

$$n2^n \stackrel{?}{=} 6(n-1)2^{n-1} - 12(n-2)2^{n-2} + 8(n-3)2^{n-3}$$

divide by 2^n

$$n \stackrel{?}{=} 3(n-1) - 3(n-2) + n-3$$

✓

(iii) Similar

claim proved.

Now the general solution to * is

$$h_n = a2^n + bn2^n + cn^2 2^n \qquad n = 0, 1, 2, \ldots$$

To find a, b, c use **

$n=0$: $a = 5$

$n=1$: $a2 + b2 + c \cdot 2 = 0$

$n=2$: $a4 + b8 + c16 = -12$

Get $a = 5$ $\quad b = -6 \quad c = 1$

So $h_n = (5 - 6n + n^2)2^n$

$= (n-5)(n-1)2^n \qquad n = 0, 1, 2, \ldots$ □

Ex Solve the recurrence

$$h_n = 2h_{n-1} + 4h_{n-2} - 8h_{n-3} \qquad n \geq 3 \qquad *$$

subject to

$$h_0 = -2, \qquad h_1 = -6 \qquad h_2 = 0 \qquad **$$

Sol Factor char poly

$$x^3 - 2x^2 - 4x + 8 = (x^2-4)(x-2)$$
$$= (x+2)(x-2)^2$$

Roots are

$$2,\ 2,\ -2$$

Each of the following is sol to $*$:

$$h_n = 2^n$$
$$h_n = n2^n$$
$$h_n = (-2)^n$$

Gen sol to $*$ is

$$h_n = (a+bn)2^n + c(-2)^n \qquad n = 0, 1, 2, \ldots$$

Use ** to solve for a, b, c

Get

$a = -3, \quad b = 1, \quad c = 1$

Sol to *, ** is

$$h_n = (n-3)2^n + (-2)^n \qquad n = 0, 1, 2, \dots$$

□

We now show how to use gen functions to solve a linear recurrence

ex Solve the recurrence

$$h_n = 3h_{n-2} - 2h_{n-3} \qquad n \geq 3$$

$$h_0 = 1, \quad h_1 = 0, \qquad h_2 = 0$$

Sol Consider gen function

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

Obs

$$\sum_{n=2}^{\infty} h_{n-2} x^n = \sum_{n=0}^{\infty} h_n x^{n+2} = x^2 g(x)$$

$$\sum_{n=3}^{\infty} h_{n-3} x^n = \sum_{n=0}^{\infty} h_n x^{n+3} = x^3 g(x)$$

$$g(x) - 3x^2 g(x) + 2x^3 g(x) =$$

$$h_0 + h_1 x + h_2 x^2 - 3(h_0 x^2)$$

$$= 1 - 3x^2$$

so

$$g(x)\left(1-3x^2+2x^3\right)=1-3x^2$$

So

$$g(x)=\frac{1-3x^2}{1-3x^2+2x^3}$$

factor denom

$$=\frac{1-3x^2}{(1-x)^2(1+2x)}$$

partial fraction decomp

$$=\frac{8}{9}\frac{1}{(1-x)^2}-\frac{14}{9}\frac{x}{(1-x)^2}+\frac{1}{9}\frac{1}{1+2x}$$

$$=\frac{8}{9}\sum_{n=0}^{\infty}(n+1)x^n-\frac{14}{9}\sum_{n=0}^{\infty}n\,x^n+\frac{1}{9}\sum_{n=0}^{\infty}(-2)^n x^n$$

$$=\sum_{n=0}^{\infty}\left(\frac{8(n+1)-14n+(-2)^n}{9}\right)x^n$$

$$=\sum_{n=0}^{\infty}\underbrace{\frac{8-6n+(-2)^n}{9}}_{=h_n}x^n$$

$$h_n=\frac{8-6n+(-2)^n}{9}\qquad n=0,1,2,\ldots$$

$\square$

7.5 Nonhomogeneous Recurrence Relations

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Given a sequence $\{h_n\}_{n=0}^{\infty}$ of real numbers

Suppose it satisfies a rec such as

$$h_n = a h_{n-1} + b h_{n-2} + c \qquad n \geq 2$$

where

a, b indep of n

$c =$ dep on n

Sol depends on nature of function c

We illustrate with examples

Ex Fix $a, c \in \mathbb{R}$ Assume

$$h_n = a\, h_{n-1} + c \qquad n \geq 1$$

Find h_n

Sol (Examine small n)

n	h_n
0	h_0
1	$a h_0 + c$
2	$a(a h_0 + c) + c = a^2 h_0 + (1+a)c$
3	$a(a^2 h_0 + (1+a)c) + c$
	$= a^3 h_0 + (1+a+a^2)c$
$\vdots$	$\vdots$
n	$a^n h_0 + (1+a+a^2+\dots+a^{n-1})c$
	$= a^n h_0 + \dfrac{1-a^n}{1-a} c$

Ex Solve the recurrence

$$h_n = 4h_{n-1} - 4h_{n-2} + 2 \qquad n \geq 2$$

$$h_0 = 3, \qquad h_1 = 6$$

Sol (reduce to hom case)

$$h_n = 4h_{n-1} - 4h_{n-2} + 2$$

$n \to n-1$ $$h_{n-1} = 4h_{n-2} - 4h_{n-3} + 2$$

subtract

$$h_n = 5h_{n-1} - 8h_{n-2} + 4h_{n-3} \qquad n \geq 3$$

Now solve using prev methods

Factor char poly

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2(x-1)$$

Gen sol is

$$h_n = (an+b)2^n + c\,1^n \qquad n = 0, 1, 2 \ldots$$

Find c

$$2 = h_n - 4h_{n-1} + 4h_{n-2}$$

$$= c - 4c + 4c$$

$$= c$$

$$h_n = (an+b)2^n + 2$$

Use init conds to find a, b

$a = 1$ $b = 1$

$$h_n = (n+1)2^n + 2 \qquad n = 0, 1, 2 \ldots$$

$\square$

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Ex Solve the recurrence

$$h_n = 6h_{n-1} - 9h_{n-2} + 2n \qquad n \geq 2 \qquad *$$

$$h_0 = 1, \qquad h_1 = 0$$

Sol (Red to hom case)

Note the sequence

$$a_n = n$$

satisfies

$$a_n - 2a_{n-1} + a_{n-2} = 0 \qquad n \geq 2$$

Use this to get rid of "$2n$" term in $*$

Write

$$h_n - 6h_{n-1} + 9h_{n-2} = 2n$$

coef	h_n	h_{n-1}	h_{n-2}	h_{n-3}	h_{n-4}	cnst
1	1	-6	9			$2n$
-2		1	-6	9		$2(n-1)$
1			1	-6	9	$2(n-2)$
	1	-8	22	-24	9	0

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$$h_n - 8h_{n-1} + 22h_{n-2} - 24h_{n-3} + 9h_{n-4} = 0 \qquad n \geq 4 \qquad \star$$

Factor charpoly

$$x^4 - 8x^3 + 22x^2 - 24x + 9 = (x-3)^2(x-1)^2$$

Gen sol to $\star$ is

$$h_n = (an+b)3^n + cn + d \qquad n = 0, 1, 2, \ldots$$

Find c, d

$$2n = h_n - 6h_{n-1} + 9h_{n-2}$$

$$= cn + d - 6\left(c(n-1) + d\right) + 9\left(c(n-2) + d\right)$$

n coef: $\quad 2 = c - 6c + 9c$

$$= 4c$$

$$c = 1/2$$

$$\rightarrow \quad d = 3/2$$

So far

$$h_n = (an+b)3^n + \frac{n+3}{2}$$

Find a, b

Use init. conds $h_0 = 1$, $h_1 = 0$

to get

$$a = -1/6, \qquad b = -1/2$$

Now

$$h_n = -\frac{n+3}{6}3^n + \frac{n+3}{2}$$

$$= \frac{(3-3^n)(n+3)}{6} \qquad n = 0, 1, 2 \ldots$$

$\square$

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7.5 cont

Ex Solve the rec

$$h_n = 2h_{n-1} + 5^n \qquad n \geq 1$$

$$h_0 = 3$$

Sol Note $h_1 = 11$

$$h_n - 2h_{n-1} = 5^n$$

$$h_{n-1} - 2h_{n-2} = 5^{n-1}$$

$$0 = h_n - 2h_{n-1} - 5(h_{n-1} - 2h_{n-2})$$

$$= h_n - 7h_{n-1} + 10h_{n-2} \qquad n \geq 2$$

char poly

$$x^2 - 7x + 10 = (x-5)(x-2)$$

Gen sol

$$h_n = a5^n + b2^n$$

Using $h_0 = 3$, $h_1 = 11$

$$a = 5/3 \qquad b = 4/3$$

$$h_n = \frac{5^{n+1} + 2^{n+2}}{3} \qquad n = 0, 1, 2 \ldots$$

$\square$

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2

Ex Solve the rec

$$h_n = 5h_{n-1} + 5^n \qquad n \geq 1$$

$$h_0 = 3$$

Sol Note

$$h_1 = 5 \cdot 3 + 5$$

$$= 20$$

$$h_n - 5h_{n-1} = 5^n \qquad (1)$$

$$h_{n-1} - 5h_{n-2} = 5^{n-1} \qquad -(5)$$

$$0 = h_n - 5h_{n-1} - 5(h_{n-1} - 5h_{n-2})$$

$$= h_n - 10h_{n-1} + 25h_{n-2}$$

char poly

$$x^2 - 10x + 25 = (x-5)^2$$

Gen sol

$$h_n = (an+b)5^n \qquad n = 0, 1, 2.$$

Using $h_0 = 3$, $h_1 = 20$ get

$$a = 1 \qquad b = 3$$

$$h_n = (n+3)5^n \qquad n = 0, 1, 2, \ldots$$

$\square$

ex Solve the recurrence

$$h_n = 3h_{n-1} + n \qquad n \geq 1$$

$$h_0 = 2$$

Using gen functions

Sol Define

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

Obs

$$\sum_{n=1}^{\infty} h_{n-1} x^n = \sum_{n=0}^{\infty} h_n x^{n+1} = x\, g(x)$$

Recall

$$\sum_{n=0}^{\infty} n x^n = \sum_{n=1}^{\infty} n x^n$$

$$= x \sum_{n=1}^{\infty} n x^{n-1}$$

$$= x \sum_{n=0}^{\infty} (n+1) x^n$$

$$= \frac{x}{(1-x)^2}$$

Obs

$$g(x) - 3x\,g(x) - \frac{x}{(1-x)^2} = h_0 = 2$$

$$g(x)(1-3x) = \frac{x}{(1-x)^2} + 2$$

$$g(x) = \frac{x}{(1-x)^2(1-3x)} + \frac{2}{1-3x}$$

red using partial frac

$$= \frac{11}{4(1-3x)} + \frac{x-3}{4(1-x)^2}$$

$$= \frac{11}{4}\sum_{n=0}^{\infty} 3^n x^n + \frac{1}{4}\sum_{n=0}^{\infty} n x^n - \frac{3}{4}\sum_{n=0}^{\infty} (n+1) x^n$$

$$= \sum_{n=0}^{\infty} \frac{11\cdot 3^n - 2n - 3}{4} x^n$$

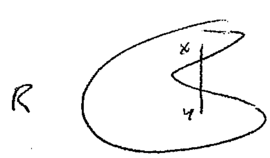
$$h_n = \frac{11\cdot 3^n - 2n - 3}{4} \qquad n = 0, 1, 2, \ldots$$

— o —

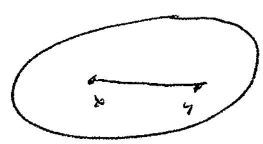
7.6 A geometry example

Given a subset R of the plane

R is <u>convex</u> whenever $\forall x, y \in R$ the line segment $\overline{xy}$ is contained in R



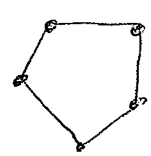
Not convex



Convex

Consider convex polygon

ex 5-gon



A <u>diagonal</u> is a line segment joining nonadjacent corners:

For convex n-gon

\# diagonals is $\binom{n}{2} - n = \frac{n(n-3)}{2}$

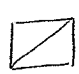
Problem For $n \geq 2$ let

h_n = # ways to divide a convex $(n+1)$-gon into triangular regions, by inserting diagonals that do not intersect

For notational conv def $h_1 = 1$

Find h_n.

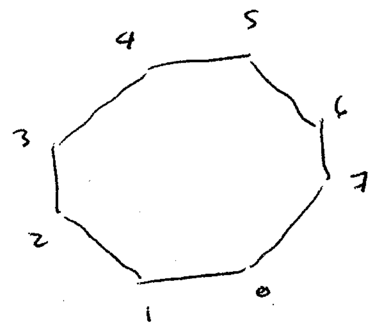
Ex

n	desc	h_n
1		1
2		1
3	 	2
4		5
⋮		⋮

To find h_n, find a recurrence it satisfies

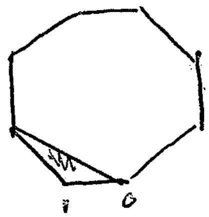
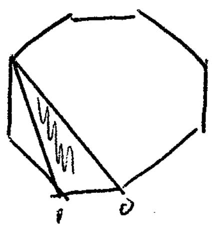
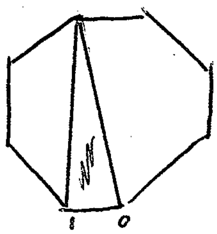
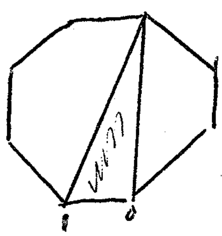
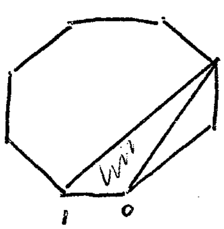
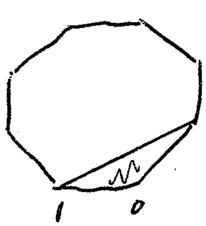
ex $n=7$

Label the vertices of convex 8-gon $0,1,\dots,7$



Consider triangular division of this 8-gon edge 01 is contained in a triangle $01t$ some t $(2 \le t \le 7)$

Cases

t	desc	# choices
2		$h_6 = h_1 h_6$
3		$h_5 = h_2 h_5$
4		$h_3 h_4$
5		$h_4 h_3$
6		$h_5 h_2$
7		$h_6 h_1$

$$h_7 = \sum_{k=1}^{6} h_k h_{7-k}$$

In gen for $n \geq 2$

$$h_n = \sum_{k=1}^{n-1} h_k h_{n-k}$$

To solve this rec use gen function

$$g(x) = \sum_{n=1}^{\infty} h_n x^n$$

Note $g(x)$ has no constant term so

$$g(0) = 0$$

Obs

$$\left(g(x)\right)^2 = \left(\sum_{r=1}^{\infty} h_r x^r\right)\left(\sum_{s=1}^{\infty} h_s x^s\right)$$

$$= \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} h_r h_s x^{r+s}$$

$$= \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} h_k h_{n-k}\right) x^n$$

$$= \sum_{n=2}^{\infty} h_n x^n$$

$$= g(x) - h_1 x \qquad h_1 = 1$$

$$= g(x) - x$$

$$(g(x))^2 - g(x) + x = 0$$

Solve for $g(x)$ using quadratic formula

$$g(x) = \frac{1 + \sqrt{1-4x}}{2} \quad \text{or} \quad g(x) = \frac{1 - \sqrt{1-4x}}{2}$$

Which sign is correct?

take $x = 0$

$g(0) = 0$

$$0 \stackrel{?}{=} \frac{1 + \sqrt{1-0}}{2} \qquad 0 = \frac{1 - \sqrt{1-0}}{2}$$

NO yes

$$g(x) = \frac{1 - \sqrt{1-4x}}{2}$$

$$= \frac{1 - (1-4x)^{1/2}}{2}$$

Apply Newton binom thm

$$= \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

$$= -\frac{1}{2} \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n x^n$$

For $n \geq 1$

$$h_n = -\frac{1}{2}\binom{1/2}{n}(-4)^n$$

$$= \frac{1}{n}\binom{2n-2}{n-1}$$

ex $n=6$

$$-\frac{1}{2}\binom{\frac{1}{2}}{6}(-4)^6 = -1/2 \quad \frac{\frac{1}{2}\ -\frac{1}{2}\ -\frac{3}{2}\ -\frac{5}{2}\ -\frac{7}{2}\ -\frac{9}{2}}{6\ 5\ 4\ 3\ 2\ 1} \quad -4\ -4\ -4\ -4\ -4\ -4$$