Lecture 17 Friday Oct 12 10/12/12 1

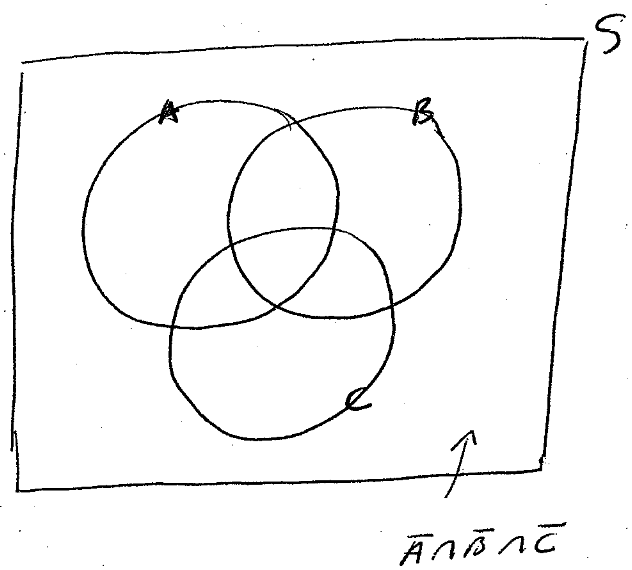
$\boxed{\text{Math 475 exam I Monday Oct 15 ch 2–5. HW sols at math.wisc.edu/~terwilli/teaching.html (bottom)}}$

Ch 6 Inclusion / Exclusion + Applications

6.1 Incl / Excl

Ex Given a finite set S

Given subsets $A, B, C \subseteq S$



Find $|\bar{A} \cap \bar{B} \cap \bar{C}|$

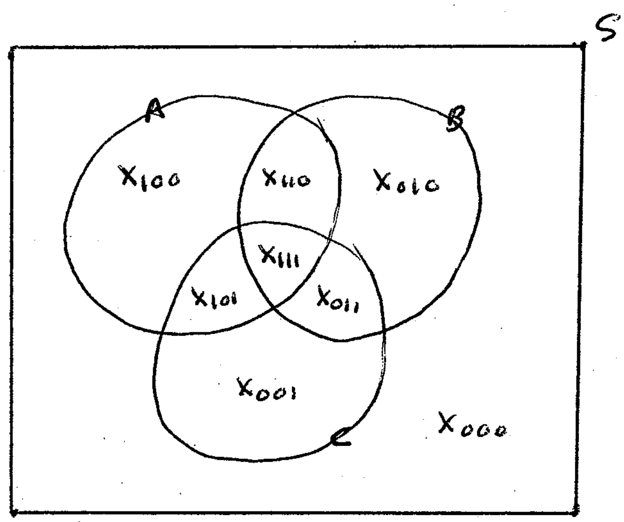
in terms of

$$\left.\begin{array}{c} |S| \\ |A| \quad |B| \quad |C| \\ |A \cap B| \quad |A \cap C| \quad |B \cap C| \\ |A \cap B \cap C| \end{array}\right\} *$$

10/12/12

2

<u>Sol</u> Define variables x_{ijk} as shown



So $x_{100} = |A \wedge \bar{B} \wedge \bar{C}|$ etc.

We desire x_{000}

Write $*$ in terms of x_{ijk}

	x_{000}	x_{100}	x_{010}	x_{001}	x_{110}	x_{101}	x_{011}	x_{111}
$\lvert S\rvert$	1	1	1	1	1	1	1	1
$\lvert A\rvert$	0	1	0	0	1	1	0	1
$\lvert B\rvert$	0	0	1	0	1	0	1	1
$\lvert C\rvert$	0	0	0	1	0	1	1	1
$\lvert A\cap B\rvert$	0	0	0	0	1	0	0	1
$\lvert A\cap C\rvert$	0	0	0	0	0	1	0	1
$\lvert B\cap C\rvert$	0	0	0	0	0	0	1	1
$\lvert A\cap B\cap C\rvert$	0	0	0	0	0	0	0	1

Each row above gives equation. For instance

$$|C| = x_{001} + x_{101} + x_{011} + x_{111}$$

Now "backsolve" to get the x_{ijk} in terms of *

$$X_{111} = |A \cap B \cap C|$$

$$X_{011} = |B \cap C| - |A \cap B \cap C|$$

$$X_{101} = |A \cap C| - |A \cap B \cap C|$$

$$X_{110} = |A \cap B| - |A \cap B \cap C|$$

$$X_{001} = |C| - X_{101} - X_{011} - X_{111}$$

$$= |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$X_{010} = |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

$$X_{100} = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$X_{000} = |S| - (\text{all the above } x_{ijk})$$

$$=$$

term	$\|S\|$	$\|A\|$	$\|B\|$	$\|C\|$	$\|A \cap B\|$	$\|A \cap C\|$	$\|B \cap C\|$	$\|A \cap B \cap C\|$
coef	1							-1
							-1	1
						-1		1
					-1			1
				-1		1	1	-1
			-1		1		1	-1
		-1			1	1		-1
coef	1	-1	-1	-1	1	1	1	-1

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = X_{000}$$

$$= |S| - |A| - |B| - |C|$$

$$+ |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

$\square$

the above example is "$n = 3$" case of Inclusion / Exclusion

General Case

Given finite set S

Given subsets $A_1, A_2, \ldots, A_n$ of S

For each subset $s \subseteq \{1, 2, \ldots, n\}$ define

$$A_s = \bigcap_{i \in s} A_i$$

For example $n = 6$

s	A_s
$\{2, 4\}$	$A_2 \cap A_4$
$\{2, 3, 6\}$	$A_2 \cap A_3 \cap A_6$
ϕ	S

theorem (Inclusion/Exclusion)

With the above notation

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| = \sum_{s} |A_s| (-1)^{|s|} \quad *$$

where the sum is over all the subsets s of $\{1, 2, \ldots, n\}$

pf Fix $\theta \in S$

Show that the contribution of θ to each side of $*$ is the same.

Define a subset $\Delta \subseteq \{1, 2, \ldots, n\}$ by

$$\Delta = \{ i \mid \theta \in A_i \}$$

So $\Delta = \phi$ iff $\theta \in \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n$

For LHS of $*$:

$$\theta \text{ contributes } \begin{cases} 1 & \text{if } \Delta = \phi \\ 0 & \text{if } \Delta \neq \phi \end{cases}$$

Consider RHS of $*$:

For $1 \le i \le n$

$\theta \in A_i$ if $i \in \Delta$

$\theta \notin A_i$ if $i \notin \Delta$

So for $s \subseteq \{1, 2, \ldots, n\}$

$\theta \in A_s$ if $s \subseteq \Delta$ (Here θ contributes 1 to $|A_s|$)

$\theta \notin A_s$ if $s \nsubseteq \Delta$ (... 0 ...)

Contrib of θ to RHS of $*$

$$= \sum_{s \subseteq \Delta} (-1)^{|s|}$$

$$= \sum_{i=0}^{|\Delta|} \sum_{\substack{s \subseteq \Delta \\ |s| = i}} (-1)^{|s|}$$

$$= \sum_{i=0}^{|\Delta|} \binom{|\Delta|}{i} (-1)^i$$

$$= \begin{cases} (1-1)^{|\Delta|} = 0 & \text{if } \Delta \neq \phi \\ 1 & \text{if } \Delta = \phi \end{cases}$$

So contribution of θ to each side of $*$ is same ✓ $\square$

Ex Find the number of integers among

$1, 2, 3, \ldots, 100$

that are not divisible by 4 or 5 or 6

Sol Define a set

$$S = \{1, 2, \ldots, 100\}$$

Define

$$A = \{ i \in S \mid 4 \text{ divides } i \}$$

$$B = \{ i \in S \mid 5 \text{ divides } i \}$$

$$C = \{ i \in S \mid 6 \text{ divides } i \}$$

We seek $|\bar{A} \cap \bar{B} \cap \bar{C}|$

Prime factorization

$$4 = 2 \times 2$$

$$5 = 5$$

$$6 = 2 \times 3$$

set X	elements in X are divisible by
A	4
B	5
C	6
$A \cap B$	$4 \times 5 = 20$
$A \cap C$	$2 \times 2 \times 3 = 12$
$B \cap C$	$5 \times 6 = 30$
$A \cap B \cap C$	$2 \times 2 \times 3 \times 5 = 60$

Set	size	reason
S	100	
A	25	$100 = 4 \times 25$
B	20	$100 = 5 \times 20$
C	16	$100 = 6 \times 16 + 4$
$A \cap B$	5	$100 = 20 \times 5$
$A \cap C$	8	$100 = 12 \times 8 + 4$
$B \cap C$	3	$100 = 30 \times 3 + 10$
$A \cap B \cap C$	1	$100 = 60 \times 1 + 40$

By Incl / Excl

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 100 - 25 - 20 - 16 + 5 + 8 + 3 - 1$$

$$= 54$$

$\square$

6.2 Combinations of multisets

Ex Find the number of integral solutions to

$$x + y + z = 20$$

$$0 \le x \le 7 \qquad 0 \le y \le 8 \qquad 0 \le z \le 9$$

Sol. Let S = set of all nonneg integral sols to $x+y+z=20$

Define

A = set of elements in S with $x \ge 8$

B = ... $y \ge 9$

C = ... $z \ge 10$

We seek $|\bar{A} \cap \bar{B} \cap \bar{C}|$

set X	elements in X satisfy	$\lvert X \rvert$
S	$0 \le x \quad 0 \le y \quad 0 \le z$	$\binom{22}{2}$
A	$8 \le x \quad 0 \le y \quad 0 \le z$	$\binom{14}{2}$
B	$0 \le x \quad 9 \le y \quad 0 \le z$	$\binom{13}{2}$
C	$0 \le x \quad 0 \le y \quad 10 \le z$	$\binom{12}{2}$
$A \cap B$	$8 \le x \quad 9 \le y \quad 0 \le z$	$\binom{5}{2}$
$A \cap C$	$8 \le x \quad 0 \le y \quad 10 \le z$	$\binom{4}{2}$
$B \cap C$	$0 \le x \quad 9 \le y \quad 10 \le z$	$\binom{3}{2}$
$A \cap B \cap C$	$8 \le x \quad 9 \le y \quad 10 \le z$	0

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = \binom{22}{2} - \binom{14}{2} - \binom{13}{2} - \binom{12}{2}$$

$$+ \binom{5}{2} + \binom{4}{2} + \binom{3}{2}$$

Ex Find the number of **20**-combinations for the multiset

$$\{7\cdot a,\ 8\cdot b,\ 9\cdot c\}$$

Sol Each 20-combin is a multiset

$$\{x\cdot a,\ y\cdot b,\ z\cdot c\}$$

such that

$$x+y+z=20$$

$$0\le x\le 7$$

$$0\le y\le 8$$

$$0\le z\le 9$$

# of 20-combinations

= # of integral solutions to

$$x+y+z=20$$

$$0\le x\le 7, \qquad 0\le y\le 8 \qquad 0\le z\le 9$$

= Same ans as prev problem

$\square$

Lecture 18 Wednesday Oct 17

6.3 Derangements

We introduce derangements with an example

Ex n people put their business cards into a box. After shuffling, each person withdraws one card at random. Find the probability that no one gets their card back.

Sol Label people $1, 2, \ldots, n$

For $1 \le i \le n$ let c_i = card drawn by person i

So $c_1 c_2 \cdots c_n$ is perm of $\{1, 2, \ldots, n\}$

Define sample space

P = set of all perms of $\{1, 2, \ldots, n\}$

Each outcome in P equally likely

$|P| = n!$

Event D : the set of perms $c_1 c_2 \cdots c_n$ in P such that $c_i \neq i$ for $1 \le i \le n$

Find Prob(D)

$$\text{Prob}(D) = \frac{|D|}{|P|}$$

Find $|D|$.

Use incl/excl

For $1 \le i \le n$ define

A_i = set of perms $c_1 c_2 \cdots c_n$ in P such that
$c_i = i$

So

$$D = \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n$$

For $S \subseteq \{1, 2, \dots, n\}$ define

$$A_S = \bigcap_{i \in S} A_i$$

= set of perms $c_1 c_2 \cdots c_n$ in P such that
$c_i = i$ for all $i \in S$

By incl/excl

$$|D| = \sum_{\substack{S \\ S \subseteq \{1,2,\dots,n\}}} |A_S| (-1)^{|S|}$$

For $s \subseteq \{1,2,\dots,n\}$ find $|A_s|$

To construct a perm $c_1c_2\cdots c_n \in A_s$ we proceed in stages:

stage	to do	# choices
1	choose c_i for $i \in s$ $[\ c_i = i\ ]$	1
2	choose c_i for $i \notin s$ $[\text{arb perm of } \{1,2,\dots,n\} \setminus s]$	$(n-\lvert s\rvert)!$

$$|A_s| = (n-|s|)!$$

Now

$$|D| = \sum_{\substack{s \\ s \subseteq \{1,2,\dots,n\}}} |A_s| (-1)^{|s|}$$

$$= \sum_{k=0}^{n} \sum_{\substack{s \\ s \subseteq \{1,2,\dots,n\} \\ |s|=k}} |A_s| (-1)^{|s|}$$

$$= \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{= \frac{n!}{k!(n-k)!}} (n-k)! \, (-1)^k$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$\text{Prob}(O) = \frac{|O|}{|S|}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \qquad \square$$

Note Ref to prev example

what happens to $\text{Prob}(O)$ as $n \to \infty$?

Recall from calculus

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

[e = Euler constant
$e^{i\pi} = -1 \quad i^2 = -1$
$e \approx 2.718\ldots$]
sum converges for all x

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$\frac{1}{e} = e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

$$\lim_{n \to \infty} \Pr(O) = \frac{1}{e}$$

— o —

Def A perm $c_1 c_2 \cdots c_n$ of $\{1, 2, \ldots, n\}$ is a <u>derangement</u> whenever $c_i \neq i$ for $1 \leq i \leq n$.

Define

$$D_n = \# \text{ of derangements of } \{1, 2, \ldots, n\}$$

— o —

We saw

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

n	1	2	3	4	$\cdots$
D_n	0	1	2	9	$\cdots$

Ex Relate D_n, D_{n-1}

Sol

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$D_{n-1} = (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}$$

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + \frac{(-1)^n}{n!}$$

$$= \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}$$

So

$$D_n = n D_{n-1} + (-1)^n \qquad n = 2, 3, \ldots$$

Ex Relate D_n, D_{n-1}, D_{n-2}

Sol

$$D_n = n D_{n-1} + (-1)^n$$

$$D_{n-1} = (n-1) D_{n-2} + (-1)^{n-1}$$

$$= (n-1) D_{n-2} - (-1)^n$$

Add

$$D_n + D_{n-1} = n D_{n-1} + (n-1) D_{n-2}$$

So

$$D_n = (n-1)\left(D_{n-1} + D_{n-2} \right)$$

$n = 2, 3, \ldots$

$D_0 = \underline{1}$

Recall D_n = # derangements of $\{1,2,\ldots,n\}$

Show combinatorically that

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \qquad n = 2,3,\ldots$$

$$D_0 = 1$$

<u>Sol</u> Construct a derangement $c_1 c_2 \cdots c_n$ in stages

- choose $c_1 \in \{2,3,\ldots,n\}$ ($n-1$ choices)

  WLOG $c_1 = 2$

- choose $c_2 c_3 \cdots c_n$

  Case $c_2 = 1$:

  $c_3 c_4 \cdots c_n$ is derangement of $\{3,4,\ldots,n\}$

  # choices is D_{n-2}

  Case $c_2 \neq 1$:

$$\begin{array}{c} \\ 2 \\ \hline 1 \end{array} \quad \overbrace{\begin{array}{ccccc} \neq 2,1 & \neq 2,3 & \cdots & \neq 2,n \\ \hline 2 & 3 & & n \end{array}}^{\text{fill blanks with } 1,3,4,\ldots,n}$$

view as new position "1"

choices is D_{n-1}

So

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \qquad \square$$

Lecture 19 Friday Oct 19

6.4 Perms with forbidden positions

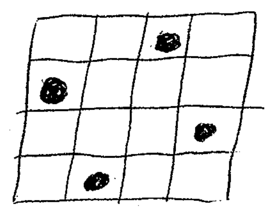
Given perm $s_1 s_2 \cdots s_n$ of $\{1, 2, \ldots, n\}$

View as placing n nonattacking rooks on $n \times n$ chessbd

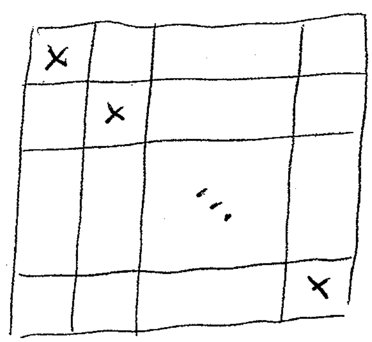
ith rook in row i, column s_i $(1 \le i \le n)$

ex $n = 4$

3142 $\longleftrightarrow$



Rook placement is a derangement whenever the rooks are not in the forbidden positions X:



We now consider other forbidden positions

To illustrate the techique we consider small example

Ex Find the number of ways to place 3 nonattacking rooks on the following 3×3 chessbd with forbidden positions

×	×	
	×	×
		×

Sol We can see ans $= 1$

But let us solve this using incl/excl

let $S =$ set of all nonattacking rook placements ignoring forb positions

$$|S| = 3!$$

let $A_i =$ rook placements in S where rook i is in a forb position from row i

we seek $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$

By incl/excl

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - |A_1| - |A_2| - |A_3|$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

set X	desc of $\lvert X \rvert$	$\lvert X \rvert$
S		$3! = 6$
A_1	# x's in row 1 times 2!	4
A_2	# x's in row 2 times 2!	4
A_3	# x's in row 3 times 2!	2
$A_1 \cap A_2$	(# ways to put non attacking rooks in X positions of rows 1, 2) times 1!	3
$A_1 \cap A_3$	(# ways to put nonattacking rooks in X-pos of rows 1,3) times 1!	2
$A_2 \cap A_3$	(# ways to put nonattacking rooks in x-pos of rows 2,3) times 1!	1
$A_1 \cap A_2 \cap A_3$	# ways to put nonattacking rooks in x-pos of rows 1,2,3	1

$$
\begin{aligned}
|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= 6 - 4 - 4 - 2 + 3 + 2 + 1 - 1 \\
&= 6 - 10 + 6 - 1 \\
&= 1
\end{aligned}
$$

Consider partial sums

$|S| = 3!$

$|A_1| + |A_2| + |A_3| = (\text{total } \# \text{ x's})\ 2!$

$= (\# \text{ ways to put 1 rook in X-pos})\ 2!$

$|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$

$= (\# \text{ ways to put 2 nonattacking rooks in X-pos})\ 1!$

$|A_1 \cap A_2 \cap A_3| = (\# \text{ ways to put 3 nonattacking rooks in X-pos})\ 0!$

For $0 \le i \le 3$ define

$r_i = \#$ ways to put i nonattacking rooks in X-pos

So $r_0 = 1$

Then

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = r_0\, 3! - r_1\, 2! + r_2\, 1! - r_3\, 0!$$

$$= \sum_{i=0}^{3} r_i\, (3-i)!\, (-1)^i$$

General case

Given $n \times n$ chessbd with forbidden positions

For $0 \le i \le n$ let r_i = # ways to place i nonattacking rooks in the forbidden positions

then by incl/excl

ways to place n nonattacking rooks on chessbd not in forb positions

$$= \sum_{i=0}^{n} r_i (n-i)! \; (-1)^i$$

Ex Find # ways to place 5 nonattacking rooks on chessbd

	1	2	3	4	5
1	×			×	
2			×		
3					
4	×				×
5		×			×

Sol. For $0 \le i \le 5$ find r_i

$r_0 = 1$

r_1 = # x's = 7

$r_2 = 4 + 5 + 4 + 2 + 1$

$= 16$

sim

$r_3 = 14$

$r_4 = 4$

$r_5 = 0$

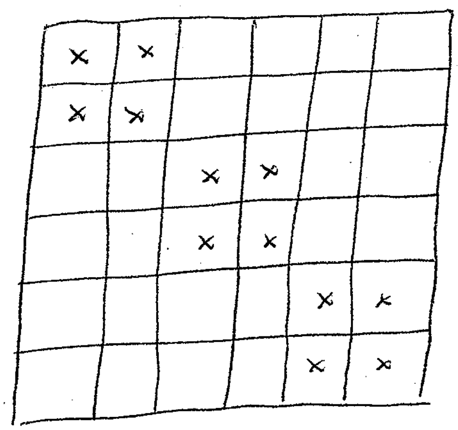
Ans $= 5! - 7 \cdot 4! + 16 \cdot 3! - 14 \cdot 2! + 4 \cdot 1!$ $\square$

Ex Find the number of perms $a_1 a_2 a_3 a_4 a_5 a_6$

of $\{1,2,3,4,5,6\}$ such that

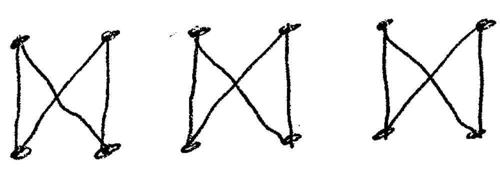
$a_1, a_2 \notin \{1,2\}$ $\quad$ $a_3, a_4 \notin \{3,4\}$ $\quad$ $a_5, a_6 \notin \{5,6\}$

Sol. View as rook placement problem with forbidden positions



For $0 \le i \le 6$ find r_i

Helps to view chessbd as defining a graph diagram



rows

cols

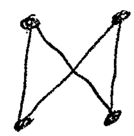
 means X appears in row r, col c

r_i = # of matchings of above graph using exactly i edges

" i-matching "

<u>subproblem</u>

For $i=0,1,2$ find # i-matchings for



i	0	1	2
# i-matchings	1	4	2

For convenience define "matching polynomial" in var X:

$$1 + 4X + 2X^2$$

For $i = 0,1,2$ coef of X^i = # i-matchings of $\bowtie$

Matching poly for $\bowtie \bowtie \bowtie$ = $(1+4X+2X^2)^3$

$$= 1 + 12X + 54X^2 + 112X^3 + 108X^4 + 48X^5 + 8X^6$$

So for $\bowtie \; \bowtie \; \bowtie$

i	0	1	2	3	4	5	6
r_i	1	12	54	112	108	48	8

So

ans = # ways to place 6 nonattacking rooks on chessbd with forb pos *

$$= \sum_{i=0}^{6} r_i (6-i)! \, (-1)^i$$

$$= 6! - 12 \cdot 5! + 54 \cdot 4! - 112 \cdot 3! + 108 \cdot 2! - 48 \cdot 1! + 8 \cdot 0!$$

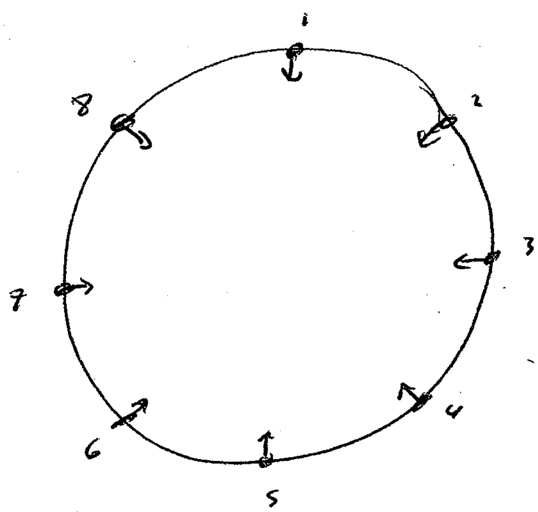
$\square$

Ex A Carousel has 8 seats each represented by different animal

8 Children are seated on the carousel facing inward so each child looks at another childs front

In how many ways can children change seats so each faces a different child?

Sol. Label seats around carousel



Seats labelled i, $i+4$ are opposite $1 \le i \le 4$

For $1 \le i \le 8$ child i moves to new seat labelled a_i

$a_1 a_2 \cdots a_8$ is perm of $\{1,2,\ldots,8\}$ s.t. a_i, a_{i+4} are not opp for $1 \le i \le 4$

We comp # such perms

let P = set of all perm of $\{1,2,\dots,8\}$

For $1 \le i \le 4$ let A_i = set of perms in P s.t. i, $i+4$ are opp

We seek

$$\left| \overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \right|$$

For $s \subseteq \{1,2,3,4\}$ def

$$A_s = \bigcap_{i \in s} A_i$$

We routinely find

$\lvert s \rvert$	0	1	2	3	4
$\lvert A_s \rvert$	$8!$	$8 \times 6!$	$8 \times 6 \times 4!$	$8 \times 6 \times 4 \times 2!$	$8 \times 6 \times 4 \times 2$

So by incl/excl

$$\text{ans} = \left| \overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \right|$$

$$= \sum_{s \subseteq \{1,2,3,4\}} |A_s| (-1)^{|s|}$$

$$= 1 \cdot 8! \quad - 4 \cdot 8 \cdot 6! \quad + 6 \cdot 8 \cdot 6 \cdot 4! \quad - 4 \cdot 8 \cdot 6 \cdot 4 \cdot 2! + 1 \cdot 8 \cdot 6 \cdot 4 \cdot 2$$

$$= 23040$$

$\square$

6.5 Another forbidden position problem

Recall Given $n \times n$ chessbd with forbidden positions X

For $0 \le i \le n$ let r_i = # ways to place i nonattacking rooks in X-positions

Then

ways to place n nonattacking rooks on chessbd not in X-pos

$$= \sum_{i=0}^{n} r_i (n-i)! (-1)^i$$

Note the r_i are unchanged if we permute the rows/cols of the chessbd

Ex let Q_n = # ways to put n nonattacking rooks on $n \times n$ chessbd with forbidden pos

$(i, i+1) \qquad 1 \le i \le n-1$

For $n=6$

	X				
		X			
			X		
				X	
					X

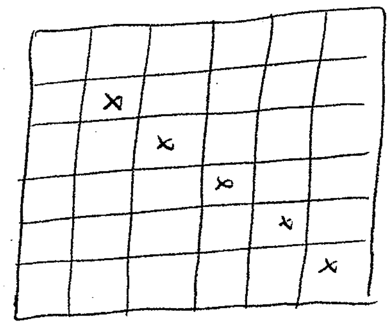
A bit easier if we permute rows of chessbd

$$1 \to 2 \to 3 \to \cdots \to n \to 1$$

Forb positions become $(\ast)$

$$(i,i) \qquad 2 \le i \le n$$

$n=6$



For $0 \le i \le n$ find r_i:

r_i = # ways to place i nonattacking rooks in X-pos

$$= \begin{cases} \binom{n-1}{i} & \text{if } i \le n-1 \\ 0 & \text{if } i = n \end{cases}$$

So

$$\varphi_n = \sum_{i=0}^{n-1} \binom{n-1}{i} (n-i)! (-1)^i$$

LEM For $n \geq 1$

$$Q_n = D_n + D_{n-1}$$

pf 1: Use above formula for Q_n and earlier formula for D_n

pf 2 (combinatorial)

Consider placement of n rooks using view *

<u>Case</u> rook in row 1 col 1

\# ways to place remaining $n-1$ rooks is D_{n-1}

<u>Case</u> no rook in row 1 col 1

\# choices is D_n

$\square$

10/22/12

4

6.6 Mobius inversion

For an integer $n \geq 1$ define

$\phi(n)$ = # of integers among $1, 2, \ldots, n$ that are $\underbrace{\text{relatively prime}}_{\text{no prime factor in common with } n}$ to n

"Euler ϕ - function"

ex $n = 6$

$1 \cancel{2} \cancel{3} \cancel{4} 5 \cancel{6}$

$\phi(6) = 2$

For $n \geq 1$ find $\phi(n)$ by incl/excl:
Consider prime factorization of n
Write

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

$p_1, p_2, \ldots, p_k$ dist primes

n_i pos integer $(1 \leq i \leq k)$

For $1 \le i \le k$ let

A_i = set of integers among $1, 2, \dots, n$ that are divisible by p_i

Then

$$\phi(n) = \left| \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_k \right|$$

For each subset

$$s \subseteq \{1, 2, \dots, k\}$$

define

$$A_s = \bigcap_{i \in s} A_i$$

and

$$p_s = \prod_{i \in s} p_i$$

A_s = set of integers among $1, 2, \dots, n$ that are divisible by p_s

$$= \{ p_s,\ 2p_s,\ 3p_s,\ \dots\ \tfrac{n}{p_s} p_s \}$$

So

$$|A_s| = \frac{n}{p_s}$$

By incl/excl

$$\phi(n) = \sum_{s \subseteq \{1, 2, \dots, k\}} |A_s| (-1)^{|s|}$$

$$= n \sum_{s \subseteq \{1, 2, \dots, k\}} p_s^{-1} (-1)^{|s|}$$

$$= n \prod_{i=1}^{k} (1 - p_i^{-1})$$

This gives

<u>Thm</u> For $n \geq 1$, the Euler ϕ-function satisfies

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - p_i^{-1}\right)$$

where $p_1, p_2, \ldots, p_k$ are the distinct primes that divide n

Ex

- $\phi(1)$
- $\phi(p^r) = p^r - p^{r-1}$ $\quad$ p prime $\quad$ $r \geq 1$
- For rel prime integers $m, n \geq 1$

$$\phi(mn) = \phi(m)\,\phi(n)$$

LEM For $n \geq 1$

$$n = \sum_{d/n} \phi(d)$$

where the sum is over all pos integers d that divide n

[ex $6 = \underset{1}{\underset{\shortparallel}{\phi(1)}} + \underset{1}{\underset{\shortparallel}{\phi(2)}} + \underset{2}{\underset{\shortparallel}{\phi(3)}} + \underset{2}{\underset{\shortparallel}{\phi(6)}}$]

pf Define

$$R_n = \{ k \mid 1 \leq k \leq n, \quad k, n \text{ rel prime} \}$$

So $\phi(n) = |R_n|$

Obs

$$\{1, 2, \dots, n\} = \bigcup_{d/n} \frac{n}{d} R_d \qquad (\text{disj union})$$

So

$$n = |\{1, 2, \dots, n\}|$$

$$= \sum_{d/n} \underset{\phi(d)}{\underset{\shortparallel}{|R_d|}}$$

$\square$

We now generalize prev lem.

Given any real valued function F defined on pos integers.

Define

$$G(n) = \sum_{d/n} F(d) \qquad n \geq 1$$

Find F in terms of G

<u>Case</u> $n = p^r$ p prime $r \geq 1$

r	$G(p^r)$
0	$F(1)$
1	$F(1) + F(p)$
2	$F(1) + F(p) + F(p^2)$

So

$$F(1) = G(1)$$

$$F(p^r) = G(p^r) - G(p^{r-1}) \qquad r \geq 1$$

<u>Case</u> $n = p^r q^s$ $\qquad p, q$ dist primes $\qquad r, s \geq 1$

$$G(n) = \sum_{d/n} F(d)$$

$$= \sum_{i=0}^{r} \sum_{j=0}^{s} F(p^i q^j)$$

One checks

$$F(p^r q^s) = G(p^r q^s) - G(p^r q^{s-1}) - G(p^{r-1} q^s) + G(p^{r-1} q^{s-1})$$

ie

$$F(n) = G(n) - G(n/q) - G(n/p) + G\left(\frac{n}{pq}\right)$$

<u>gen case</u>

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

$p_1, p_2, \dots, p_k$ dist primes
pos ints $n_1, n_2, \dots, n_k$

For $s \subseteq \{1, 2, \dots, k\}$ write

$$p_s = \prod_{i \in s} p_i$$

One checks by incl/excl

$$F(n) = \sum_{s \subseteq \{1,2,\dots,k\}} G(n/p_s)(-1)^{|s|} \qquad *$$

We now make * more elegant

Define a function μ on set of pos integers:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \quad p_1, p_2, \dots, p_k \text{ dist primes} \\ 0 & \text{else} \end{cases}$$

then * becomes

$$F(n) = \sum_{d/n} G(n/d)\, \mu(d)$$

"Mobius inversion"

$$= \sum_{d/n} G(d)\, \mu(n/d)$$

μ called <u>Mobius function</u>

ex Recall for $n \geq 1$

$$n = \sum_{d/n} \phi(d)$$ ϕ = Euler ϕ-function

Def

$$F(n) = \phi(n)$$
$$G(n) = n$$

So

$$G(n) = \sum_{d/n} F(d)$$

By Mobius inversion

$$F(n) = \sum_{d/n} G(d)\,\mu(n/d)$$

ie

$$\phi(n) = \sum_{d/n} d\,\mu(n/d)$$ $\square$

Lecture 21 Wed Oct 24

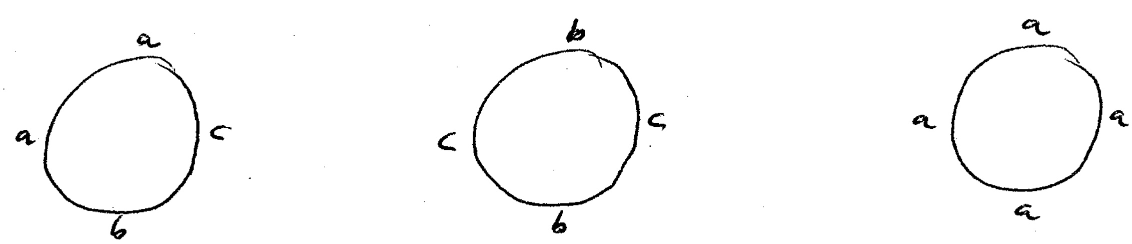
Problem Given integers $n, k \geq 1$

Find the number of circular n-permutations of the multiset

$$\underset{C(n)}{\shortparallel}$$

$$\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\} \qquad *$$

Ex $n=4$ $k=3$ $a_1 = a$ $a_2 = b$ $a_3 = c$



period: 4 2 1

First consider (ordinary) n-perms of $*$

To construct one fill in blanks:

___ ___ ... ___

k choices k choices k choices

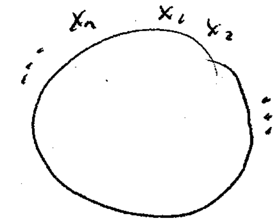
$\#$ n-perms of $*$ is k^n

Now consider function

n-perms of * $\longrightarrow$ circular n-perms of *

Wrap:

$x_1 x_2 \cdots x_n \longrightarrow$

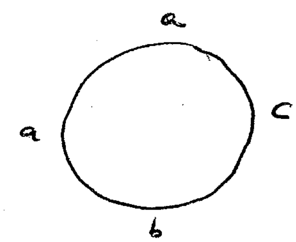


wrap around circle

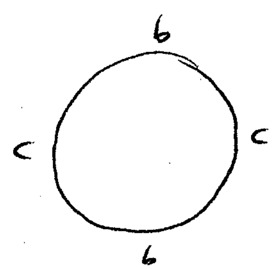
Wrap:

ex

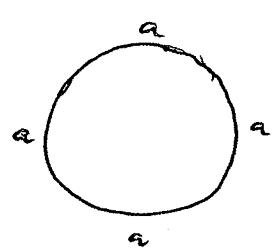
a c b a $\rightarrow$
c b a a $\rightarrow$
b a a c $\rightarrow$
a a c b $\rightarrow$



b c b c $\rightarrow$
c b c b $\rightarrow$



a a a a $\longrightarrow$



Given circular n-perm θ of $*$

$$\left| \left\{ n\text{-perms of } * \text{ that Wrap sends to } \theta \right\} \right| = \text{Period}(\theta)$$

For each pos integer d that divides n

let $C(n,d) = $ # circular n-perms of $*$ that have period exactly d

So

$$C(n) = \sum_{d/n} C(n,d) \qquad (1)$$

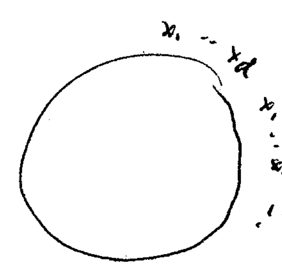
For d/n find $C(n,d)$

Construct n-perms of $*$ that yield a circular n-perm of $*$ with period divisible by d:

For any d-perm $x_1 x_2 \cdots x_d$ of $*$

Consider n-perm

$$\underline{x_1 x_2 \cdots x_d} \ \underline{x_1 x_2 \cdots x_d} \ \cdots \ \underline{x_1 x_2 \cdots x_d} \xrightarrow{\text{wrap}}$$



n/d copies

period divides d

This yields

$$k^d = \sum_{e/d} e\, C(n,e) \qquad (2)$$

[one eq for each divisor d of n]

Apply Mobius inversion to (2):

$\forall\ d/n$

$$d\,C(n,d) = \sum_{e/d} \mu(d/e)\, k^e$$

So

$$C(n,d) = \frac{1}{d} \sum_{e/d} \mu(d/e)\, k^e$$

Now by (1)

$$C(n) = \sum_{d/n} C(n,d)$$

$$= \sum_{d/n} \frac{1}{d} \sum_{e/d} \mu(d/e)\, k^e$$

$$= \frac{1}{n} \sum_{d/n} \frac{n}{d} \sum_{e/d} \mu(d/e)\, k^e$$

[change order of summation]

$$= \frac{1}{n} \sum_{e/n} k^e \underbrace{\sum_{e/d/n} n/d\, \mu(d/e)}_{\displaystyle = \sum_{D/N} D\, \mu(N/D) = \phi(N) = \phi(n/e)}$$

change vars
$D = n/d$
$N = n/e$

$$= \frac{1}{n} \sum_{e/n} \phi(n/e)\, k^e$$

We have shown

Thm For $n, k \geq 1$

the number of circular n-permutations of the multiset

$$\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$$

is

$$\frac{1}{n} \sum_{e \mid n} \phi(n/e)\, k^e$$

where ϕ = Euler ϕ-function $\square$

Recall Mobius inv

$$G(n) = \sum_{d/n} F(d) \qquad n \geq 1$$

$$F(n) = \sum_{d/n} \mu(n/d)\, G(d) \qquad n \geq 1$$

We now generalize

Given any finite set X with partial order $\leq$

Given real valued function F on X

$$F : X \to \mathbb{R} \qquad \text{(real numbers)}$$

Define a function

$$G : X \to \mathbb{R}$$

by

$$G(x) = \sum_{z \leq x} F(z) \qquad (*)$$

sum over all $z \in X$ such that $z \leq x$

Find

F in terms of G

Sol (Matrix approach)

Aside

Recall matrix mult

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$$

Recall identity

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Recall scalar mult

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Recall inverses:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad-bc \neq 0$$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

Back to X

Order the elements of X

$$X = \{ x_1, x_2, \ldots, x_N \}$$

Do this in a way that respects partial order:

$$x_i \le x_j \rightarrow i < j \qquad (1 \le i, j \le N)$$

Define $N \times N$ matrix M:

$$(i,j)\text{-entry of } M = \begin{cases} 1 & \text{if } x_i \le x_j \\ 0 & \text{if } x_i \not\le x_j \end{cases} \qquad 1 \le i, j \le N$$

M has form

$$M = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & 0 & & & 1 \end{pmatrix}$$

"upper triangular"

Note M^{-1} exists

Using functions F, G define row vectors

$$\overline{F} = (F(x_1), F(x_2), \ldots, F(x_N))$$

$$\overline{G} = (G(x_1), G(x_2), \ldots, G(x_N))$$

(*) Becomes

$$\overline{G} = \overline{F}\, M$$

↑ matrix mult.

Multiply both sides on right by M^{-1}

$$\bar{G}\, M^{-1} = (\bar{F}\, M)\, M^{-1}$$

$$= \bar{F}\,(M M^{-1})$$ matrix mult is associative

$$= \bar{F}\, I$$

$$= \bar{F}$$

So

$$\bar{G}\, M^{-1} = \bar{F} \qquad (**)$$

Using M^{-1} we define a function

$$\mu : X \times X \to \mathbb{R}$$

Cartesian prod

For $x, y \in X$

$\mu(x,y) =$ (i,j)-entry of M^{-1} where $x = x_i$, $y = x_j$

(**) becomes:

$\forall x \in X,$

$$F(x) = \sum_{y \leq x} G(y)\, \mu(y,x)$$

μ called the <u>Mobius function</u> of the poset $X, \leq$

ex Given integer $n \geq 1$

Define poset:

$$X = \{ d \mid d \text{ a pos integer that divides } n \}$$

Partial order by divisibility

Consider Mobius function of poset

$$\mu : X \times X \to \mathbb{R}$$

For $d|n$, $e|n$

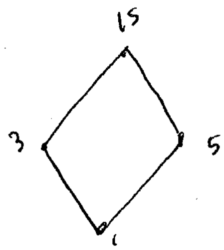
classical Mobius

$$\mu(d,e) = \begin{cases} \mu(e/d) & \text{if } d|e \\ 0 & \text{else} \end{cases}$$

We illustrate with $n = 15$

take $n = 15$
$= 3 \times 5$

$$X = \{1, 3, 5, 15\}$$



$$M = \begin{array}{c|cccc} & 1 & 3 & 5 & 15 \\ \hline 1 & \multicolumn{4}{c}{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \\ 3 & & & & \\ 5 & & & & \\ 15 & & & & \end{array}$$

$$M^{-1} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Check $MM^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

M^{-1} gives Mobius function μ

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{y} \\ & & 1 & 3 & 5 & 15 \\ \hline & 1 & 1 & -1 & -1 & 1 \\ & 3 & 0 & 1 & 0 & -1 \\ x & 5 & 0 & 0 & 1 & -1 \\ & 15 & 0 & 0 & 0 & 1 \end{array}$$

$\mu(x,y)$ entry in row x, col y

$$\mu(1,3) = -1 = \underset{\text{classical}}{\mu(3)}$$

$$\mu(1,5) = -1 = \mu(5)$$

$$\mu(1,15) = 1 = \mu(15)$$

$$\mu(3,15) = -1 = \mu(15/3) = \mu(5)$$

$$\mu(5,15) = -1 = \mu\left(\tfrac{15}{5}\right) = \mu(3)$$

etc

$\square$

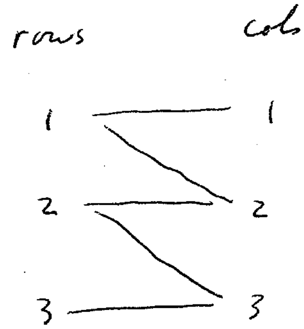
Ch 6 $\boxed{\text{Extra}}$

Given $n \times n$ matrix A with entries 0 or 1

Represent A by diagram:

$n=3$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



Observe

\# ways to put n nonattacking rooks on 1-entries of A

= \# n-matchings of diagram

Find this number.

Let R = set of rows of A = $\{1, 2, \ldots, n\}$

C = set of cols of A = $\{1, 2, \ldots, n\}$

View n-matching of diagram as a bijection

$$f : R \to C$$

such that

$$x \text{ — } f(x) \quad \forall x \in R$$

Use incl excl to find \#

Case A has all entries 1

So $x \text{ — } y \quad \forall x \in R \ \forall y \in C$

Compute \# of n-matchings 2 ways.

Calc I

$\#$ n-matchings $= n!$

Calc II

Let S = set of all functions $R \to C$

$|S| = n^n$

For $1 \le i \le n$ define

$A_i = \{ f \in S \mid i \notin \mathrm{Im}(f) \}$

↑ image of f

Obs

$\#$ n-matchings $= |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$

For $s \subseteq \{1, 2, \ldots, n\}$ define

$A_s = \bigcap_{i \in s} A_i$

$A_s = \{ f \in S \mid \mathrm{Im}(f) \subseteq \{1, 2, \ldots, n\} \setminus s \}$

$|A_s| = (n - |s|)^n$

By incl/excl

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| = \sum_{s \subseteq \{1,2,\ldots,n\}} |A_s| \, (-1)^{|s|}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (n-k)^n (-1)^k$$

Comparing Calcs I, II get

$$n! = \sum_{k=0}^{n} \binom{n}{k} (n-k)^n (-1)^k \qquad n = 0, 1, 2, \ldots$$

<u>Gen case</u> Use above technique to get formula (6.2.8) in the text.