Lecture 13 Wednesday Oct 3 10/3/12

Ch 5 Binomial Coefficients

Binomial coefs came up in Ch 2. We now consider them in depth. Recall for integers $0 \le k \le n$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \# \text{ } k\text{-subsets of } \{1,2,\ldots,n\}$$

We saw

$$\binom{n}{k} = \binom{n}{n-k}$$

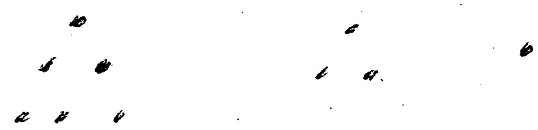
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad 1 \le n \qquad \text{"Pascal Formula"}$$

We now find more identities for binom coefs

Ex Consider a tetrahedral pyramid of stacked cannon balls of height n. How many balls?

Sol

For $n=3$ top view

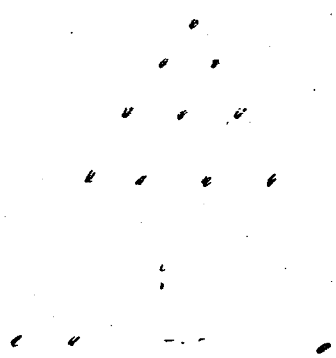


floor 1 2 3

ans = $1+3+6 = 10$

Gen n:

<u>Floor 1</u>



balls on floor 1 is

$$1+2+3+\cdots+n \underset{\text{by ind}}{=} \binom{n+1}{2}$$

Similar formula for each floor.

total # balls =

$$\binom{n+1}{2}+\binom{n}{2}+\binom{n-1}{2}+\cdots+\binom{4}{2}+\binom{3}{2}+\binom{2}{2}$$

Now simplify using Pascal formula:

$$\underbrace{\binom{n+1}{2} + \underbrace{\binom{n}{2} + \binom{n-1}{2} + \cdots + \underbrace{\binom{4}{2} + \underbrace{\binom{3}{2} + \underset{\binom{3}{3}}{\underset{\|}{\binom{2}{2}}}}_{\binom{4}{3}}}_{\binom{5}{3}}}_{\binom{n+1}{3}}}_{\binom{n+2}{3}}$$

Pyramid of height n has $\binom{n+2}{3}$ balls $\square$

In summary

$$\sum_{k=2}^{N} \binom{k}{2} = \binom{N+1}{3} \qquad N = 2, 3, \ldots$$

Interp using Pascals triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$\vdots$

$1 \quad N \quad \binom{N}{2} \quad \binom{N}{3} \quad \cdots$

$1 \quad N+1 \quad \binom{N+1}{2} \quad \boxed{\binom{N+1}{3}} \quad \cdots$

the sum of the circled entries is boxed entry

Other diagonal sums are similarly handled:

For integers $0 \le k \le n$

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

Ex For an integer $n \geq 1$ find

$$1^3 + 2^3 + 3^3 + \cdots + n^3$$

*

in closed form

Sol Write n^3 in terms of binomial coefs

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} = \frac{n^3 - 3n^2 + 2n}{6}$$

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2 - n}{2}$$

Solve for n^3

$$n^2 = 2\binom{n}{2} + n$$

$$n^3 = 6\binom{n}{3} + 3n^2 - 2n$$

$$= 6\binom{n}{3} + 6\binom{n}{2} + n$$

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$$* = \sum_{k=1}^{n} k^3$$

$$= 6 \sum_{k=1}^{n} \binom{k}{3} + 6 \sum_{k=1}^{n} \binom{k}{2} + \sum_{k=1}^{n} k$$

$$= 6 \binom{n+1}{4} + 6 \binom{n+1}{3} + \binom{n+1}{2}$$

$$= \frac{(n+1)n(n-1)(n-2)}{4} + (n+1)n(n-1) + \frac{(n+1)n}{2}$$

$$= \frac{(n+1)n\left(n^2-3n+2+4n-4+2\right)}{4}$$

$$= \binom{n+1}{2}^2$$

∎

—o—

For integers $l \geq 1$, $n \geq 1$

the sum

$$1^l + 2^l + 3^l + \cdots + n^l$$

can be similarly found . . .

We recall the binomial theorem:

thm Given variables x, y Given integer $n \geq 0$

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

pf Expand LHS

$$(x+y)^n = (x+y)(x+y)(x+y) \cdots (x+y) \qquad n \text{ copies}$$

$$= \sum a_1 a_2 \cdots a_n$$

sum is over all sequences $a_1 a_2 \cdots a_n$ such that $a_i \in \{x, y\}$ for $1 \leq i \leq n$

Each summand $a_1 a_2 \cdots a_n$ contributes $x^{n-k} y^k$ where $k = |\{i \mid 1 \leq i \leq n, a_i = y\}|$

For $0 \leq k \leq n$ the number of summands $a_1 a_2 \cdots a_n$ that contribute $x^{n-k} y^k$

$=$ # of ways to create $a_1 a_2 \cdots a_n$ using exactly k y's

$=$ # k-subsets of $\{1, 2, \ldots n\}$

$= \binom{n}{k}$

Therefore

$$\sum a_1 a_2 \cdots a_n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$\square$

Sp case: set $y=1$ in binom thm

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k \qquad \left[\text{since } \binom{n}{k} = \binom{n}{n-k}\right]$$

Ex For an integer $n \geq 1$ find

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} \qquad \circledast$$

Sol Consider

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

take the derivative with respect to x

$$n(1+x)^{n-1} = \sum_{k=0 \to 1}^{n} \binom{n}{k} k x^{k-1}$$

Now set $x=1$

$$n 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

$$= \circledast$$

$\square$

Ex For an integer $n \geq 1$ find

$$1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n}$$ *

pf Consider

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Take deriv 2 times

$$n(1+x)^{n-1}$$

$$n(n-1)(1+x)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k(k-1)x^{k-2}$$

Set $x=1$

$$n(n-1)2^{n-2} = \sum_{k=0}^{n} \binom{n}{k} \underbrace{k(k-1)}_{k^2-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} k^2 - \underbrace{\sum_{k=0}^{n} \binom{n}{k} k}_{= n2^{n-1}}$$

So

$$\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n-1)2^{n-2} + n2^{n-1}$$

$$= n(n+1)2^{n-2}$$ □

For integers $l \geq 1$, $n \geq 1$ the sum

$$\sum_{k=0}^{n} k^l \binom{n}{k}$$ can be similarly found

Ex For an integer $n \geq 0$ find

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 \qquad (*)$$

Sol try small n to see pattern

n		sum of squares
0	$\begin{matrix} 1 \\ 1 \end{matrix}$	$1 = \binom{0}{0}$
1	$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$	$2 = \binom{2}{1}$
2	$\begin{matrix} 1 & 2 & 1 \\ 1 & 4 & 1 \end{matrix}$	$6 = \binom{4}{2}$
3	$\begin{matrix} 1 & 3 & 3 & 1 \\ 1 & 9 & 9 & 1 \end{matrix}$	$20 = \binom{6}{3}$
4	$\begin{matrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 16 & 36 & 16 & 1 \end{matrix}$	$70 = \binom{8}{4}$

It appears

$$(*) = \binom{2n}{n}$$

We now prove it

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Show $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Consider the set

$$\{\pm 1, \pm 2, \dots, \pm n\}$$

$2n$ elements

Let S = set of all n-elements subsets of $\{\pm 1, \pm 2, \dots, \pm n\}$

$$|S| = \binom{2n}{n}$$

For $0 \le k \le n$ define

$$S_k = \{ x \in S \mid x \text{ contains exactly } k \text{ pos elements} \}$$

$$S = \bigcup_{k=0}^{n} S_k \qquad \text{(disjoint union)}$$

$$|S| = \sum_{k=0}^{n} |S_k|$$

For $0 \le k \le n$ to find $|S_k|$ construct $x \in S_k$ via 2-stage exp:

stage	to do	# choices
1	select pos elements of x	$\binom{n}{k}$
2	select neg els of x	$\binom{n}{n-k}$ $\left(= \binom{n}{k}\right)$

$$|S_k| = \binom{n}{k}^2$$

Result follows.

$\square$

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We now extend the definition of the binomial coefs.

Given a variable z

Given integer k

we define

$$\binom{z}{k}$$

as follows:

Case $k \geq 0$

$$\binom{z}{k} = \frac{z(z-1)(z-2)\cdots(z-k+1)}{k!}$$

[polynomial in z with degree k]

Case $k < 0$

$$\binom{z}{k} = 0$$

we can set $z \to$ any real number

ex

$$\binom{5/2}{3} = \frac{5/2 \quad 3/2 \quad 1/2}{3 \cdot 2 \cdot 1}$$

$$= 5/16$$

Pascal's formula holds for gen z, k:

$$\binom{z}{k} = \binom{z-1}{k-1} + \binom{z-1}{k}$$

(ex)

Last time we saw, for integers $0 \le k \le n$

$$\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1} \qquad *$$

Next goal: find analog that holds after $n \to z$

Modify $*$ using

$$\binom{n}{k} = \binom{n}{n-k}$$

Get

$$\binom{k}{0} + \binom{k+1}{1} + \dots + \binom{n-1}{n-k-1} + \binom{n}{n-k} = \binom{n+1}{n-k}$$

Replace $k \to n-k$

$$\binom{n-k}{0} + \binom{n-k+1}{1} + \dots + \binom{n-1}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Replace $n \to z$ z var:

$$\sum_{i=0}^{k} \binom{z-i}{k-i} = \binom{z+1}{k} \qquad **$$

<u>thm</u> ** holds for a variable z and any integer $k \geq 0$

pf Repeatedly apply Pascal formula to RHS $\square$

ex take $k=3$

$$\binom{z}{3} + \binom{z-1}{2} + \binom{z-2}{1} + \binom{z-3}{0} = \binom{z+1}{3}$$

$$\binom{z}{3} = \frac{z(z-1)(z-2)}{6}, \quad \binom{z-1}{2} = \frac{(z-1)(z-2)}{2}, \quad \binom{z-2}{1} = z-2, \quad \binom{z-3}{0} = 1, \quad \binom{z+1}{3} = \frac{(z+1)z(z-1)}{6}$$

5.3 Unimodality of Binomial Coef

Consider any row of Pascals triangle

ex row 8

$$1 \le 8 \le 28 \le 56 \le 70 \ge 56 \ge 28 \ge 8 \ge 1$$

Sequence is unimodal

A finite sequence of real numbers

$$s_0 \quad s_1 \quad \cdots \quad s_n$$

is unimodal whenever $\exists\ t \quad 0 \le t \le n$ such that

$$s_0 \le s_1 \le \cdots \le s_t \ge s_{t+1} \ge \cdots \ge s_n$$

ex the sequence

$$1, 4, 5, 9, 9, 9, 6, 3$$

is unimodal

Next goal: Show that each row in Pascal triangle is unimodal

Thm Given integer $n \geq 0$

(i) For n even

$$\binom{n}{0} < \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{n/2}$$

$$\binom{n}{n/2} > \cdots > \binom{n}{n-1} > \binom{n}{n}$$

(ii) For n odd

$$\binom{n}{0} < \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{\frac{n-1}{2}}$$

$$\binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}$$

$$\binom{n}{\frac{n+1}{2}} > \cdots > \binom{n}{n-1} > \binom{n}{n}$$

In particular row n of Pascal triangle is unimodal.

Pf For $1 \leq k \leq n$ compare

$$\binom{n}{k-1}, \quad \binom{n}{k}$$

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} \quad \overset{?}{\substack{> \\ = \\ <}} \quad 1$$

Find

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} - 1 \quad \overset{?}{\substack{> \\ = \\ <}} \quad 0$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k!}$$

$$\binom{n}{k-1} = \frac{n(n-1)\cdots(n-k+2)}{(k-1)!}$$

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n-k+1}{k}$$

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} - 1 = \frac{n-2k+1}{k}$$

Find sign of

$$n-2k+1$$

For n even

k	sign of $n-2k+1$
$1 \le k \le \frac{n}{2}$	$+$
$\frac{n}{2} < k \le n$	$-$

For n odd

k	sign of $n-2k+1$
$1 \le k \le \frac{n-1}{2}$	$+$
$k = \frac{n+1}{2}$	0
$\frac{n+1}{2} < k \le n$	$-$

$\square$

Notation

For a real number r

def

$\lfloor r \rfloor$ = greatest integer at most r

$= \max\{n \mid n \in \mathbb{Z},\ n \le r\}$

"floor of r"

$\lceil r \rceil$ = least integer at least r

$= \min\{n \mid n \in \mathbb{Z},\ n \ge r\}$

"ceiling of r"

ex For an integer n

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ \frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

$$\left\lceil \frac{n}{2} \right\rceil = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ \frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

Cor For an integer $n \geq 0$

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$$

Moreover this common value is the largest among

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

$\square$

Pf Routine using prev thm.

Sperner theorem

Motivation Given any poset $X, \leq$ $|X| < \infty$

A chain in $X, \leq$ is a subset $C \subseteq X$ such that any two elements of C are comparable

The elements $x_1, x_2, \ldots, x_t$ of a chain C can be ordered s.t.

$$x_1 < x_2 < \cdots < x_t$$

A chain C is maximal whenever C is not properly contained in another chain.

An antichain in $X, \leq$ is a subset $A \subseteq X$ such that any 2 distinct elements of A are incomparable

Ex Fix integer $n \geq 1$

X: all subsets of $\{1, 2, \ldots, n\}$

$x \leq y$ whenever $x \subseteq y$ $\forall x, y \in X$

So $|X| = 2^n$

Describe max'l chains:

Max'l chain C has form $C = \{x_i\}_{i=0}^{d}$ such that

$$|x_i| = i \qquad 0 \leq i \leq n$$

and

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$$

there is a bijection between the set of max'l chains of $X, \leq$ and the perms of $\{1, 2, \ldots, n\}$.

Given a perm $a_1 a_2 \ldots a_n$ of $\{1, 2, \ldots, n\}$

Define a subset $x_i \subseteq \{1, 2, \ldots, n\}$ by
$x_i = \{a_1, a_2, \ldots, a_i\}$ $\quad 0 \leq i \leq n$

then $\{x_i\}_{i=0}^{n}$ is a max chain in $X, \leq$.

Conversely, given max'l chain C of $X, \leq$

write $C = \{x_i\}_{i=0}^{n}$ with $|x_i| = i$ for $0 \leq i \leq n$

note $x_0 = \phi$

def

$a_1 =$ unique elm x_1

$a_2 =$ " " $x_2 \setminus x_1$

$a_3 =$ " " $x_3 \setminus x_2$

$\vdots$

$a_n =$ " " $x_n \setminus x_{n-1}$

then $a_1 a_2 \ldots a_n$ is perm of $\{1, 2, \ldots, n\}$

In summary

$$\left\{ \begin{array}{c} \text{perms} \\ \text{of } \{1, 2, \ldots, n\} \end{array} \right\} \overset{\text{bij}}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{max'l} \\ \text{chains} \\ \text{in } X, \leq \end{array} \right\}$$

So $X, \leq$ has $n!$ max'l chains.

Now consider antichains of $X, \leq$

For $0 \leq k \leq n$ define

$$A_k = \{ x \in X \mid |x| = k \}$$

So $$|A_k| = \binom{n}{k}$$

$$X = \bigcup_{k=0}^{n} A_k \qquad \text{disjoint union}$$

Obs each A_k is an antichain of $X, \leq$

Take $k = \lfloor \frac{n}{2} \rfloor$:

$X, \leq$ has an antichain with cardinality $\binom{n}{\lfloor \frac{n}{2} \rfloor}$

thm (Sperner) For the above poset $X, \leq$

each antichain has cardinality at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$

pf Given antichain A of $X, \subseteq$

Count # of ordered pairs (a, C) such that

$a \in A$ and C is max'l chain containing a

Count in 2 ways

Count 1 $n!$ choices for C

Given C at most one choice for a

$\# \leq n!$

Count 2 For $a \in A$

write $|a| = k$

there are $k!\,(n-k)!$

max'l antichains containing a

For $0 \leq k \leq n$ let

$\alpha_k =$ number of elements of A of size k

so $|A| = \sum_{k=0}^{n} \alpha_k$

So $\# = \sum_{k=0}^{n} \alpha_k \, k!\,(n-k)!$

So

$$n! \geq \sum_{k=0}^{n} \alpha_k \, k!\,(n-k)!$$

$$1 \geq \sum_{k=0}^{n} \alpha_k \frac{k!\,(n-k)!}{n!}$$

$$= \sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}}$$

$$\geq \sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

$$= \frac{|A|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

So

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq |A| \qquad \square$$

$\boxed{\text{Math 475 Exam I is next Monday Oct 15}}$

5.4 Multinomial Theorem

Recall binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Now consider $t \geq 2$ variables

$x_1, x_2, \ldots, x_t$

Expand

$$(x_1 + x_2 + \cdots + x_t)^n \qquad n = 0, 1, 2 \ldots$$

Sol

$$(x_1 + \cdots + x_t)^n = (x_1 + \cdots + x_t)(x_1 + \cdots + x_t) \cdots$$

(n factors)

$$= \sum a_1 a_2 \cdots a_n$$

sum is over all sequences $a_1 a_2 \ldots a_n$ such that

$a_i \in \{x_1, x_2, \ldots, x_t\}$ for $1 \leq i \leq n$

Each summand $a_1 a_2 \cdots a_n$ contributes

$*$

$$x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$$

where

$$n_i = \left| \{ j \mid a_j = x_i \} \right| \qquad 1 \le i \le t$$

Note $n = n_1 + n_2 + \cdots + n_t$

Given integers $n_1, n_2, \ldots, n_t \ge 0$ s.t. $n = n_1 + n_2 + \cdots + n_t$

the number of summands $a_1 a_2 \cdots a_n$ that contribute $*$

= # ways to fill n blanks

$$\_\_ \; \_\_ \; \_\_ \; \cdots \; \_\_$$

with n_1 copies x_1
n_2 " x_2
...
n_t " x_t

= # perms of multiset

$$\{ n_1 \cdot x_1, n_2 \cdot x_2, \ldots, n_t \cdot x_t \}$$

$$= \frac{n!}{n_1! \, n_2! \cdots n_t!}$$

So

thm (Multinomial thm)

$$(x_1+x_2+\cdots+x_t)^n = \sum \frac{n!}{n_1!\,n_2!\,\cdots\,n_t!}\, x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$$

where sum is over all nonnegative integral solutions $n_1, n_2, \ldots, n_t$ to $n_1+n_2+\cdots+n_t = n$

Def We abbrev

$$\binom{n}{n_1\, n_2\, \cdots\, n_t} = \frac{n!}{n_1!\,n_2!\,\cdots\,n_t!}$$

$$n = n_1+n_2+\cdots+n_t$$

$n_1, n_2, \ldots, n_t \geq 0$

"multinomial coefficient"

ex take $t=2$

$$\binom{n}{n_1\, n_2} = \frac{n!}{n_1!\,n_2!}$$

$n_1+n_2=n$

$n_1 \geq 0, \quad n_2 \geq 0$

Write $k = n_1$

$n-k = n_2$

$$\binom{n}{k\ \ n-k} = \frac{n!}{k!\,(n-k)!} = \underbrace{\binom{n}{k} = \binom{n}{n-k}}_{\text{usual binom coefs}}$$

Recall Pascal's formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

We now display a multinomial analog.

thm. Given positive integers $n_1, n_2, \ldots, n_t$ and write

$$n = n_1 + n_2 + \cdots + n_t$$

Then

$$\binom{n}{n_1\ n_2 \cdots n_t} = \binom{n-1}{n_1-1\ \ n_2 \cdots n_t} + \binom{n-1}{n_1\ \ n_2-1\ \ n_3 \cdots n_t} + \cdots$$

$$\cdots + \binom{n-1}{n_1\ n_2 \cdots n_{t-1}\ \ n_t-1}$$

$$= \sum_{i=1}^{t} \binom{n-1}{n_1 \cdots n_{i-1}\ \ n_i-1\ \ n_{i+1} \cdots n_t}$$

pf1 (Algebraic) evaluate each term using def of multinomial coeff

pf2 (combinatorial)

Consider multiset

$$\{ n_1 \cdot x_1,\ n_2 \cdot x_2,\ \ldots,\ n_t \cdot x_t \}$$

let P = set of permutations of this multiset

So $$|P| = \binom{n}{n_1\ n_2\ \cdots\ n_t}$$

Given element in P, say

$$a_1 a_2 \cdots a_n$$

1st coord a_1 is any one of $x_1, x_2, \ldots, x_t$

For $1 \le i \le t$ define

P_i = set of perms in P that have 1st coord x_i

$\{P_i\}_{i=1}^{t}$ is partition of P

$$|P| = \sum_{i=1}^{t} |P_i|$$

For $1 \le i \le t$ find $|P_i|$: To obtain an element of P_i fill in blanks:

$$\underline{x_i}\ \underbrace{\_\_\ \ \_\_\ \ \cdots\ \ \_\_}_{n-1 \text{ remaining blanks}}$$

Fill rem blanks with

n_1 copies x_1
...
n_{i-1} copies x_{i-1}
$n_i - 1$ copies x_i
n_{i+1} copies x_{i+1}
:
n_t copies x_t

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$|P_i|$ = # perms of multiset

$$\{n_1 \cdot x_1,\ n_2 \cdot x_2 \cdots n_{i-1} \cdot x_{i-1} \quad (n_i - 1) \cdot x_i \quad n_{i+1} \cdot x_{i+1} \cdots n_t \cdot x_t\}$$

$$= \binom{n-1}{n_1 \cdots n_{i-1} \quad n_i - 1 \quad n_{i+1} \cdots n_t}$$

Result follows. $\square$

5.5 Newton's binomial theorem

Recall binomial theorem

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k \qquad n=0,1,2\ldots$$

So
$$(1-z)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k z^k$$

Now consider

$$(1-z)^{-n} \qquad n=0,1,2\ldots$$

View I

First note

$$\frac{1}{1-z} = 1+z+z^2+\cdots$$

where z is any real number such that $|z|<1$

check: let
$$S = 1+z+z^2+\cdots$$
$$zS = z(1+z+z^2+\cdots)$$
$$= z+z^2+\cdots$$
$$= S-1$$

so
$$1 = S(1-z)$$

so $S = \dfrac{1}{1-z}$

Now

$$(1-z)^{-n} = \frac{1}{(1-z)^n}$$

$$= \frac{1}{1-z} \ \frac{1}{1-z} \cdots \frac{1}{1-z}$$

$$= (1+z+z^2+\cdots)(1+z+z^2+\cdots) \cdots$$

(n factors)

$$= \sum a_1 a_2 \cdots a_n$$

sum over all products $a_1 a_2 \cdots a_n$

such that $a_i \in \{1, z, z^2, \ldots\}$ for $1 \le i \le n$

Each summand $a_1 a_2 \cdots a_n$ contributes

$$z^k$$

where

$$k = k_1 + k_2 + \cdots + k_n$$

$$a_i = z^{k_i} \qquad 1 \le i \le n$$

For an integer $k \ge 0$ the number of summands $a_1 a_2 \cdots a_n$ that contribute z^k

= # non neg integral sols $k_1, k_2, \ldots k_n$ to

$$k = k_1 + k_2 + \cdots + k_n$$

$$= \binom{k+n-1}{n-1} = \binom{k+n-1}{k}$$

So

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k \qquad |z|<1$$

<u>View II</u> If we retain the form of binomi theorem but allow neg exponents, answer "should be"

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k z^k$$

Check:

$$\binom{-n}{k}(-1)^k \overset{?}{=} \binom{n+k-1}{k} \qquad k=0,1,2\ldots$$

$$LHS = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}(-1)^k$$

$$= \frac{n(n+1)\cdots(n+k-1)}{k!}$$

$$RHS = \frac{(n+k-1)(n+k-2)\cdots(n+1)n}{k!}$$

$= LHS$ ✓

So

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k}(-1)^k z^k \qquad n=0,1,2\ldots \quad |z|<1$$

or

$$(1+z)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} z^k$$

We have shown:

For an integer n (pos or neg)

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k \qquad |z| < 1$$

In fact much more is true:

thm (Newton) For any real number α

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k \qquad |z| < 1$$

[pf requires advanced calculus]

Ex Find $\sqrt{3}$ (decimal expansion)

Sol $\sqrt{3} = 3^{1/2}$

$= (4-1)^{1/2}$

$= \left(4\left(1-\frac{1}{4}\right)\right)^{1/2}$

$= 2\left(1-\frac{1}{4}\right)^{1/2}$

$= 2(1-z)^{\frac{1}{2}}$ $z = \frac{1}{4}$

$$= 2\sum_{k=0}^{\infty}\binom{1/2}{k}(-1)^k z^k$$

Fn $n=0,1,2..$ def

$$a_n = 2\sum_{k=0}^{n}\binom{1/2}{k}(-1)^k z^k$$

$$\lim_{n\to\infty} a_n = \sqrt{3}$$

For large n

a_n is good approx to $\sqrt{3}$

— o —

Lecture 16 Wednesday Oct 10

Math 475 Exam I is Monday Oct 15 ch 2–5

5.6 More on posets

Given a nonempty finite set X with partial order $\leq$

Recall

For $C \subseteq X$,

C is chain whenever any 2 elements of C are comparable

For $A \subseteq X$

A is antichain whenever any 2 distinct elements of C are incomparable

For a chain C and antichain A

$$|C \cap A| \leq 1$$

Since for distinct $x, y \in C \cap A$

x, y both comparable and noncomparable cnt

Suppose we partition X into antichains $\{A_i\}_{i=1}^{\alpha}$

So
$$X = \bigcup_{i=1}^{\alpha} A_i \qquad \text{(disj union)}$$

For all chains C

$$\begin{aligned} C &= C \cap X \\ &= C \cap \left(\bigcup_{i=1}^{\alpha} A_i \right) \\ &= \bigcup_{i=1}^{\alpha} (C \cap A_i) \end{aligned}$$

So
$$|C| = \sum_{i=1}^{\alpha} \underbrace{| C \cap A_i |}_{\leq 1} \leq \alpha$$

Therefore

$$\mathrm{Max}\{|C| \mid C \text{ a chain of } X\} \leq \mathrm{Min}\{\alpha \mid \{A_i\}_{i=1}^{\alpha} \text{ is antichain partition of } X\}$$

(✱)

Suppose we partition X into chains $\{C_i\}_{i=1}^{r}$

So

$$X = \bigcup_{i=1}^{r} C_i \qquad \text{(disj union)}$$

For all antichains A.

$$\begin{aligned} A &= A \cap X \\ &= A \cap \left(\bigcup_{i=1}^{r} C_i \right) \\ &= \bigcup_{i=1}^{r} (A \cap C_i) \end{aligned}$$

So

$$|A| = \sum_{i=1}^{r} \underbrace{|A \cap C_i|}_{\leq 1}$$

$$\leq r$$

Therefore

$$\text{Max}\{|A| \,/\, A \text{ an antichain of } X\} \leq \text{Min}\{r \,/\, \{C_i\}_{i=1}^{r} \text{ is chain partition of } X\}$$

$(**)$

Next goal: show equality in (*) and (**)

theorem

$$\mathrm{Max}\{|C| \,/\, C \text{ a chain of } X\}$$

$$= \mathrm{Min}\{\alpha \,/\, \{A_i\}_{i=1}^{\alpha} \text{ is antichain partition of } X\}$$

pf Suffices to display a chain C of X and an antichain partition $\{A_i\}_{i=1}^{\alpha}$ of X such that

$$|C| = \alpha$$

Recall an element $x \in X$ is minimal whenever $\nexists\, y \in X$ s.t. $y < x$.

Define

A_1 = set of all minimal elements of X

A_2 = set of all min elements of $X \setminus A_1$

A_3 = set of all min elements of $X \setminus \{A_1 \cup A_2\}$

...

$A_\alpha \neq \phi$ and $A_{\alpha+1} = \phi$ for some α

$\{A_i\}_{i=1}^{\alpha}$ is antichain partition of X

By construction, for $2 \le i \le \alpha$ and $x \in A_i$

$\exists\, y \in A_{i-1}$ such that $y < x$

We now construct a chain C with $|C| = \alpha$

Pick $x_\alpha \in A_\alpha$

$\exists$ $x_{\alpha-1} \in A_{\alpha-1}$ such that

$$x_{\alpha-1} < x_\alpha$$

$\exists$ $x_{\alpha-2} \in A_{\alpha-2}$ such that

$$x_{\alpha-2} < x_{\alpha-1}$$

$$\vdots$$

$\exists$ $x_1 \in A_1$ such that

$$x_1 < x_2$$

Now

$$x_1 < x_2 < \cdots < x_\alpha$$

is a chain.

Take $C = \{x_i\}_{i=1}^{\alpha}$

$\square$

Example integer $n \geq 1$

X = set of all subsets of $\{1, 2, \dots, n\}$

$x \leq y$ whenever $x \subseteq y$ $(\forall x, y \in X)$

We saw earlier

each max chain in X has size $n+1$

By the theorem we can partition X into $n+1$ antichains.

For $0 \leq i \leq n$ define

$$A_i = \{ x \in X \mid |x| = i \}$$

$\{A_i\}_{i=0}^{n}$ is antichain partition of X.

Theorem (Dilworth)

$$\text{Max}\left\{ |A| \;/\; A \text{ an antichain of } X \right\} = m$$

$$= \text{Min}\left\{ r \;/\; \{C_i\}_{i=1}^{r} \text{ is chain partition of } X \right\}$$

pf Suffices to display a partition of X into m chains

We argue by induction on $|X|$

Result is trivial for $|X| = 1$

Now assume $|X| > 1$

We consider two cases

Case I $\exists$ antichain A of X that has $|A| = m$,

but $A \neq$ set of minimal els of X

$A \neq$ set of maximal els of X

Define

$$A^+ = \left\{ x \in X \;/\; \exists\, a \in A \text{ such that } a \leq x \right\}$$

$$A^- = \left\{ x \in X \;/\; \exists\, a \in A \text{ such that } a \geq x \right\}$$

So $A \subseteq A^+$

$A \subseteq A^-$

$A =$ set of min elements of A^+

$A =$ set of maximal elements of A^-

Observe:

(i) $A^+ \neq X$ (Since X has a min element y that is not in A. Note $y \notin A^+$)

So $|A^+| < |X|$

(ii) $A^- \neq X$ (Since X has a maximal element z that is not in A. Note $z \notin A^-$)

So $|A^-| < |X|$

(iii) $A^+ \cap A^- = A$ (Since $\forall x \in A^+ \cap A^-$ $\exists a \in A$

s.t. $a \le x$ and $\exists b \in A$ s.t. $x \le b$

So $a \le x \le b$

But elements a, b of A are incomparable unless $a = b$ so

$a = x = b$

Now $x = a \in A$)

(iv) $A^+ \cup A^- = X$ (If $A^+ \cup A^- \neq X$ then pick

$x \in X \setminus (A^+ \cup A^-)$

Then $A \cup \{x\}$ is antichain of size m+1 contr)

By constr

A is maximal antichain of A^+

By induction A^+ has chain partition $\{E_i\}_{i=1}^m$

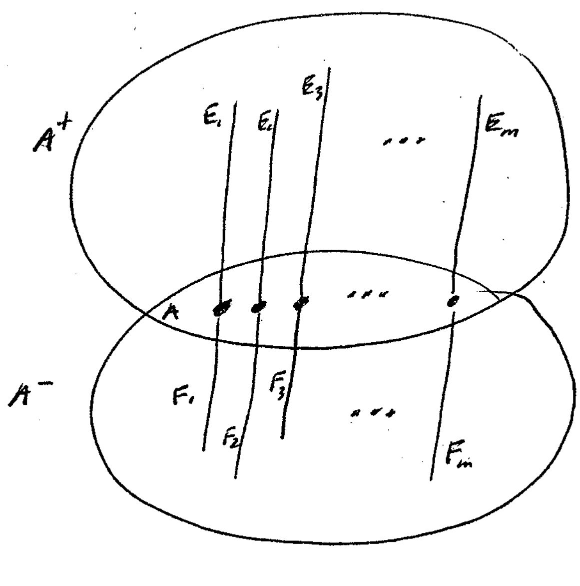
Each E_i contains unique element of A

Also by induction A^- has chain partition $\{F_i\}_{i=1}^m$

Each F_i contains unique element of A

Relabeling the F_i if necessary

E_i, F_i contain same el of A for $1 \le i \le m$



$X = A^+ \cup A^-$

For $1 \le i \le m$ glue E_i, F_i together to get single chain.

This gives m chains that partition X ✓

Case II   the only antichains of X that have size m, consist of the min els of X or the max els of X

let $x =$ min el of X

let $y =$ max'l el of X with $x \le y$ (poss $y = x$)

Consider poset

$$X \setminus \{x, y\}$$

For this poset
$m-1 =$ size of max'l antichain

By induction this poset has partition into $m-1$ chains.

These $m-1$ chains, together with the chain $x \le y$, gives a partition of X into m chains. ✓ $\square$

Example integer $n \geq 1$

X = set of all subsets of $\{1, 2, \ldots, n\}$

$x \leq y$ whenever $x \subseteq y$

For $0 \leq i \leq n$ def

$A_i = \{ x \in X \mid |x| = i \}$ Antichain

$|A_i| = \binom{n}{i}$

By Sperner thm

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

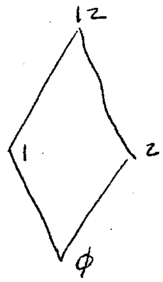
is max'l size of any antichain

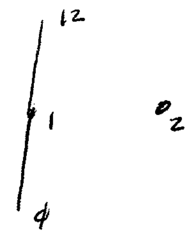
$A_{\lfloor \frac{n}{2} \rfloor}$ is max'l antichain

We now partition X into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains start with $n=2$

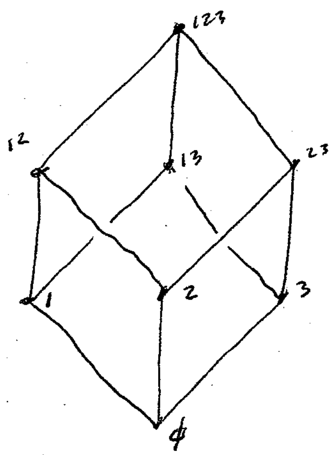
chain decomp

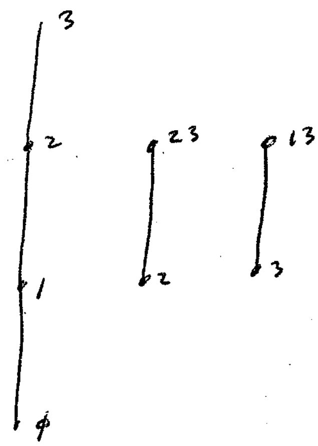
X:





$n=3$





$n=4$ Use previous chain decomp as follows:

Given a chain

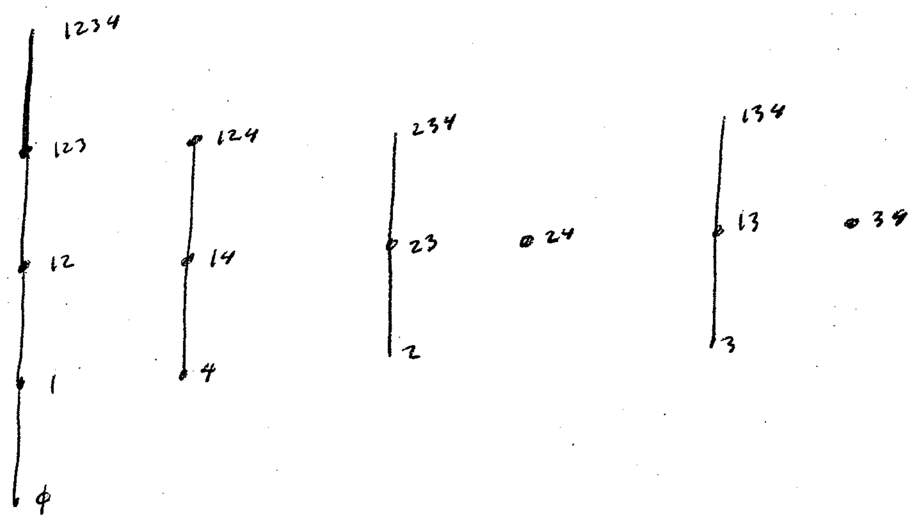
$$X_1 < X_2 < \cdots < X_\ell$$

in previous chain decomp

Create 2 chains if $\ell \geq 2$ and 1 chain if $\ell = 1$:

chain 1: $X_1 < X_2 < \cdots < X_\ell < X_\ell \cup \{4\}$

chain 2 (if $\ell \geq 2$): $X_1 \cup \{4\} < X_2 \cup \{4\} < \cdots < X_{\ell-1} \cup \{4\}$



For $n \geq 5$ chain decomp is obtained from $(n-1)$-chain decomp in sim way.

We have now constructed a chain

decomp $\{C_i\}_{i=1}^{r}$ of X

show $\quad r = \binom{n}{\lfloor \frac{n}{2} \rfloor}$

Given chain C_i

Write

C_i : $\quad x_1 < x_2 < \cdots < x_\ell$

By construction $\quad |x_1| + |x_\ell| = n \quad$ "symmetric"

So

C_i contains unique element of $A_{\lfloor \frac{n}{2} \rfloor}$

So

$$r = |A_{\lfloor \frac{n}{2} \rfloor}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

— o —