Lecture 9 Monday Sept 24 9/24/12

Ch 4 Generating Permutations and Combinations

4.1 Generating permutations

Prob: List all the permutations of $\{1,2,\dots,n\}$

ex $n=3$

123
132
213
231
312
321

For large n, hard to keep track!

We now present a listing method with nice properties

$n=2$ 12

21

$n=3$ In the $n=2$ List, insert 3 in all possible ways as follows:

123

132

312

321

231

213

$n=4$

$$
\begin{array}{ccccccc}
1234 & \rightarrow & 1243 & \rightarrow & 1423 & \rightarrow & 4123 \\
 & & & & & & \downarrow \\
1324 & \leftarrow & 1342 & \leftarrow & 1432 & \leftarrow & 4132 \\
\downarrow & & & & & & \\
3124 & \rightarrow & 3142 & \rightarrow & 3412 & \rightarrow & 4312 \\
 & & & & & & \downarrow \\
3214 & \leftarrow & 3241 & \leftarrow & 3421 & \leftarrow & 4321 \\
\downarrow & & & & & & \\
2314 & \rightarrow & 2341 & \rightarrow & 2431 & \rightarrow & 4231 \\
 & & & & & & \downarrow \\
2134 & \leftarrow & 2143 & \leftarrow & 2413 & \leftarrow & 4213
\end{array}
$$

$n \geq 5$ similar

For $n \geq 2$ the list of permutations of $\{1, 2, \ldots, n\}$ is obtained from the list of permutations for $\{1, 2, \ldots, n-1\}$ by inserting "n" in all possible ways, as shown above.

Nice properties

Given a permutation $a_1 a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$

A <u>transposition</u> of $a_1 a_2 \cdots a_n$ switches adjacent terms

$$\begin{array}{ccccccccc} a_1 & a_2 & \cdots & a_i & a_{i+1} & \cdots & a_n \\ \downarrow & \downarrow & \cdots & \downarrow & \times & \downarrow\cdots & \downarrow \\ a_1 & a_2 & \cdots & a_{i+1} & a_i & \cdots & a_n \end{array}$$

"ith transposition"

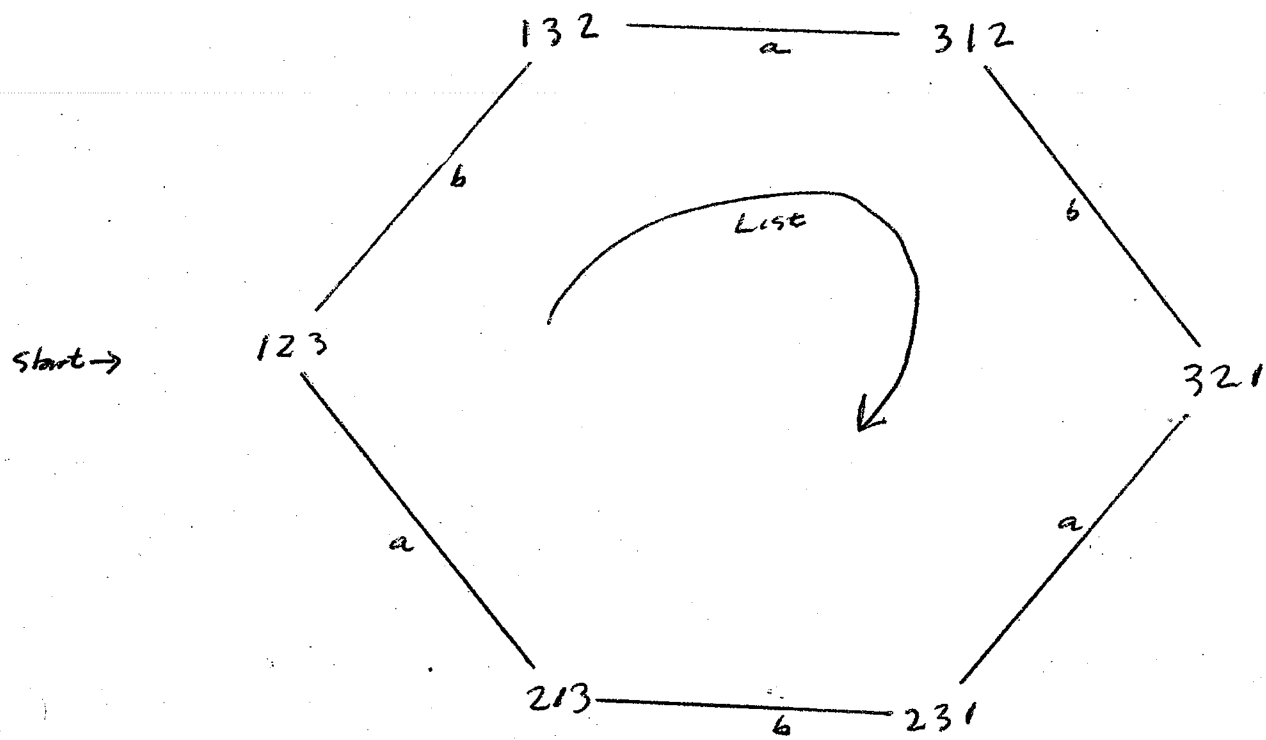
$a_1 a_2 \cdots a_n$ has $n-1$ transpositions

ex $n=4$

$$\begin{tikzcd} & 1423 \arrow[dl, "1st\ trans"'] \arrow[d, "2nd\ trans"] \arrow[dr, "3d\ trans"] & \\ 4123 & 1243 & 1432 \end{tikzcd}$$

* In our list of perms of $\{1, 2, \ldots, n\}$ each perm in list is a transposition of the preceding perm. Also, the first perm is a transposition of last perm.

ex $n=3$



a = 1st transposition

b = 2nd transposition

n=4

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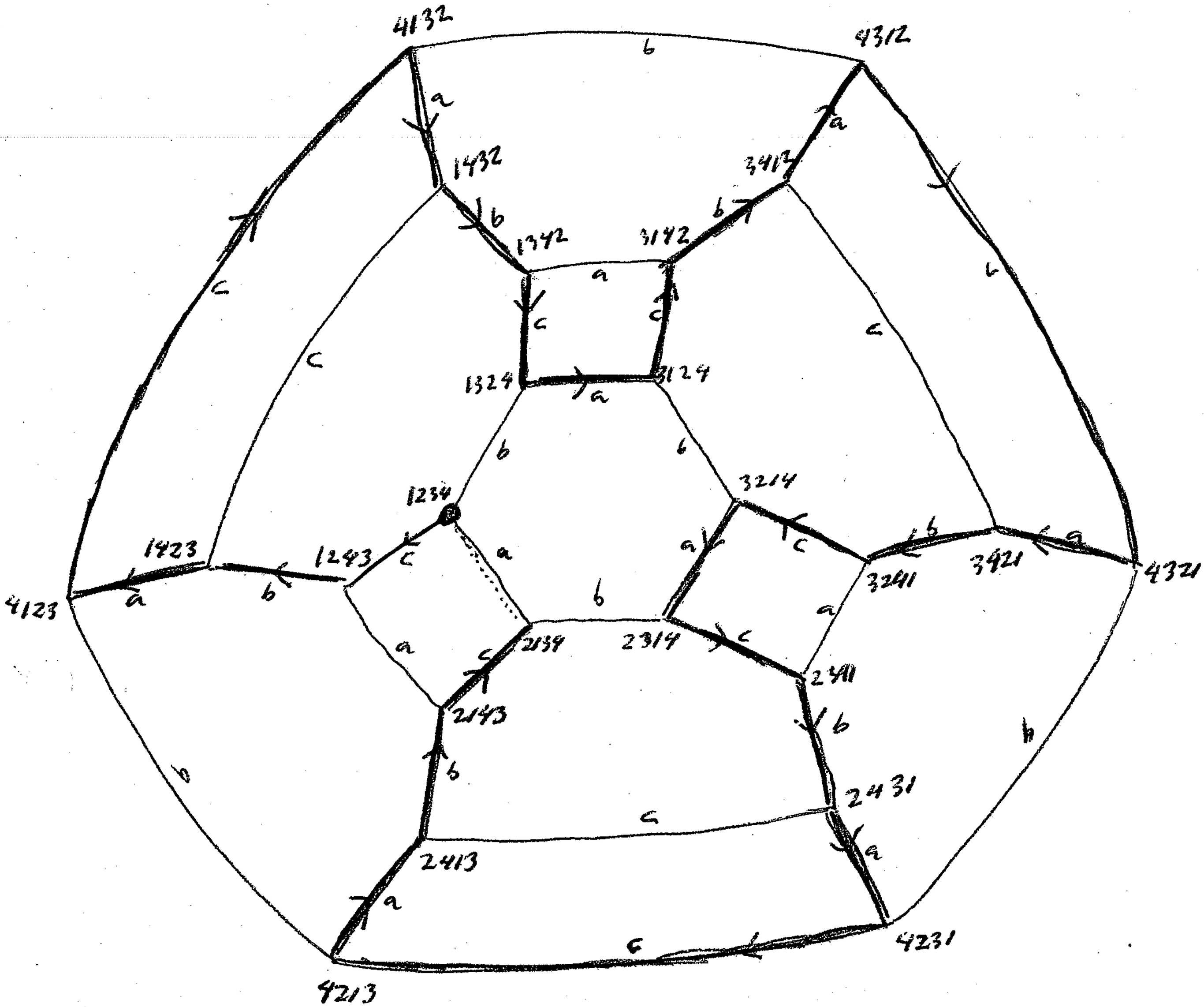

Listing order
follows path

a = 1st transposition
b = 2nd "
c = 3d "

View as polytope in
3 dimensions

<u>Problem</u> Given perm $a_1 a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$ what is <u>next</u> perm in our list?

- Appears difficult to answer without reproducing entire list up to $a_1 a_2 \cdots a_n$
- To remove this difficulty we add information as follows.
- For $1 \le k \le n$ assign a direction to a_k

$$\overrightarrow{a_k} \quad \text{or} \quad \overleftarrow{a_k}$$

- For $1 \le k \le n$ a_k called <u>mobile</u> if its arrow points to an adjacent smaller member

ex $n = 5$

$$\overrightarrow{4}\ \overleftarrow{5}\ \overleftarrow{1}\ \overleftarrow{3}\ \overleftarrow{2}$$

mobile

the following algorithm generates all the permutations of $\{1, 2, \ldots, n\}$ in the order that we discussed earlier

- start with $\overleftarrow{1}\ \overleftarrow{2} \cdots \overleftarrow{n}$

while there exists a mobile integer do:

- Find the Largest mobile integer m
- switch m with the adjacent integer it points to
- change the direction of each integer greater than m

ex $n=3$

$$\overleftarrow{1}\ \overleftarrow{2}\ \overleftarrow{3}$$

$$\overleftarrow{1}\ \overleftarrow{3}\ \overleftarrow{2}$$

$$\overleftarrow{3}\ \overleftarrow{1}\ \overleftarrow{2}$$

$$\overrightarrow{3}\ \overleftarrow{2}\ \overleftarrow{1}$$

$$\overleftarrow{2}\ \overrightarrow{3}\ \overleftarrow{1}$$

$$\overleftarrow{2}\ \overleftarrow{1}\ \overrightarrow{3}$$

4.2 Inversions of Permutations

For an integer $n \geq 1$

there are $n!$ perms of $\{1,2,\ldots,n\}$

Consider Cartesian product

$$\{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-2\} \times \cdots \times \{0,1,2\} \times \{0,1\} \times \{0\} \qquad (*)$$

The set $(*)$ has $n!$ elements.

Next goal: display a bijection between $(*)$ and the set of all perms of $\{1,2,\ldots,n\}$

— o —

Given a perm $a_1 a_2 \cdots a_n$ of $\{1,2,\ldots,n\}$

an <u>inversion</u> of this perm is an ordered pair (a_k, a_l)

such that

$$k < l \quad \text{and} \quad a_k > a_l$$

"a_k and a_l are out of order"

ex $n=4$

perm: 4132

inversions: $(4,1)$ $(4,3)$ $(4,2)$ $(3,2)$

Given a permutation $a_1a_2\cdots a_n$ of $\{1,2,\dots,n\}$

For $1 \le i \le n$ let

b_i = # of inversions that have i as 2nd coord

= # of elements among $a_1, a_2, \dots, a_n$ that are larger than i and appear to left of i

The sequence $(b_1, b_2, \dots, b_n)$ is called the <u>inversion sequence</u> for $a_1a_2\cdots a_n$.

Note $b_1+b_2+\cdots+b_n$ = total number of inversions for $a_1a_2\cdots a_n$

ex $n=4$ perm 4132

i	1	2	3	4
b_i	1	2	1	0

$1+2+1+0 = 4$ = # inversions for 4132

LEM Given a perm of $\{1, 2, \ldots, n\}$

with inversion sequence $(b_1, b_2, \ldots, b_n)$

then

$$0 \le b_1 \le n-1$$

$$0 \le b_2 \le n-2$$

$$0 \le b_3 \le n-3$$

$$\vdots$$

$$0 \le b_{n-1} \le 1$$

$$0 = b_n$$

pf For $1 \le i \le n$ show

$$0 \le b_i \le n-i.$$

Call the perm $a_1 a_2 \ldots a_n$

b_i = # elements among $a_1, a_2, \ldots, a_n$ that are larger than i and to left of i

$\le$ # elements among $a_1, a_2, \ldots, a_n$ that are larger than i

$\left[\text{choices are } \underbrace{i+1, i+2, \ldots, n-1, n}_{n-i \text{ poss}} \right]$

$\le n-i$

$\square$

We can recover a perm of $\{1,2,\dots,n\}$ from its inversion sequence as follows.

ex $n=6$

i	1	2	3	4	5	6
b_i	4	0	2	1	1	0

Find orig perm $a_1 a_2 a_3 a_4 a_5 a_6$

Method 1

6

6 5

6 4 5

6 4 3 5

2 6 4 3 5

2 6 4 3 1 5

Method 2

_ _ _ _ 1 _

2 _ _ _ 1 _

2 _ _ 3 1 _

2 _ 4 3 1 _

2 _ 4 3 1 5

2 6 4 3 1 5

In summary we have

Thm: Given an integer $n \geq 1$

the function which sends a permutation of $\{1,2,\dots,n\}$ to its inversion sequence is a bijection from the set of all perms of $\{1,2,\dots,n\}$ to the set

$$\{0,1,\dots,n-1\} \times \{0,1,\dots,n-2\} \times \cdots \times \{0,1,2\} \times \{0,1\} \times \{0\}. \qquad (*)$$

pf Cardinality of set $(*)$ is $n!$

$= \# \text{ perms of } \{1,2,\dots,n\}$

So suffices to show function is 1-1

function is 1-1 since each perm of $\{1,2,\dots,n\}$ is determined by its inversion sequence. $\square$

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Given a perm $a_1 a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$

with inversion sequence $(b_1, b_2, \ldots, b_n)$

Recall

$b_1 + b_2 + \cdots + b_n$ = total # of inversions

"inversion number"

"length"

• the inversion number

is equal to the minimum number of transpositions required

to bring $a_1 a_2 \cdots a_n$ to $123 \cdots n$.

Reason: Let ℓ = inversion number of $a_1 a_2 \cdots a_n$

At most ℓ transpositions required:

use b_1 transpositions to move "1" to position 1

$1 * * \cdots *$

use b_2 " "2" to position 2

$12 * * \cdots *$

"

use b_n " "n" to position n

$12 \cdots n$

At least ℓ transpositions required: each transposition changes the inversion number by 1 (± 1)

□

ex $n=6$ perm 624135

inversion sequence

i	1	2	3	4	5	6
b_i	3	1	2	1	1	0

inversion number $= 3+1+2+1+1+0$

$= 8$

Bring 624135 to 123456 using 8 transpositions:

perm	to do	# transpositions used
624135	move "1"	
621435		3
612435		
162435	move "2"	1
126435		
126345	move "3"	2
123645		
123465	move "4"	1
123456	move "5"	1
	move "6"	0

4.3 Generating combinations

Problem. Given a set S with $|S|=n$

List all the subsets of S

Ex $n=3$ $S=\{1,2,3\}$

One List is

ϕ $\{1\}$ $\{2\}$ $\{3\}$ $\{12\}$ $\{13\}$ $\{23\}$ $\{123\}$

For large n gets complicated

We now present a Listing method with nice properties

From now on take

$$S=\{x_{n-1}, x_{n-2}, \ldots, x_2, x_1, x_0\}$$

Step 1

We identify each subset Ω of S with a sequence $a_{n-1}a_{n-2}\cdots a_1 a_0$ of zeros and ones:

For $0\le i\le n-1$

$$a_i=\begin{cases}1 & \text{if } x_i\in\Omega\\ 0 & \text{if } x_i\notin\Omega\end{cases}$$

ex $n = 6$

$$S = \{ x_5, x_4, x_3, x_2, x_1, x_0 \}$$

Take

$$\Omega = \{ x_4, x_2, x_1 \}$$

x_5	x_4	x_3	x_2	x_1	x_0
0	1	0	1	1	0

__Step 2__ View each sequence of zeros and ones as the "base 2" representation of a nonnegative integer

ex the sequence 010110 corresponds to the integer

$$0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$$

$$= 16 + 4 + 2$$

$$= 22$$

ex Find the "base 2" representation of 57

Sol

Recall

i	5	4	3	2	1	0
2^i	32	16	8	4	2	1

take out 32

$$\begin{array}{r} 57 \\ -\ 32 \\ \hline 25 \end{array}$$

take out 16

$$\begin{array}{r} 25 \\ -\ 16 \\ \hline 9 \end{array}$$

take out 8

$$\begin{array}{r} 9 \\ -\ 8 \\ \hline 1 \end{array}$$

answer = 111001

Step 3

the List

$0, 1, 2, 3, \ldots, 2^n - 1$

of integers, each expressed in base 2, effectively gives all the subsets of an n-element set "squashed order"

ex $n=3$

m	m in base 2	corresp subset of $\{x_2, x_1, x_0\}$
0	000	ϕ
1	001	$\{x_0\}$
2	010	$\{x_1\}$
3	011	$\{x_1, x_0\}$
4	100	$\{x_2\}$
5	101	$\{x_2, x_0\}$
6	110	$\{x_2, x_1\}$
7	111	$\{x_2, x_1, x_0\}$

squashed order

ex For $n=7$

Consider subset

$$\{x_5, x_4, x_2, x_1, x_0\}$$

of

$$S = \{x_6, x_5, x_4, x_3, x_2, x_1, x_0\}$$

Find the next subset in the squashed order.

Sol. Convert to base 2:

x_6	x_5	x_4	x_3	x_2	x_1	x_0
0	1	1	0	1	1	1

add 1:

$$\begin{array}{r} 0110111 \\ +\quad 1 \\ \hline 0111000 \end{array}$$

Convert back:

$$\{x_5, x_4, x_3\}$$

$\square$

We just showed how to list the subsets of an n-element set.

We now give an alt approach using <u>Gray codes</u>

<u>Problem</u> For $n \geq 1$ list all the n-tuples of zeros and ones such that each n-tuple differs from previous one in exactly one coordinate

Such a list is called a <u>Gray code of order n</u>

ex $n = 3$

000
001
011
010
110
111
101
100

Def. A Gray code is <u>cyclic</u> whenever 1st term and last term differ in exactly one coord

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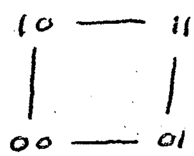
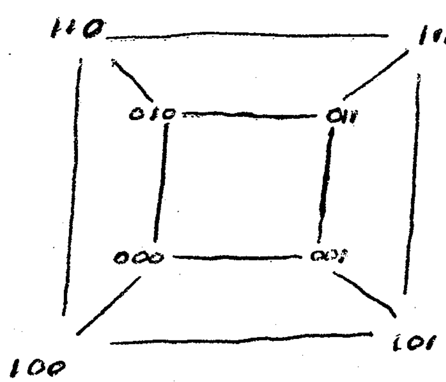
Geometric interpretation of Gray codes

For an integer $n \geq 1$ we define a graph called the n-cube

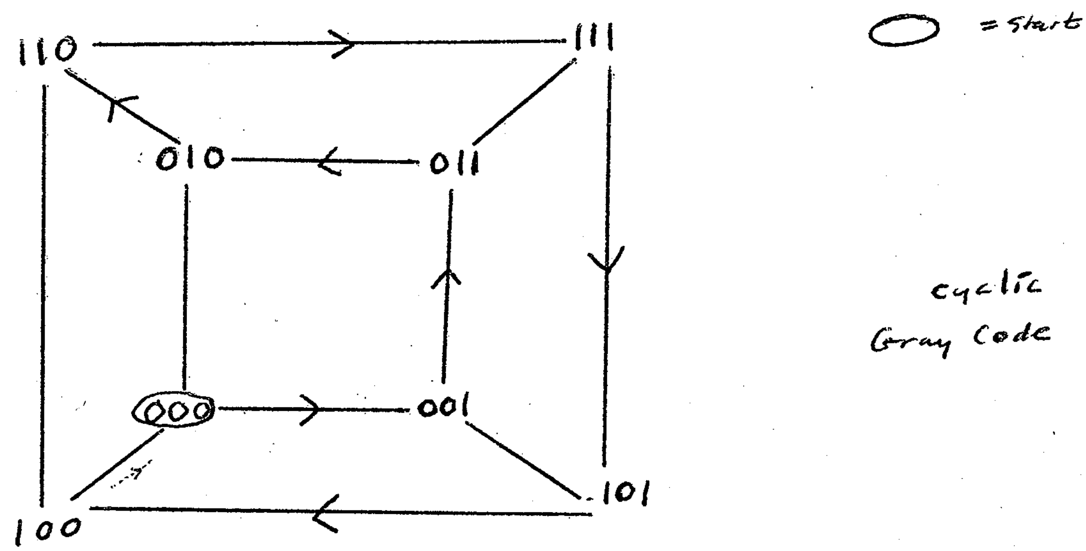
- The vertex set X consists of all n-tuples of 0's and 1's. So $|X| = 2^n$
- Two vertices declared adjacent whenever they differ in exactly one coordinate

n	n-cube
1	0 — 1
2	
3	
⋮	⋮

Observe • A Gray Code of order n is a path through the n-cube that visits each vertex exactly once.

• The Gray Code is cyclic whenever the last vertex is adjacent to the first vertex.

ex $n=3$



cyclic Gray Code

We now describe a special cyclic Gray code called the

Reflected Gray Code

ex display the reflected Gray codes of order 4

coords

3	2	1	0	
0	0	0	0	
0	0	0	1	reflect
0	0	1	1	
0	0	1	0	reflect
0	1	1	0	
0	1	1	1	
0	1	0	1	
0	1	0	0	reflect
1	1	0	0	
1	1	0	1	
1	1	1	1	
1	1	1	0	
1	0	1	0	
1	0	1	1	
1	0	0	1	
1	0	0	0	

Algorithm to obtain the reflected Gray code of order n:

- Reflected Gray code of order $n=1$ is $\begin{matrix} 0 \\ 1 \end{matrix}$

- For $n \geq 2$ the reflected Gray code of order n is obtained from the reflected Gray code of order $n-1$ by creating 2 copies, with 2nd copy in inverted order.

  For each term in Copy 1 add a leading 0

  For each term in Copy 2 add a leading 1

— o —

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We continue to discuss the reflected Gray codes

Problem Given a term in the reflected Gray code of order n:

$$g_{n-1} \cdots g_2 g_1 g_0$$

(i) what is next term?
(ii) what is preceding term?

If we list the codewords for small values of n the following pattern emerges.
we just need to specify which coord to change:
For (i), (ii)

(i) For $\sum_{i=0}^{n-1} g_i$ even change g_0

For $\sum_{i=0}^{n-1} g_i$ odd change g_s for the unique integer $s \geq 1$ such that $g_{s-1} = 1$ and each of $g_0, g_1, \ldots, g_{s-2}$ is 0

$$\ldots ** \boxed{}\, 100 \cdots 0$$

↑ change

(ii) For $\sum_{i=0}^{n-1} g_i$ odd change g_0

For $\sum_{i=0}^{n-1} g_i$ even change g_s for the unique integer $s \geq 1$ such that $g_{s-1} = 1$ and each of $g_0, g_1, \dots, g_{s-2}$ is 0

Ex Consider reflected Gray code of order 8

For the term

i	7	6	5	4	3	2	1	0
g_i	1	0	1	0	0	1	1	0

(i) Find next term

(ii) Find preceding term

Note $\sum g_i = 4$ is even

(i) change g_0:

ans = 10100111

(ii) Find s such that $g_s g_{s-1} \cdots g_1 g_0 = \boxed{}\,10\cdots00$

$s=2$: $\boxed{1}\,1\,0$

↑ change

ans = 10100010

∎

4.4 Generating r-subsets

<u>Problem</u> Given a finite set S.

Given integer r $1 \le r \le |S|$

List all the r-subsets of S.

Sol 1: List all the subsets of S using the method of Section 4.2.

Discard each term that does not have cardinality r.

Sol 2: Use <u>Lexicographical</u> <u>order</u> as described below.

Ex $|S| = 26$ $r = 3$

View S as the letters of the alphabet

$S = \{a, b, c, \ldots z\}$

View each 3-subset of S as a word of length 3

and list them in alphabetical order:

abc	⋮	Lex order
abd	ayz	
abe	bcd	
⋮	⋮	
abz	byz	
acd	cde	
ace	⋮	
⋮	~~x~~yz	
acz		
ade		
⋮		

Ex In Lex order, list all the 3-subsets of $\{1,2,3,4,5\}$

Sol Consider alphabet with "letters"

$$1 < 2 < 3 < 4 < 5$$

In alphabetical order List all words of form

xyz $\quad 1 \le x < y < z \le 5$

123
124
125
134
135
145
234
235
245
345

$\downarrow$ Lex order

$\square$

Ex Consider the 5-subsets of

$\{1,2,3,4,5,6,7,8,9\}$

in Lex order.

What is position of

$1\,3\,4\,\cancel{7}\,8$ (*)

ie if (*) is the mth term, what is m?

[start counting with $m=1$]

Sol strategy: count the terms that come after (*).

	types of 5-subsets		# choices
No restr:	abcde	$1 \le a < b < c < d < e \le 9$	$\binom{9}{5}$
come after (*)	abcde	$2 \le a < b < c < d < e \le 9$	$\binom{8}{5}$
	1bcde	$4 \le b < c < d < e \le 9$	$\binom{6}{4}$
	13cde	$5 \le c < d < e \le 9$	$\binom{5}{3}$
	134de	$8 \le d < e \le 9$	$\binom{2}{2} = 1$
	134~~7~~e	$9 \le e \le 9$	$\binom{1}{1} = 1$

$$m = \binom{9}{5} - \binom{8}{5} - \binom{6}{4} - \binom{5}{3} - \binom{2}{2} - \binom{1}{1}$$

thm For $1 \le r \le n$ consider r-subsets of

$$\{1, 2, \dots, n\}$$

in lex order.

The pos of the r-subset

$$\{a_1, a_2, \dots, a_r\} \qquad 1 \le a_1 < a_2 < \dots < a_r \le n$$

is

$$\binom{n}{r} - \binom{n-a_1}{r} - \binom{n-a_2}{r-1} - \dots - \binom{n-a_{r-1}}{2} - \binom{n-a_r}{1}$$

$\square$

pf Same ideas as in prec example

Problem Given a finite set S

Given integer r $1 \le r \le |S|$

List all the r-permutations of S

Sol: First List the r-subsets of S in Lex order

For each subset list all the permutations of its elements using the method of Section 4.1

ex List all the 3-perms of $\{1, 2, 3, 4\}$.

Sol First List 3-subsets:

123
124
134
234

Expand: $\longrightarrow$

123	132	312	321	231	213
124	142	412	421	241	214
134	143	413	431	341	314
234	243	423	432	342	324

$\square$

Lecture 12 Monday Oct 1

4.5 Partial Orders and Equivalence Relations

Consider a set X

Consider the Cartesian product

$$X \times X = \{ (a,b) \mid a,b \in X \}$$

A subset

$$R \subseteq X \times X$$

is called a <u>relation</u> on X

For $a,b \in X$ write

aRb whenever $(a,b) \in R$ "a is related to b"

$a \not R b$ ... $(a,b) \notin R$ "a not related to b"

Ex 1 $X =$ set of integers

$\forall a,b \in X$

aRb whenever $a-b$ is even

Ex 2 $X =$ set of all subsets of $\{1,2,\ldots,n\}$

$\forall a,b \in X$

aRb whenever $a \subseteq b$

Ex 3 $X = \{1, 2, \ldots, 100\}$

$\forall a, b \in X$

aRb whenever $a < b$

We now list some important conditions on relations. Given a relation R on a set X,

Condition	Meaning	examples
Reflexive	xRx (for all $x \in X$)	1, 2
Irreflexive	$x \not R x$ (for all $x \in X$)	3
Symmetric	xRy implies yRx (for all $x, y \in X$)	1
Antisymmetric	For distinct $x, y \in X$, xRy implies $y \not R x$	2, 3
Transitive	xRy and yRz implies xRz (for all $x, y, z \in X$)	1, 2, 3

Def Given a relation R on a set X.

- Call R a <u>partial order</u> whenever R is reflexive, antisymmetric, transitive.

- Call R a <u>strict partial order</u> whenever R is Irreflexive, antisymmetric, transitive.

- The set X together with a partial order on X is called a <u>partially ordered set</u> (or <u>poset</u>)

- For $x, y \in X$, call x, y <u>comparable</u> whenever

  xRy or yRx

  Call x, y <u>incomparable</u> whenever

  $x \not R y$ and $y \not R x$

- Call R a <u>total order</u> whenever R is a partial order and x, y are comparable for all $x, y \in X$

Ex Given integer $n \geq 1$

$X = \{1, 2, \dots, n\}$

Given permutation $a_1 a_2 \cdots a_n$ of X

Define a relation R on X by:

$\forall\ x, y \in X$

xRy whenever $x = y$ or x comes before y among $a_1 a_2 \dots a_n$

Then R is a total order (check)

Ex Given integer $n \geq 1$

$X = \{1, 2, \dots, n\}$

Given total order R on X

Call an element $x \in X$ minimal whenever there does not exist $y \in X$ s.t. $y \neq x$ and yRx

Observe

X has unique minimal element a_1

$X \setminus \{a_1\}$ $\cdots$ a_2

$X \setminus \{a_1, a_2\}$ $\cdots$ a_3

$\vdots$

Then $a_1 a_2 \cdots a_n$ is permutation of X.

The previous 2 examples show:

Thm Given integer $n \geq 1$
$X = \{1, 2, \ldots, n\}$.

There is a 1–1 correspondance between the total orders on X and the permutations of X.

— o —

A generic partial order on a set X is usually denoted $\leq$.

For $x, y \in X$ write $x < y$ whenever $x \neq y$ and $x \leq y$.

Def Given a set X with partial order $\leq$

For $x, y \in X$ we say <u>y covers x</u> whenever

$x < y$ and there does not exist $z \in X$ such that

$$x < z < y$$

In this case write

$$x <_c y$$

Ex Take $X = \{1, 2, 3, \ldots, 10\}$

Define a partial order $\leq$ on X by

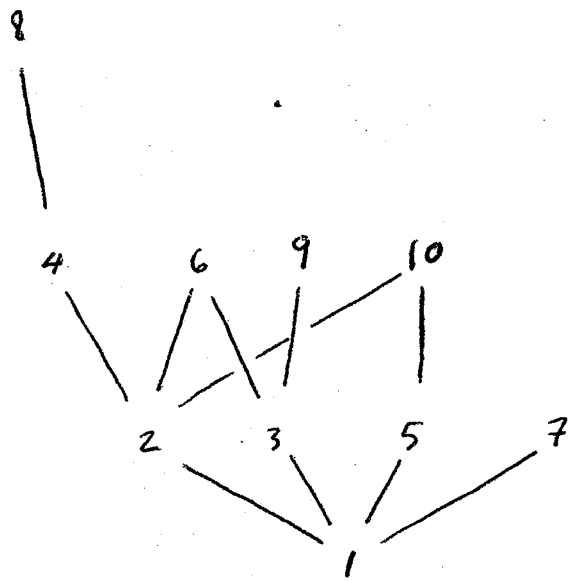
$x \leq y$ whenever $x \mid y$ (x divides y)

Describe the cover relation $<_c$

Sol. For $x, y \in X$ draw

$$\begin{tikzcd} y \arrow[d, dash] \\ x \end{tikzcd}$$

whenever $x <_c y$



"Hasse diagram"

Given a set S

Let $\mathcal{P}(S)$ denote the set of all subsets of S

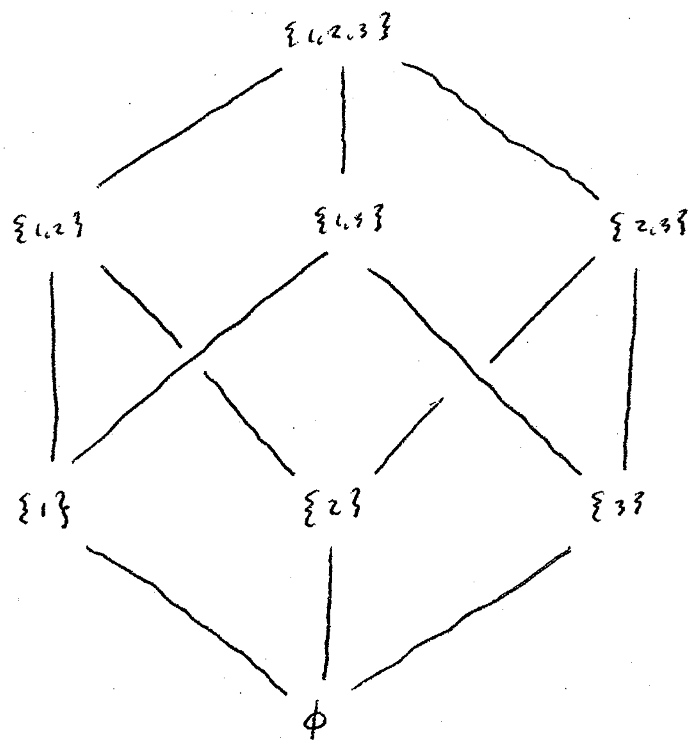
So $|\mathcal{P}(S)| = 2^n \quad n = |S|$

Obs The containment relation $\subseteq$ is a partial order on $\mathcal{P}(S)$

Ex Take $S = \{1,2,3\}$

$X = \mathcal{P}(S)$

Describe the Hasse diagram for the poset $X, \subseteq$



It's the 3-cube!

Def Given set X

Given 2 partial orders on X:

$\leq_1$ $\leq_2$

$\leq_2$ is called an <u>extension</u> of $\leq_1$ whenever

$x \leq_1 y$ implies $x \leq_2 y$ $\forall x, y \in X$

Def Given a partial order $\leq$ on a set X

A <u>linear extension</u> of $\leq$ is an extension of $\leq$

that is a total order

Thm Let $\leq$ denote a partial order on a finite set X.

Then $\leq$ has at least one linear extension.

pf Let $n = |X|$

Let x_1 denote a minimal element of X

let x_2 ... $X \setminus \{x_1\}$

Let x_3 ... $X \setminus \{x_1, x_2\}$

$\vdots$

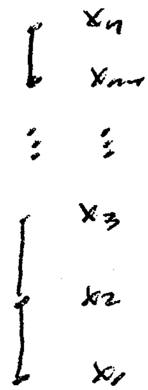
This gives ordering $x_1, x_2, \ldots, x_n$ of elements of X

Define a total order on X as follows.

In this total order

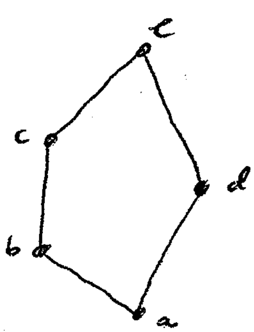
x_i covers x_{i-1} for $2 \le i \le n$

So Hasse diagram is



One checks the total order is a linear ext of $\le$ $\square$

Ex Consider poset with Hasse diagram



Find all the linear extensions.

Sol $a < b < c < d < e$

$a < b < d < c < e$

$a < d < b < c < e$

$\square$

We now discuss equivalence relations

Given set X

A relation R on X is an <u>equivalence</u> <u>relation</u>

whenever R is

reflexive, symmetric, transitive

Ex $S = \{1, 2, \ldots n\}$

$X = \mathcal{P}(S)$

Define a relation R on X by

$x R y$ whenever $|x| = |y|$

Then R is an equiv relation

— o —

Ex Given a set X

Partition X into nonempty subsets

$$X = X_1 \cup X_2 \cup \cdots \cup X_r \qquad \text{(disjoint union)}$$

Define a relation R on X as follows:

For $x, y \in X$

xRy whenever x, y are in the same part of the partition

Then R is an equiv. relation (check)

Ex Given an equiv relation R on a set X

For $x \in X$ define

$$[x] = \{ y \in X \mid xRy \} \qquad \text{"the equivalence class of } x\text{"}$$

Obs $\forall x, y \in X$

$$[x] = [y] \quad \text{if } xRy$$

or

$$[x] \cap [y] = \phi \quad \text{if } x \not R y$$

Moreover the set of distinct equiv classes give a partition of X into non empty subsets.

— o —

A generic equivalence relation is usually denoted by $\sim$