Ch 3 The pigeonhole principle (PP)

3.1 PP: simple form

thm (PP) For an integer $n \geq 1$, if we put $n+1$ objects in n boxes, then some box gets at least 2 objects.

Ex. Among 13 people there are 2 that have birthdays in same month.

Ex Given $2n$ people consisting of n married couples. We pick $n+1$ people. Then among them are a married couple.

To motivate the next application of PP we make some observations.

- For $a, b \in \mathbb{Z}$ at least one of

$$a, \quad b, \quad a+b$$

is even

- For $a, b, c \in \mathbb{Z}$ at least one of

$$a, \ b, \ c, \ a+b, \ b+c, \ a+b+c$$

is divisible by 3

- For $a, b, c, d \in \mathbb{Z}$ at least one of

$$\begin{array}{cccc} a & b & c & d \\ a+b & b+c & c+d & \\ a+b+c & b+c+d & & \\ a+b+c+d & & & \end{array}$$

is divisible by 4

In general we have the following.

LEM Given integers

$$a_1, a_2, \dots, a_m \qquad (m \geq 1)$$

at least one of the sums

$$a_{i+1} + \cdots + a_{j-1} + a_j \qquad 0 \leq i < j \leq m$$

is divisible by m.

pf For $x, y \in \mathbb{Z}$ write

$$x \equiv y \ (m) \qquad \text{"} x \text{ equals } y \text{ modulo } m \text{"}$$

whenever m divides $x-y$.

obs: $\forall x \in \mathbb{Z}$ $\exists$ unique $r \in \{0, 1, 2, \dots, m-1\}$

such that

$$x \equiv r \ (m)$$

Indeed r is remainder when we divide x by m.

Consider $m+1$ sums

$$0$$
$$a_1$$
$$a_1 + a_2$$
$$a_1 + a_2 + a_3$$
$$\vdots$$
$$a_1 + a_2 + \cdots + a_m$$

Each sum is equal to exactly one of $\{0, 1, 2, \ldots, m-1\}$

mod m.

By PP $\exists$ i, j $(0 \leq i < j \leq m)$ such that

$$a_1 + a_2 + \cdots + a_i \equiv a_1 + a_2 + \cdots + a_j \quad (m)$$

Now

$$0 \equiv \underbrace{a_1 + a_2 + \cdots + a_j - (a_1 + a_2 + \cdots + a_i)}_{\substack{\shortparallel \\ a_{i+1} + \cdots + a_{j-1} + a_j}} \quad (m)$$

So

m divides $a_{i+1} + \cdots + a_{j-1} + a_j$ $\square$

LEM A student has 37 days to prepare for an exam.

- On each day 1, 2, ..., 37 she studies a certain number of hours (varies with day).
- She requires at most 60 study hours in total.
- She studies at least 1 hour each day.

Then $\exists$ a sequence of consecutive days during which she studies exactly 13 hours.

Pf For $1 \le i \le 37$ let

b_i = # hours she studies on day i

So $b_i \geq 1$

Consider the numbers

$$\{ b_1 + b_2 + \cdots + b_i + 13 \}_{i=0}^{36} \cup \{ b_1 + b_2 + \cdots + b_j \}_{j=1}^{37} \qquad (*)$$

We have $37 \times 2 = 74$ numbers

All among

$$1, 2, \ldots, 72$$

By PP numbers $(*)$ are not distinct

So $\exists$ i, j $(0 \leq i < j \leq 37)$ such that

$$b_1 + b_2 + \cdots + b_i + 13 = b_1 + b_2 + \cdots + b_j$$

Now

$$13 = b_{i+1} + \cdots + b_j$$

During the days $i+1, \ldots, j$ she studies exactly 13 hours $\square$

Def

Given sets X, Y

Given function $f: X \to Y$

(i) f is called <u>1-1</u> whenever:

$\forall$ distinct $x, x' \in X$

$f(x) \neq f(x')$

(ii) f is called <u>onto</u> whenever:

$\forall y \in Y \; \exists x \in X$ such that

$f(x) = y$

(iii) f is a <u>bijection</u> whenever f is 1-1 and onto.

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In terms of functions PP is saying the following:

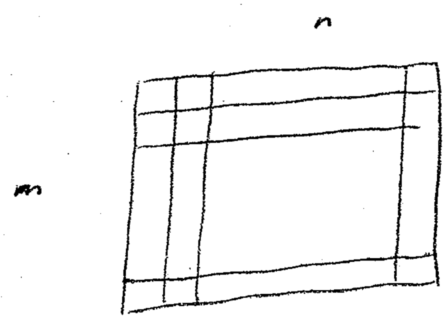
- (PP V2) Given finite sets X, Y
  Given a function $f: X \to Y$
  If $|X| > |Y|$ then f is not 1-1.

Related statements:

- Assume $|X| = |Y|$ If f is onto then f is 1-1
- Assume $|X| = |Y|$ If f is 1-1 then f is onto

Ex

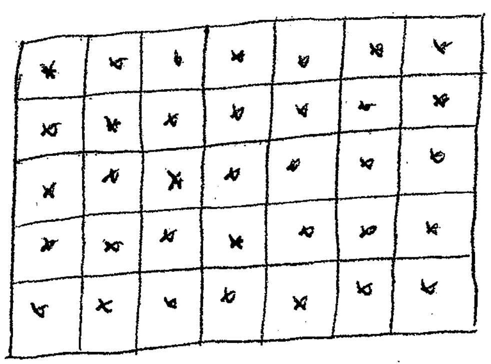
Given $m \times n$ chessboard



start at NW square and walk "SE"

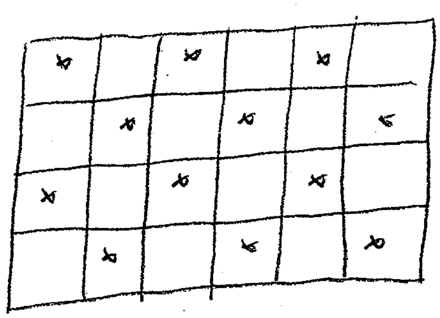
Will you eventually traverse every square?

Case $m = 5$ $n = 7$



yes

Case $m = 4$ $n = 6$



NO

Call m, n relatively prime whenever they have no prime factor in common.

5, 7 rel prime

4, 6 not rel prime since 2 divides both

The Chinese remainder theorem says that you will traverse every square of the $m \times n$ chessbd whenever m, n are relatively prime.

We sharpen notation

Label the rows of the chessbd $0, 1, \ldots, m-1$

Label the cols of the chessbd $0, 1, \ldots, n-1$

For an integer $N \geq 1$ define the set

$$\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$$

View chessbd as Cartesian product

$$\mathbb{Z}_m \times \mathbb{Z}_n$$

Consider a function

$$f: \begin{array}{ccc} \mathbb{Z}_{mn} & \longrightarrow & \mathbb{Z}_m \times \mathbb{Z}_n \\ t & \longrightarrow & \text{Location on chessbd after } t \text{ steps of the walk} \end{array}$$

obs:

f is onto $\Leftrightarrow$ the walk will traverse each square of the chessbd

For $t \in \mathbb{Z}_{mn}$ describe $f(t)$

Write $f(t) = (a, b)$

Meaning:
after t steps, located in row a, col b of chessbd

obs:
a = remainder when divide t by m
b = remainder when divide t by n

We now formally state Chinese Remainder Thm.

thm (Chinese Remainder thm)

Given rel prime pos integers m, n

Given $a \in \mathbb{Z}_m$ and $b \in \mathbb{Z}_n$

then $\exists \; x \in \mathbb{Z}_{mn}$ such that

$$x \equiv a \; (m) \quad \text{and} \quad x \equiv b \; (n)$$

pf Consider our function

$$f: \begin{array}{ccc} \mathbb{Z}_{mn} & \longrightarrow & \mathbb{Z}_m \times \mathbb{Z}_n \\ t & \longrightarrow & \left(\begin{array}{c}\text{remainder when divide} \\ t \text{ by } m\end{array}, \begin{array}{c}\text{remainder when} \\ \text{divide } t \text{ by } n\end{array}\right) \end{array}$$

show f is onto.

Obs $|\mathbb{Z}_m \times \mathbb{Z}_n| = |\mathbb{Z}_m| \, |\mathbb{Z}_n|$

$$= mn$$

$$= |\mathbb{Z}_{mn}|$$

By PP suf to show f is 1-1

Suppose f is not 1-1.

then $\exists \quad x, x' \in \mathbb{Z}_{mn}$ $(x < x')$ such that

$$f(x) = f(x')$$

Call this common value (r, s)

So

$$x \equiv r \ (m) \qquad x \equiv s \ (n)$$

$$x' \equiv r \ (m) \qquad x' \equiv s \ (n)$$

So $$x' - x \equiv 0 \ (m) \qquad x' - x \equiv 0 \ (n)$$

So $$m \mid x' - x$$

$$n \mid x' - x$$

But m, n rel prime so

$$mn \mid x' - x$$

Now $$x' - x \geq mn$$

But $$0 \leq x < x' \leq mn - 1$$

So $$x' - x \leq mn - 1$$

Cont.

Therefore f is 1-1.
Result follows. $\square$

Here is another application of PP

LEM Fix integer $n \geq 1$

Pick a subset

$$S \subseteq \{1, 2, \ldots, 2n\}, \qquad |S| = n+1$$

Then $\exists$ distinct $i, j \in S$ such that

$$i \mid j \quad \text{or} \quad j \mid i$$

pf For an integer $x > 0$

factor x into primes:

$$x = 2^{n_1} \underbrace{3^{n_2} 5^{n_3} \cdots}_{\text{call this product } x_{odd}\text{, the } \underline{\text{odd part of } x}}$$

x_{odd} is the unique odd integer such that

$x = x_{odd}$ times a power of 2

Obs

$$1 \leq x_{odd} \leq x$$

x	1	2	3	4	5	6	7	8	9	10	...
x_{odd}	1	1	3	1	5	3	7	1	9	5	...

For integers $x, y > 0$ assume

$$x_{odd} = y_{odd}$$

then $\quad x / y$ or y / x

Reason: write

$$x = m 2^r$$
$$y = m 2^s$$

$(m = x_{odd} = y_{odd})$

If $r \le s$ then x / y

If $s \le r$ then y / x

For $1 \le i \le 2n$ $\quad i_{odd}$ is contained in

$$\{1, 3, 5, \dots, 2n-1\} \qquad (*)$$

Set $(*)$ has cardinality n

Recall $|S| = n+1$, so by PP

$\exists$ distinct $i, j \in S$ such that

$$i_{odd} = j_{odd}$$

Now $\quad i / j$ or j / i $\quad \square$

3.2 Pigeonhole principle: strong form

We consider more versions of PP.

thm Given n boxes

For $1 \le i \le n$ pick integer $q_i \ge 1$

We put

$$q_1 + q_2 + \cdots + q_n - n + 1$$

objects in the boxes. Then:

Box 1 gets at least q_1 objects

or Box 2 " q_2 "

or ...

Box n " q_n "

pf Suppose not. Then

Box 1 gets $\le q_1 - 1$ objects

and Box 2 " $\le q_2 - 1$ "

and ...

and Box n " $\le q_n - 1$ objects

Total # objects must be $\le$

$$(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) = q_1 + q_2 + \cdots + q_n - n$$
$$< q_1 + q_2 + \cdots + q_n - n + 1, \text{ Cont. } \square$$

Another version of PP

Thm Given integers

$$m_1, m_2, \ldots, m_n \qquad (1 \le n < \infty)$$

Given integer q

Assume that the average value of $m_1, m_2, \ldots, m_n$ is greater than $q-1$. Then $\exists\ i\ (1 \le i \le n)$ such that

$$m_i \ge q$$

pf Compute average:

$$\frac{m_1 + m_2 + \cdots + m_n}{n} > q-1 \qquad (*)$$

Suppose the theorem is false. Then

$$m_i < q \qquad (1 \le i \le n)$$

m_i, q are integers so

$$m_i \le q-1 \qquad (1 \le i \le n)$$

Now
$$\frac{m_1 + m_2 + \cdots + m_n}{n} \le \frac{q-1 + q-1 + \cdots + q-1}{n}$$
$$= \frac{n(q-1)}{n}$$
$$= q-1$$

Contradicts $*$. So theorem is true. $\square$

Given an integer $n \geq 1$

Consider a permutation $a_1 a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$

ex $n = 5$

2 5 4 1 3

Consider "decreasing subsequences" and "increasing subsequences"

r	decreasing subsequences length r					
3	541	543				
2	21	54	51	53	41	43
1	2	5	4	1	3	

r	increasing subsequences length r			
2	25	24	23	13
1	2	5	4	1 3

For a sequence $a_1 a_2 \ldots a_n$ by a <u>subsequence of length r</u> we mean $a_{i_1} a_{i_2} \cdots a_{i_r}$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$

this subsequence is <u>decreasing</u> if $a_{i_1} > a_{i_2} > \cdots > a_{i_r}$

" " <u>increasing</u> if $a_{i_1} < a_{i_2} < \cdots < a_{i_r}$

Thm Given integer $r \geq 0$

Let $n = r^2 + 1$

Then Each perm of $\{1, 2, \ldots, n\}$ has either

(i) at least one increasing subsequence of length $r+1$

or

(ii) at least one decreasing subsequence of length $r+1$

Pf. We assume (i) fails and obtain (ii)

Call the permutation $a_1 a_2 \cdots a_n$

For $1 \le i \le n$ define

l_i = max'l length of an increasing subsequence of $a_1 a_2 \cdots a_n$ that ends at a_i

ex $r=3$ $n=10$

a_i	3	10	6	2	5	9	1	8	4	7
l_i	1	2	2	1	2	3	1	3	2	3

By assumption

$$l_i \le r \qquad 1 \le i \le n$$

Note: for $1 \le i < j \le n$ such that $a_i < a_j$ we have

$$l_i < l_j$$

Since for any increasing subsequence of length l_i that ends at a_i we could append a_j to get an increasing subsequence of length l_i+1 that ends at a_j.

For $1 \le \ell \le r$ define

$$S_\ell = \{ i \mid 1 \le i \le n,\ a_i = \ell \}$$

By the note, the elements of S_ℓ form a <u>decreasing subsequence</u>

ex

ℓ	decreasing subsequence induced by S_ℓ
1	3 2 1
2	10 6 5 4
3	9 8 7

$S_1, S_2, \ldots, S_r$ partition $\{1, 2, \ldots, n\}$

So $$\sum_{\ell=1}^{r} |S_\ell| = n = r^2+1$$

By PP $\exists\, \ell$ $(1 \le \ell \le r)$ such that

$$|S_\ell| \ge r+1$$

Therefore $\exists$ decreasing subsequence of length $\ge r+1$. $\square$

The prev thm can be generalized as follows.

thm Given integer $r \geq 0$

let $n = r^2+1$

Given a sequence of real numbers

$a_1 a_2 \cdots a_n$

then $\exists$ subsequence

$a_{i_1} a_{i_2} \cdots a_{i_{r+1}}$

of length $r+1$ such that either

(i) $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_{r+1}}$ "weakly inc"

or

(ii) $a_{i_1} > a_{i_2} > \cdots > a_{i_{r+1}}$ decreasing

pf We assume (i) fails and get (ii)

For $1 \leq i \leq n$ define

l_i = max'l length of a weakly inc subsequence that ends at a_i

By assumption $l_i \leq r$ $1 \leq i \leq n$

For $1 \leq l \leq r$ define

$S_l = \{ i \mid 1 \leq i \leq n, \ l_i = l \}$

Elements in S_l form dec subsequence

$S_1, S_2, \ldots, S_r$ partition $\{1, 2, \ldots, n\}$

$$\sum_{l=1}^{r} |S_l| = n = r^2+1$$

By PP $\exists \, l$ $(1 \leq l \leq r)$ s.t. $|S_l| \geq r+1$

So $\exists$ dec subsequence of length $\geq r+1$ $\square$

3.3 Ramsey theory

We will be discussing the complete graph K_n

A graph is a pair (X, E) where

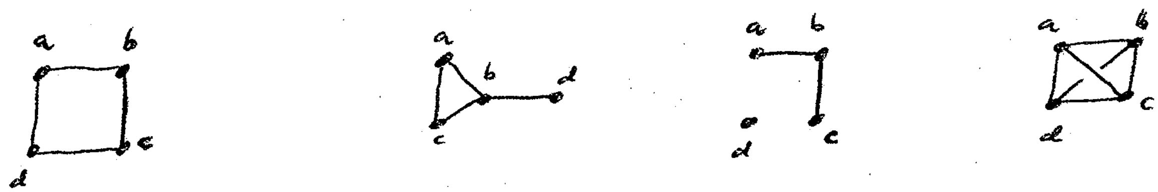
(i) X is a finite set (the vertices)

(ii) E is a set of 2-element subsets of X (the edges)

Vertices $x, y \in X$ are adjacent whenever $x \neq y$ and $xy \in E$.

When we draw a graph, adjacent vertices are connected by an arc.

Ex Some graphs with vertex set $X = \{a, b, c, d\}$:



$E = \{ab, bc, cd, da\}$

Graph on right is complete. Complete means any pair of distinct vertices are adjacent

Let K_n denote the complete graph on n vertices.

K_1 K_2 K_3 K_4 K_5 ...

K_n has $\binom{n}{2}$ edges

Def. Given a graph (X, E)

By a <u>subgraph</u> of (X, E) we mean a graph (X', E') where

$X' \subseteq X$

$E' = \{ xy \mid x, y \in X', \; xy \in E \}$

LEM For the complete graph K_6

Suppose we color each edge red or blue.

Then either:

(i) There exists a K_3 subgraph with all edges red "a red K_3"

or

(ii) There exists a K_3 subgraph with all edges blue "a blue K_3"

pf We assume (i) fails and show (ii)

Pick a vertex x of K_6

5 edges contain x

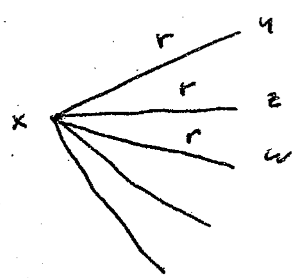
By Pigeonhole principle either

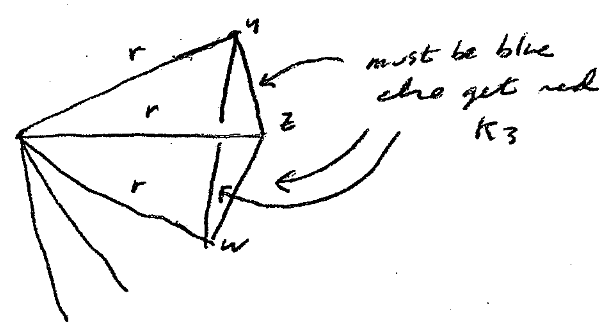
- at least 3 of these edges are red

or

- at least 3 of these edges are blue

WLOG at least 3 are red





Now the subgraph with vertices y, z, w is a blue K_3 $\square$

Def Given an integer $t \geq 2$

For $2 \leq m, n \leq t$

$$K_t \to K_m, K_n$$

means:

No matter how we color the edges of K_t red or blue, there exists a red K_m subgraph or a blue K_n subgraph

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Ref to above Def, obs

$$K_t \to K_m, K_n$$

implies

$$K_s \to K_m, K_n \qquad \text{for all integers } s \geq t$$

<u>Problem</u> Given integers $m, n \geq 2$

- Does $K_t \to K_m, K_n$ for suf large t?
- If so, find the minimal t such that $K_t \to K_m, K_n$.

This minimal t is called the <u>Ramsey number</u> $r(m,n)$

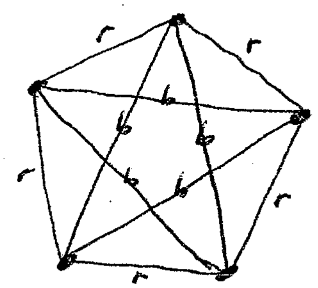
obs $r(m,n) = r(n,m)$

LEM $r(3,3) = 6$

pf We showed $K_6 \to K_3, K_3$

Also $K_5 \not\to K_3, K_3$

Reason:



$\square$

LEM $r(n,2) = n$ $n = 2, 3, \ldots$

pf Routine $\square$

(Ramsey thm)

Thm Given integers $m, n \geq 2$. Then

$$K_t \to K_m, K_n$$

for suf large t.

pf By induction on $m+n$

We may assume $m \geq 3$, $n \geq 3$; else we are done by prev lemma.

By induction

$r(m, n-1)$ exists

and

$r(n-1, m)$ exists.

Define

$$t = r(m, n-1) + r(m-1, n)$$

show

$$K_t \to K_m, K_n$$

Color each edge of K_t red or blue

Pick vertex x of K_t

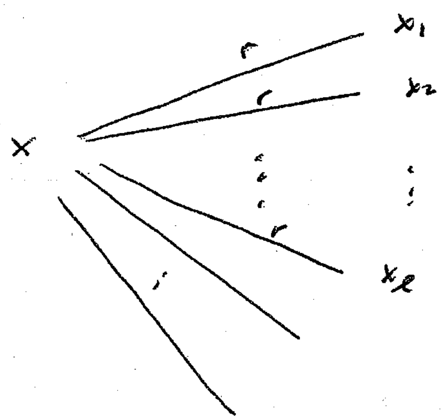
$t-1$ edges of K_t contain x

By PP either

(i) at least $r(m-1, n)$ of these edges are red

or (ii) at least $r(m, n-1)$ of these edges are blue

Assume Case (i): the other case similar.



$\ell = r(m-1, n)$

By def

$$K_\ell \to K_{m-1}, K_n$$

Consider K_ℓ subgraph with vertices $x_1, x_2, \dots, x_\ell$.

Either:

- it has a red K_{m-1} subgraph

  [which together with x forms a red K_m subgraph of K_t]

or

- it has a blue K_n subgraph

  [which is a blue K_n subgraph of K_t]

Therefore K_t has a red K_m subgraph or a blue K_n subgraph.

So $K_t \to K_m, K_n$ □

the above thm shows that

$$r(m,n) \text{ exists for all integers } m,n \geq 2.$$

thm For integers $m,n \geq 2$

$$r(m,n) \leq \binom{m+n-2}{m-1}$$

pf By induction on $m+n$

<u>Case $m=2$</u>

$$r(2,n) = n = \binom{n}{1}$$ ✓

<u>Case $n=2$</u>

$$r(m,2) = m = \binom{m}{m-1}$$ ✓

<u>Case $m \geq 3, n \geq 3$</u>

Define

$$t = r(m,n-1) + r(m-1,n)$$

By the proof of the prev thm

$$K_t \to K_m, K_n$$

So

$$r(m,n) \leq t$$

Now

$$
\begin{aligned}
r(m,n) &\le t \\
&= r(m,n-1) + r(m-1,n) \\
&\qquad\qquad\qquad\qquad \text{by ind} \\
&\le \binom{m+n-1-2}{m-1} + \binom{m-1+n-2}{m-1-1} \\
&= \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2} \\
&= \binom{m+n-2}{m-1} \qquad \square
\end{aligned}
$$

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Generalizations of Ramsey's thm.

Given an integer $n \geq 2$

Given integers $2 \leq n_1, n_2 \ldots n_\ell \leq n$

$K_n \longrightarrow K_{n_1}, K_{n_2}, \ldots, K_{n_\ell}$

means

No matter how we color the edges of K_n

with ℓ colors $C_1, C_2, \ldots, C_\ell$

$\exists$ K_{n_1} subgraph with all edges colored C_1

or $\exists$ K_{n_2} --- C_2

or ...

or $\exists$ K_{n_ℓ} subgraph --- C_ℓ

For integers $n_1, n_2, \ldots, n_\ell \geq 2$

Ramsey number

$r(n_1, n_2 \ldots, n_\ell)$ = min n such that

$K_n \rightarrow K_{n_1}, K_{n_2}, \ldots, K_{n_\ell}$

Ex $r(3,3,3) \leq 17$

pf Suffices to show $K_{17} \to K_3, K_3, K_3$

Color the edges of K_{17} red or blue or green

Show $\exists$ K_3 subgraph that is red or blue or green

Pick a vertex x of K_{17}

K_{17} has 16 edges containing x

By PP either

- at least 6 of these are red ← WLOG

or • at least 6 of these are blue

or • at least 6 of these are green

Pick vertices $\{x_i\}_{i=1}^{6}$ that are connected to x by a red edge

Consider the K_6 subgraph with vertex set $\{x_i\}_{i=1}^{6}$

<u>Case</u> this K_6 contains a red edge. then the 2 vertices involved together with x give a red K_3

<u>Case</u> this K_6 does not contain a red edge. then since $r(3,3)=6$ this K_6 contains a K_3 subgraph that is blue or green.

In either case K_{17} contains a K_3 subgraph that is red or blue or green.

Therefore $K_{17} \to K_3, K_3, K_3$ $\square$

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In fact $r(3,3,3) = 17$