# Math 475: Introduction to Combinatorics
## Lecture 1, MWF 12:05–12:55 p.m., B115 Van Vleck
## Syllabus for Semester I, 2012/2013

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Text: Introductory Combinatorics, 5th Edition, by Richard A. Brualdi.

**Course Content**: We will cover roughly chapters 1–8 and 14 in the text. The main topics include permutations and combinations; pigeon-hole principle; partial orders; Dilworth's theorem; the inclusion-exclusion principle; recurrence relations and generating functions; difference sequences; Catalan numbers; Stirling numbers; partition numbers; counting equivalence classes in the presence of symmetries.

**Exams and Grades**: The course grade is based on two in-class exams and the final exam. Each of the in-class exams is worth 100 points, and the final exam is worth 150 points. In addition the graded homework is worth 50 points. Here is the exam schedule:

- Exam 1: Monday, October 15
- Exam 2: Monday, November 26
- Final exam: Thursday, Dec 20, at 7:45–9:45 a.m.

**Homework**: At the end of the syllabus there is a list of exercises to be handed in. These will be marked by a graduate student assigned to me. Your work on these exercises should be well presented, in good English. A clear explanation is just as important as the correct answer. It is suggested, but not required, to type your answers using for example Latex. It is fine to form a study group and discuss the exercises with your classmates; however the work you hand in should be your own and not copied from someone else. When you turn in your homework it must be a paper copy; please do not email your homework to me or the grader. Late homework will not be accepted.

**Calculator Policy**: During an exam no books, notes, calculators, cell phones, pagers, or any electronic devices will be allowed.

**How to prepare for the exams**: The list of exercises at the end of this syllabus is the minimal homework requirement; it is recommended that you do many more exercises on your own. For each exam including the final, the exam problems will be based on, and in many cases taken verbatim from, the exercises that appear at the end of the relevant chapters in the text. These exercises might not appear in the table at the end of the syllabus. Generally speaking, the more exercises from the text that you work out and understand, the easier the exam problems will seem.

**Rough Schedule:**

| Week | M | W | F |
|---|---|---|---|
| Sept 3 | Labor day | Ch 1 | Ch 2 |
| Sept 10 | Ch 2 | Ch 2 | Ch 2 |
| Sept 17 | Ch 3 | Ch 3 | Ch 3 |
| Sept 24 | Ch 4 | Ch 4 | Ch 4 |
| Oct 1 | Ch 4 | Ch 5 | Ch 5 |
| Oct 8 | Ch 5 | Ch 5 | Ch 6 |
| Oct 15 | Exam I | Ch 6 | Ch 6 |
| Oct 22 | Ch 6 | Ch 6 | Ch 7 |
| Oct 29 | Ch 7 | Ch 7 | Ch 7 |
| Nov 5 | Ch 7 | Ch 8 | Ch 8 |
| Nov 12 | Ch 8 | Ch 8 | Ch 8 |
| Nov 19 | Ch 8 | Rev | Thanksgiving |
| Nov 26 | Exam II | Ch 14 | Ch 14 |
| Dec 3 | Ch 14 | Ch 14 | Ch 14 |
| Dec 10 | Ch 14 | Ch 14 | Ch 14 |

**Homework assignments:**

| Chapter | Exercises | Tent. Due date |
|---|---|---|
| 1 | 3, 4, 7, 14, 17, 30, 31, 36, 43 | Not to hand in |
| 2 | 4b, 5b, 6, 7, 9, 11, 13, 14, 15, 19b, 26, 27 | Friday, Sept. 14 |
| 2 | 30, 38, 39, 41, 42, 45, 55b, 60, 61, 63 | Friday, Sept. 21 |
| 3 | 4, 5, 9, 10, 14, 16, 18, 20, 27, 28 | Friday, Sept. 28 |
| 4 | 7b, 8, 15a, 15c, 16a, 16c, 17, 23, 24, 29, 33 | Friday, Oct. 5 |
| 5 | 5, 6, 7, 12, 13, 16, 18, 23, 24, 25, 27, 28, 29 | Friday, Oct. 12 |
| 5 | 8, 9, 19, 30, 31, 32, 34, 39, 40, 43, 46, 48 | Not to hand in |
| 6 | 2, 9, 12, 13, 16, 17, 21, 23, 24a, 26, 27, 28, 31 | Friday, Oct. 26 |
| 7 | 8, 9, 11, 14, 15, 17, 18, 19, 21, 29, 30 | Friday Nov. 2 |
| 7 | 26, 27, 28, 32, 34, 35, 36, 40, 43, 47, 48, 50 | Friday, Nov. 9 |
| 8 | 1, 2, 3, 6, 7, 8, 12, 13, 14, 15, 19, 20, 26 | Friday, Nov. 16 |
| 14 | 1, 11, 12, 13, 14, 18, 20, 22, 23, 24, 25, 29 | Friday Dec 7 |
| 14 | 40, 41, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53 | Friday Dec 14 |

For the textbook a list of errata is located at `http://www.math.wisc.edu/~brualdi`

Errata for: Richard Brualdi, Introductory Combinatorics 5th Edition

Chapter 1
(i) page 24, problem 38.
An implicit assumption is that $n$ points are colored red
and $n$ points are colored blue.

Chapter 2
(i) page 56, middle.
..the set $S$ of consisting of..
should be
..the set $S$ consisting of..

Chapter 3
(i) page 83, problem 4.
An implicit assumption is that the chosen integers are mutually
distinct.

Chapter 4
(i) page 112, first example.
In the display of the 3-permutations,
the entry 312 at the bottom left should be 213.
(ii) page 122, problem 39.
The partially ordered set $(X,\subseteq)$ should be
$(\mathcal P(X),\subseteq)$.

Chapter 5
(i) page 142, middle.
..with $|S|=k$ ..
should be
..with $|A|=k$ ..
(ii) page 157, 158, problems 26, 27, 28.
There should be no reference to $k$ in the opening sentence,
since $k$ is the index of summation.
Also, in problem 26, the expression $2n+1$ should be $2n+2$.
(iii) page 159, problem 44.
In the displayed equation the right-hand side should be $(-3)^n$.
(iv) page 159, problem 45.
In the displayed equation the right-hand side should be $(-4)^n$.

Chapter 6
(i) page 201, problem 30.
..for each type of letter..
should be
..for each type of letter except $d$..
Without this adjustment the answer is zero.
(ii) page 202, problem 33. The quantity
$a(n,k)$ is defined incorrectly.
In the correct definition, $a(n,k)$ is the
number of ways to place the $k$ nonattacking rooks
in the forbidden positions.

Chapter 7 OK

Chapter 8 OK

Chapter 14 OK

# Math 475 Introduction to Combinatorics

Paul Terwilliger

- Hand out syllabus

Text: Introductory Combinatorics 5th Ed
Richard Brualdi

- Will cover ch 1–9, 14
- Book is a classic – we will follow it closely
- ~weekly hw assignments will be graded (see syllabus)
- 1st assignment:

  Read Ch 1 and do hw listed in syl

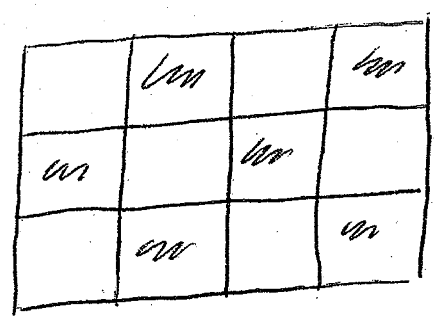
- Today: a few topics from ch 1
- Friday: start ch 2

# Perfect covers of chessboards

Given positive integers $m, n$

Consider $m \times n$ chessboard

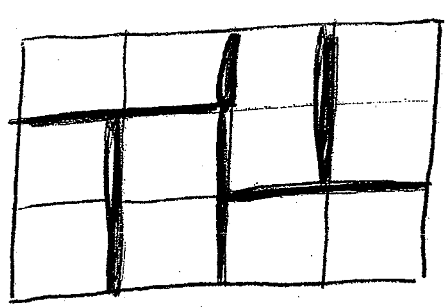
ex $m = 3$ $n = 4$



A <u>domino</u> consists of 2 squares

Consider a partition of the chessbd into dominoes
"perfect cover"

ex



Assume our $m \times n$ chess bd has a perfect cover by dominoes

let $T =$ # dominoes used

Obs

$$2T = mn = \text{\# squares in chessbd.}$$

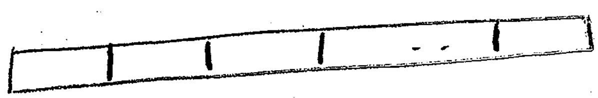
So $mn$ is even

Assume $mn$ is even.

How many perfect covers by dominoes are there?

ex $m=1$ $n$ even

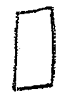
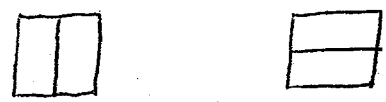
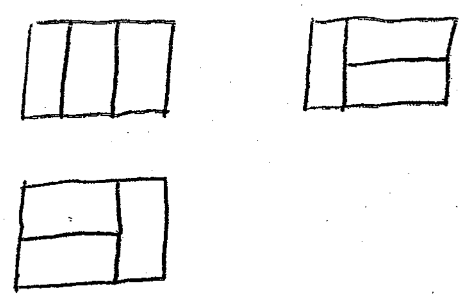
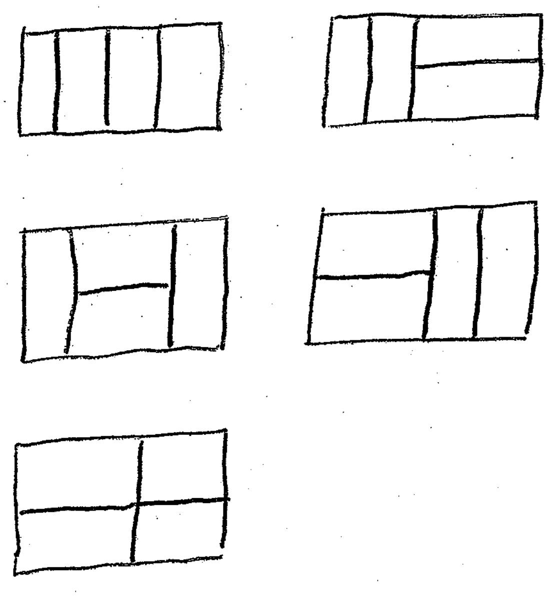
Unique perfect cover is



Try $m=2$

For $n = 1, 2, \ldots$ let $h_n =$ # perfect covers of $2 \times n$ chessbd by dominoes

Find $h_n$

| n | examples | $h_n$ |
|---|---|---|
| 1 |  | 1 |
| 2 |  | 2 |
| 3 |  | 3 |
| 4 |  | 5 |

LEMMA 1 The sequence $h_1, h_2, \ldots$ is uniquely determined by

$$h_n = h_{n-1} + h_{n-2} \qquad n \geq 3$$

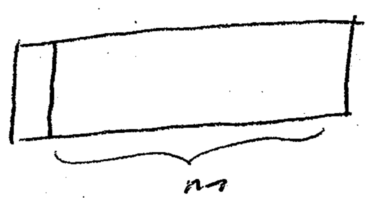
$$h_1 = 1, \qquad h_2 = 2$$

"Fibonacci sequence"

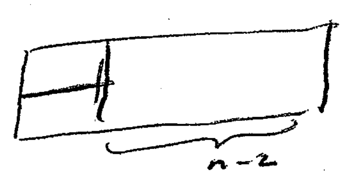
pf Given a perfect cover of a $2 \times n$ chessbd by dominoes

Consider dominoes that cover column 1

Either 1 vertical domino
or 2 horiz dominoes




or




#choices $h_{n-1}$ $h_{n-2}$

So $h_n = h_{n-1} + h_{n-2}$

$\square$

EX 2 A $2 \times 10$ chessbd has how many perfect covers by dominoes?

Sol

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $h_n$ | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |

ans = 89

Ex Given integer $b \geq 2$

Suppose $\exists$ perfect cover of $m \times n$ chessbd
using $1 \times b$ tiles

let $T$ = # tiles used

Observe

$Tb = mn$ = # squares in chessbd

So $b$ is a factor of $mn$ "$b$ divides $mn$" or "$b \mid mn$"

LEM 3 In fact $b \mid m$ or $b \mid n$

pf: Divide $m$ by $b$ and consider remainder $r$
" $n$ " $b$ ... $s$

So $m = pb + r$ $p, r \in \mathbb{Z}$, $0 \leq r \leq b-1$

$n = qb + s$ $q, s \in \mathbb{Z}$ $0 \leq s \leq b-1$

Observe

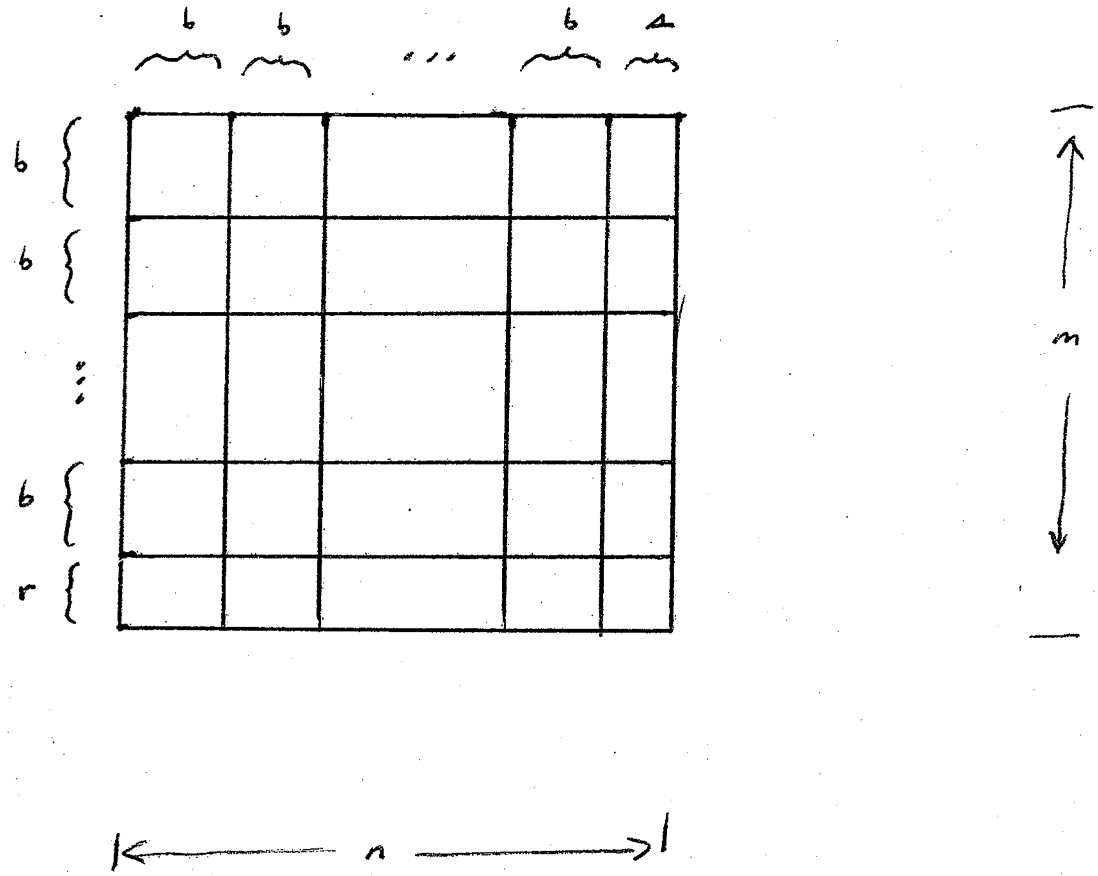
$r = 0$ iff $b \mid m$

$s = 0$ iff $b \mid n$

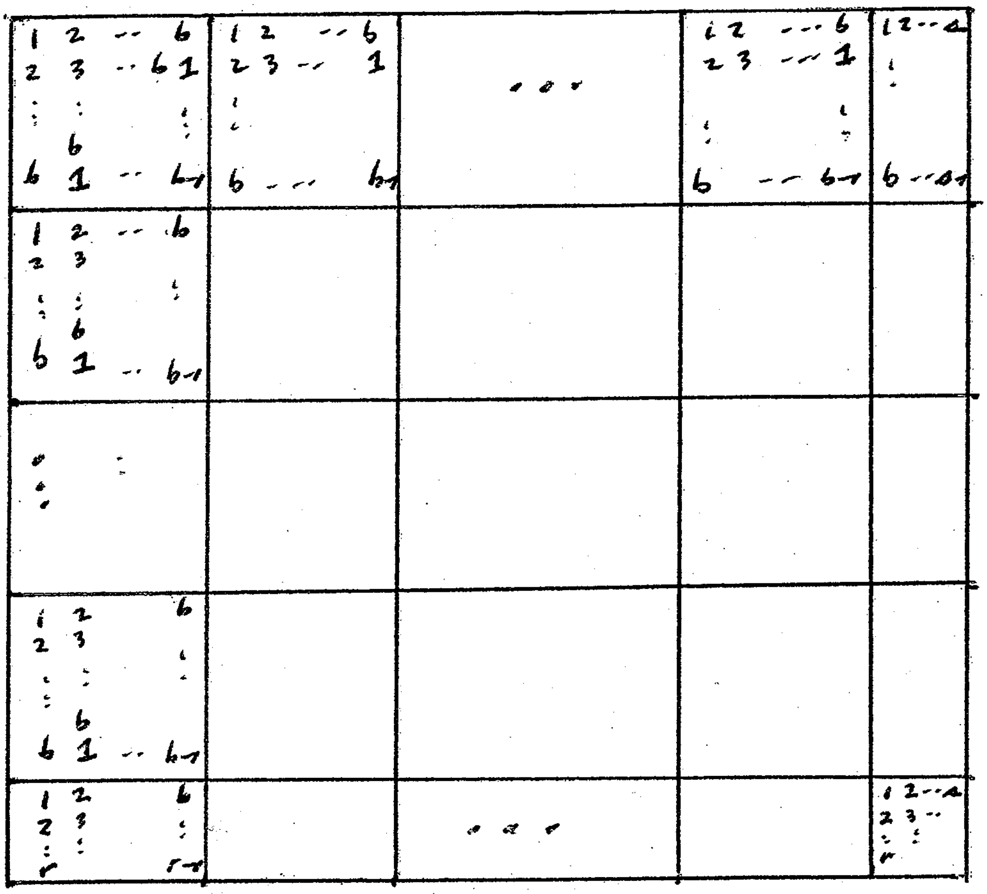
Suf to show $r = 0$ or $s = 0$

I will assume $r \neq 0$, $s \neq 0$ and get contradiction.

Partition the squares of the chessbd as follows



In chessbd give each square a label from $1, 2, \ldots, b$ as follows:



Consider our perfect cover of chessbd by 1x6 tiles

Obs: Each 1x6 tile covers one each of
1, 2, ..., 6 (in some order)

let $T$ = # tiles

$T$ = # of squares in chessbd labelled 1
= ... 2
= ... :
= ... 6

Consider our partition of the chessbd

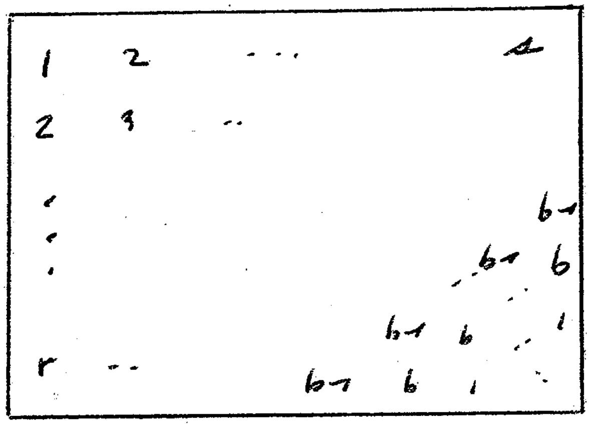
- For each 6x6 block exactly 6 squares labelled 1
-- 2
--:
-- 6

- For each $r\times 6$ block exactly $r$ squares labelled 1
-- -- 2
: :
-- 6

- For each $6\times s$ block exactly $s$ squares labelled 1
... 2
:
... 6

For the unique $r\times s$ block
must have
# squares labelled 1
= # squares labelled 2
= ...
= # squares labelled 6

the $r \times s$ block is labelled as follows:




We may now argue

$r \neq 0, s \neq 0 \longrightarrow$ top left square in $r \times s$ block exists

$\longrightarrow$ in $r \times s$ block at least one square gets label 1
label $b$

$\longrightarrow$ . . .

Now By const in $r \times s$ block

$$\#\text{ squares labelled } b-1 = (\#\text{ squares labelled } b) + 1$$

Contradiction.

We conclude $r = 0$ or $s = 0$ $\square$

Given a perfect matching of an $m \times n$ chessbd by $1 \times b$ tiles

Call it trivial whenever all the tiles are oriented the same way:

all horizontal

or all vertical.

Cor 4. Given an integer $b \geq 2$ and a $m \times n$ chessboard TFAE:

(i) $\exists$ perfect cover of chessbd using $1 \times b$ tiles

(ii) $\exists$ trivial perfect cover of chessbd using $1 \times b$ tiles

pf (i) $\to$ (ii) By Lem 3 $b \mid m$ or $b \mid n$

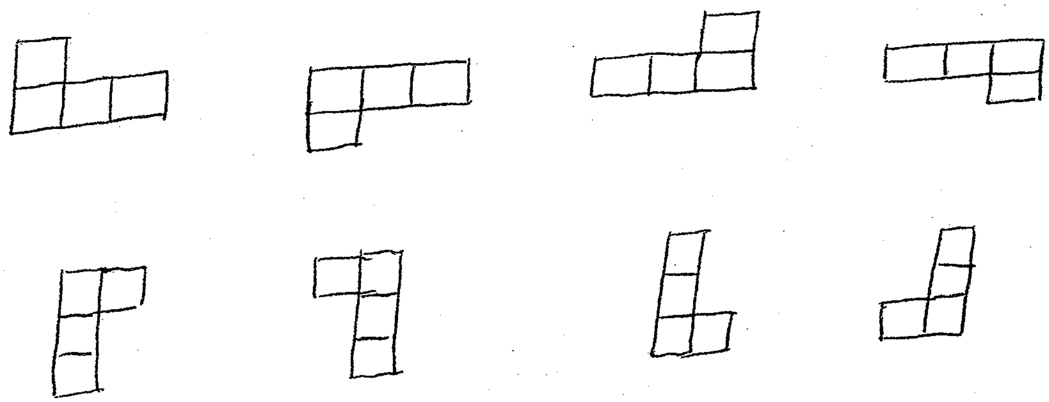
WLOG $b \mid m$

obs $\exists$ perfect cover of chessbd with all tiles vertical

(ii) $\to$ (i) clear $\square$

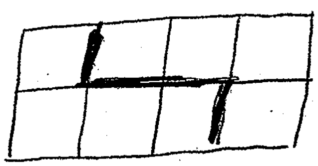
## tetromino coverings of chessbds

A tetromino consists of an L-shape of 4 squares



Which $m \times n$ chessbds have a perfect covering by tetrominoes?

ex $m=2$ $n=4$



Given $m \times n$ chessbd suppose such a perfect cover exists

let $T = \#$ tiles

$$4T = mn$$

So $4 \mid mn$

This condition is nec but not suf, since $m=2, n=2$ is counterex.

LEM For a $m \times n$ chessbd suppose $\exists$ perfect cover by tetrominoes. Then $8 \mid mn$

pf Let $T$ = # tiles used

$$4T = mn$$

WLOG $m$ even write $m = 2r$

$$4T = 2rn$$
$$2T = rn$$

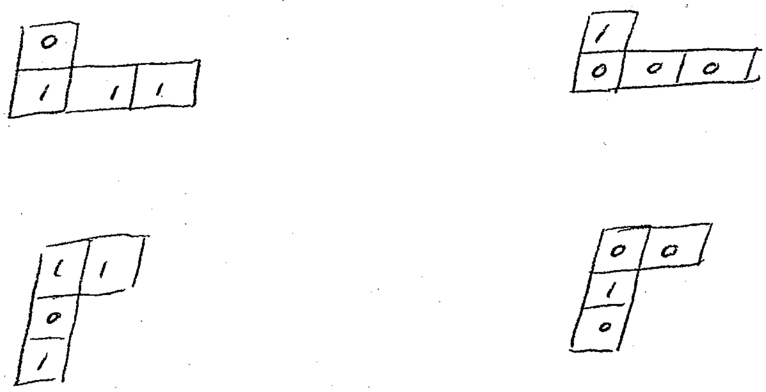
For $m \times n$ chessbd Label each square as follows:

$$\begin{array}{cc} & n \\ m & \boxed{\begin{array}{cccc} 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ & & \vdots & \\ 1 & 1 & 1 & \cdots \end{array}} \end{array}$$

Sum of Labels $= rn$
$= \text{even}$

Consider tetrominoes in the perfect cover

possible Labellings:



In all cases sum of labels is odd

So

$$\text{even} = \text{sum of all labels on } m \times n \text{ chessbd}$$
$$= \sum_{i=1}^{T} \Big( \underbrace{\text{sum of labels on } i\text{th tetro}}_{\text{odd}} \Big)$$

So $T$ is even

We saw $2T = rn$

So $4 \mid rn$

$m = 2r$

So $8 \mid mn$ $\square$

## Matchings

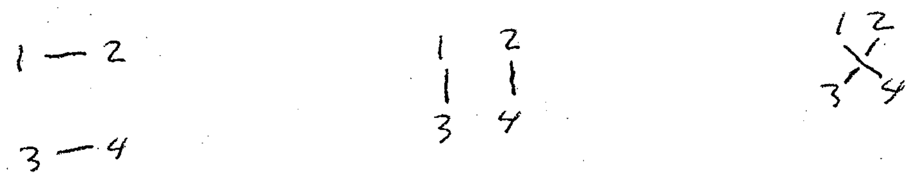
Given a positive integer $n$

Suppose $2n$ people at a party pair off to form $n$ teams of 2

In how many ways can this be done?

Call this number $M_n$

Ex. $n=2$ Label the persons $1, 2, 3, 4$




$M_2 = 3$

LEM $M_n = (2n-1)(2n-3)(2n-5)\cdots 3\cdot 1$ $n=1,2,\dots$

pf Label the persons $1, 2, \dots, n$.

Consider the partner for person 1: # choices $= 2n-1$

Remove person 1 and their partner

$2n-2$ persons left.

# ways to pair these off is $M_{n-1}$

So

$$M_n = (2n-1) M_{n-1} \qquad n = 2, 3 \ldots$$

Obs

$$M_1 = 1$$

Result follows by induction on $n$ $\square$

Lec 2 Friday Sept 7

Ch 2 Permutations and combinations

2.1 4 basic counting principles

Given a set $S$

A <u>partition</u> of $S$ is a decomposition of $S$ into subsets such that each element of $S$ is contained in exactly one of the subsets.

Ex
$$S = \{1, 2, 3, a, b\}$$
$$A = \{1, a\}$$
$$B = \{2, b\}$$
$$C = \{3\}$$

$A, B, C$ partition $S$

For a set $S$ define

$$|S| = \text{number of elements in } S$$

"cardinality of $S$" "size of $S$"

For above example

$$|S| = 5 = 2+2+1$$
$$= |A| + |B| + |C|$$

This illustrates

<u>Addition Principle</u> Suppose a set $S$ is partitioned into subsets $S_1, S_2, \ldots, S_k$.

Then

$$|S| = |S_1| + |S_2| + \cdots + |S_k|$$

Given sets $A, B$

their Cartesian product denoted $A \times B$

is the set of all ordered pairs $(a, b)$ with $a \in A$

and $b \in B$

Ex $\quad A = \{1, 2, 4\}$ $\quad B = \{r, s\}$

els of $A \times B$ are in table:

| A \ B | r | s |
|---|---|---|
| 1 | (1,r) | (1,s) |
| 2 | (2,r) | (2,s) |
| 4 | (4,r) | (4,s) |

Obs

$$|A \times B| = 6 = 3 \times 2 = |A|\,|B|$$

In general for any finite sets $A, B$

$$|A \times B| = |A|\,|B|$$

Ex For an Integer $n \geq 1$ Find the number of sequences

$$a_1 a_2 \cdots a_n \qquad a_i \in \{0, 1\} \quad 1 \leq i \leq n$$

Sol Def $A = \{0, 1\}$

View set of sequences as Cartesian product

$$\underbrace{A \times A \times \cdots \times A}_{n \text{ copies}}$$

$$\# \text{sequences} = |A|^n$$
$$= 2^n$$

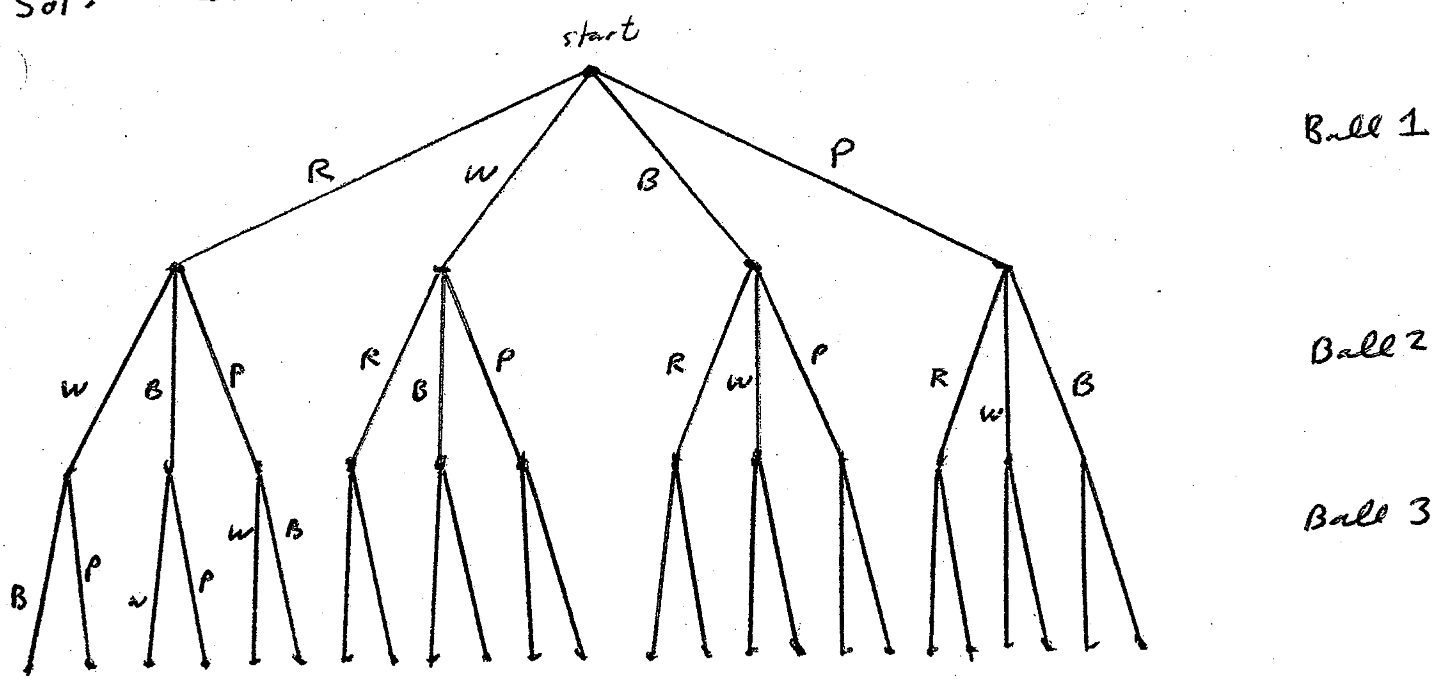
Prob: A box contains

1 Red ball
1 white ball
1 blue ball
1 purple ball

3 balls are withdrawn, one at a time, without replacement and colors noted.

How many outcomes are possible?

Sol: Illustrate with "tree diagram"




Each outcome is represented by a path in the tree diagram, from "start" to a "leaf"

$$\# \text{outcomes} = 4 \times 3 \times 2 = 24$$

Above examples illustrate the

## Multiplication principle

Consider a multistage experiment with $k$ stages

Assume:

- stage 1 has $n_1$ possible outcomes
- stage 2 has $n_2$ possible outcomes (regardless of result of stage 1)
- stage 3 has $n_3$ possible outcomes (regardless of results of stages 1, 2)

$\vdots$

- stage $k$ has $n_k$ possible outcomes (regardless of results of prev stages)

Then the total number of outcomes is

$$n_1 \times n_2 \times \cdots \times n_k$$

Ex For an integer $n \geq 0$

the set $\{1, 2, \ldots, n\}$ has how many subsets?

Sol. View the situation as an $n$-stage experiment

in which we select a subset

| stage | to do | # choices |
|---|---|---|
| 1 | decide is "1" in subset | 2 |
| 2 | decide is "2" in subset | 2 |
| $\vdots$ | | $\vdots$ |
| $n$ | decide is "n" in subset | 2 |

(Y or N)

By mult principle the # subsets is $2^n$

Given sets $S \subseteq \mathcal{U}$

the complement of $S$ in $\mathcal{U}$ is

$$\overline{S} = \{ x : x \in \mathcal{U} \text{ and } x \notin S \}$$

We have partition

$$\mathcal{U} = S \cup \overline{S} \qquad \text{(disjunion)}$$

So $$|\mathcal{U}| = |S| + |\overline{S}|$$

In other words

Subtraction principle: For $|\mathcal{U}| < \infty$ and $S \subseteq \mathcal{U}$,

$$|\overline{S}| = |\mathcal{U}| - |S|$$

Ex A product code consists of a string of 4 dist letters taken from the set

$$\{A, H, Q, T, Z\}$$

How many such codes contain the letter "H"?

Sol View

$U$ = set of all product codes

$S$ = set of product codes that <u>do not</u> contain H

Find $|\bar{S}|$ using

$$|\bar{S}| = |U| - |S|$$

Find $|U|$:

Construct prod code by filling in blanks left to right

$$\underset{\substack{\uparrow \\ 5 \text{ choices}}}{\underline{\qquad}} \quad \underset{\substack{\uparrow \\ 4 \text{ choices}}}{\underline{\qquad}} \quad \underset{\substack{\uparrow \\ 3 \text{ choices}}}{\underline{\qquad}} \quad \underset{\substack{\uparrow \\ 2 \text{ choices}}}{\underline{\qquad}}$$

By mult principle

$$|U| = 5 \times 4 \times 3 \times 2$$
$$= 120$$

Find $|S|$:

$\underset{\uparrow}{\_\_\_\_}$ 4 choices $\quad \underset{\uparrow}{\_\_\_\_}$ 3 choices $\quad \underset{\uparrow}{\_\_\_\_}$ 2 choices $\quad \underset{\uparrow}{\_\_\_\_}$ 1 choice

$$|S| = 4 \times 3 \times 2 \times 1$$
$$= 24$$

$$|\overline{S}| = 120 - 24$$
$$= 96$$

$\square$

<u>Division principle</u> Given a finite set $\mathcal{U}$ partition $\mathcal{U}$ into subsets with equal cardinality. Then

$$\text{the number of subsets} = \frac{|\mathcal{U}|}{\text{cardinality of each subset}}$$

Ex Given integers $3 \le k \le n$

We create a pearl necklace as follows.

Given $n$ distinct pearls

Necklace uses $k$ of these pearls.

How many necklaces are possible?

[two necklaces are same if they differ by rot/refl]

Sol: Call the pearls $1, 2, \ldots, n$.

Define

$\mathcal{U}$ = set of all sequences of $k$ pearls taken from $\{1, 2, \ldots, n\}$

First find $|\mathcal{U}|$:

To select an element of $\mathcal{U}$, fill in blanks left to right.

$$\underset{\substack{\uparrow \\ n \text{ choices}}}{\underset{1}{\_\_\_}} \quad \underset{\substack{\uparrow \\ n-1 \text{ choices}}}{\underset{2}{\_\_\_}} \quad \cdots \quad \underset{\substack{n-k+1 \\ \text{choices}}}{\underset{k}{\_\_\_}}$$

By multiplication principle

$$|\mathcal{U}| = n(n-1)\cdots(n-k+1)$$

Define a binary relation $\sim$ on $\mathcal{U}$ as follows.

$\forall\, x, y \in \mathcal{U}$

$x \sim y$ whenever $y$ is obtained from $x$ by rot/refl

$\sim$ is an equiv relation.

Each equiv classes consists of exactly $2k$ elements

$$\begin{array}{ll} a_1 a_2 \cdots a_k & a_k a_{k-1} \cdots a_1 \\ a_2 a_3 \cdots a_k a_1 & a_1 a_k \cdots a_2 \\ a_3 \cdots a_k a_1 a_2 & a_2 a_1 a_k \cdots a_3 \\ \vdots & \vdots \\ a_k a_1 \cdots a_{k-1} & a_{k-1} \cdots a_1 a_k \end{array}$$

Each necklace corresponds to an equiv class of $\sim$

$$\#\text{necklaces} = \#\text{ equiv classes for } \sim$$

$$= \frac{|\mathcal{U}|}{\text{card of each equiv class}}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{2k}$$

## 2.2 Permutations of sets

Given a set $S$

Given integer $r \geq 0$

An <u>r-permutation</u> of $S$ is a sequence

$$a_1 a_2 \cdots a_r$$

where $a_1, a_2, \ldots, a_r$ are distinct elements of $S$

By a <u>permutation</u> of $S$ we mean an r-perm where $r = |S|$

Ex $\qquad S = \{a, b, c\}$

the permutations of $S$ are

abc $\qquad$ cba
bca $\qquad$ acb
cab $\qquad$ bac

the 2-perms of $S$ are

ab $\qquad$ ba $\qquad$ ca
ac $\qquad$ bc $\qquad$ cb

the 1 perms of $S$ are

a $\qquad$ b $\qquad$ c

No r-perms of $S$ for $r > 3$

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Ex Given finite set $S$ write $n = |S|$

For $0 \le r \le n$ find the number of $r$-permutations of $S$

Sol Select an $r$-perm $a_1 a_2 \cdots a_r$

in stages

| stage | to do | # choices |
|---|---|---|
| 1 | pick $a_1$ | $n$ |
| 2 | pick $a_2$ | $n-1$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $r$ | pick $a_r$ | $n-r+1$ |

By multiplication principle

$$\# \text{ r-perms of } S = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{\text{call this } P(n,r)}$$

For an integer $n \geq 0$ define

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1$$ "n factorial"

By convention $0! = 1$

Obs

$$P(n,r) = \frac{n!}{(n-r)!}$$

Obs

$$\# \text{ perms of } S = P(n,n)$$
$$= n!$$

Ex At a party there are 8 women and 11 men. How many ways are there to form 8 couples consisting of 1 man and 1 woman?

Sol. Label women 1, 2, ..., 8

Proceed in stages:

| stage | to do | # choices |
|---|---|---|
| 1 | select partner for w1 | 11 |
| 2 | ... w2 | 10 |
| ⋮ | ⋮ | |
| 8 | ... w8 | 4 |

By mult principle

$$\# \text{ways} = 11 \times 10 \times \cdots \times 4$$
$$= P(11, 8)$$

□

Ex Given set $S$ of people $|S| = n$

For $1 \le r \le n$ seat $r$ people from $S$ around circular table. Only care about who sits to your left/right

"circular $r$-perm"

[ $r=4$    $\begin{matrix} & a & \\ d & & b \\ & c & \end{matrix}$ $=$ $\begin{matrix} & d & \\ c & & a \\ & b & \end{matrix}$ $\neq$ $\begin{matrix} & d & \\ a & & c \\ & b & \end{matrix}$ ]

How many arrangements possible?

Sol (Sim to necklace problem)

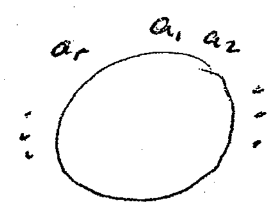
Let $\mathcal{U}$ = set of all $r$-perms of $S$

$|\mathcal{U}| = P(n,r)$

Given $r$-perm $a_1 a_2 \cdots a_r$ in $\mathcal{U}$

wrap $a_1 a_2 \cdots a_r$ around table clockwise

$a_1 a_2 \cdots a_r \Longrightarrow$



the following r-perms get sent to the same table arrangement:

$$a_1 a_2 \cdots a_r$$
$$a_2 a_3 \cdots a_r a_1 \qquad (*)$$
$$a_3 \cdots a_r a_1 a_2$$
$$\vdots$$
$$a_r a_1 \cdots a_{r-1}$$

Define a binary relation $\sim$ on $\mathcal{U}$ as follows:

$\forall\, x, y \in \mathcal{U}$

$x \sim y$ whenever $x, y$ give same table arrangement

$\sim$ is equiv relation

Each equiv class has form $(*)$ with $r$ elements

Each table arrangement corresponds to an equiv class of $\sim$

$$\#\text{ table arrangements} = \#\text{ equiv classes of } \sim = \frac{|\mathcal{U}|}{\text{card of each equiv class}} = \frac{P(n,r)}{r}$$

$\square$

In summary

Thm Given a finite set $S$ with $|S| = n$

For $1 \le r \le n$ the number of circ r-permutations of $S$

is $\dfrac{P(n,r)}{r} = \dfrac{n!}{r\,(n-r)!}$

In particular taking $r = n$, the number of

circular permutations of $S$ is

$$\frac{n!}{n} = (n-1)!$$

## 2.3 Combinations of Sets

Given a set $S$ $|S| = n$

For $0 \leq r \leq n$

By an <u>r-combination</u> (or <u>r-subset</u>) of $S$

we mean a subset of $S$ with cardinality $r$.

View as selection of $r$ objects from $S$ where selection order does not matter.

Ex Find the number of r-combinations of $S$.

Sol. Call this number $C(n,r)$

To find it, we count the number of r-perms of $S$

as follows.

We construct an r-perm from $S$ using this 2-stage procedure:

| stage | to do | # choices |
|---|---|---|
| 1 | select an r-combination X from S | $C(n,r)$ |
| 2 | order the elements of X | $r!$ |

By multiplication principle

$$\# \text{ r-perms of } S = C(n,r)\, r!$$

$$\| $$

$$P(n,r)$$

So

$$C(n,r) = \frac{P(n,r)}{r!}$$

$$= \frac{n!}{r!\,(n-r)!}$$ $\square$

Other notations for $C(n,r)$:

"binomial coefficient"

$_nC_r$ $\qquad\qquad \binom{n}{r}$

we will use

Convention

$$\binom{n}{r} = 0 \quad \text{if} \quad r < 0 \text{ or } r > n$$

Note

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

$$\binom{n}{r} = \binom{n}{n-r} \qquad 0 \le r \le n$$

Recall "Pascal triangle"

the numbers

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\vdots & & & \vdots & & & \ddots
\end{array}
$$

are

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & \vdots & & & \vdots & & & \vdots &
\end{array}
$$

thm For $1 \le k \le n-1$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

pf 1 Use formula $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

pf 2 Consider set

$$S = \{1, 2, \ldots, n\}$$

Partition the set of $k$-subsets of $S$ into 2 types:

| type of $k$-subset of $S$ | # choices |
|---|---|
| $k$-subsets of $S$ that contain "1" | $\binom{n-1}{k-1}$ ← # ways to select remaining $k-1$ elements from $\{2, \ldots, n\}$ |
| $k$-subsets of $S$ that do not contain "1" | $\binom{n-1}{k}$ ← # ways to select $k$ elements from $\{2, 3, \ldots, n\}$ |

So by addition principle

$$\binom{n}{k} = \# \text{ } k\text{-subsets of } S$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \square$$

Here is another formula for binom coefficients

thm For an integer $n \geq 0$

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

pf. Consider the set

$$S = \{1, 2, \ldots, n\}$$

$2^n$ = # subsets of $S$

= # 0-subsets of $S$

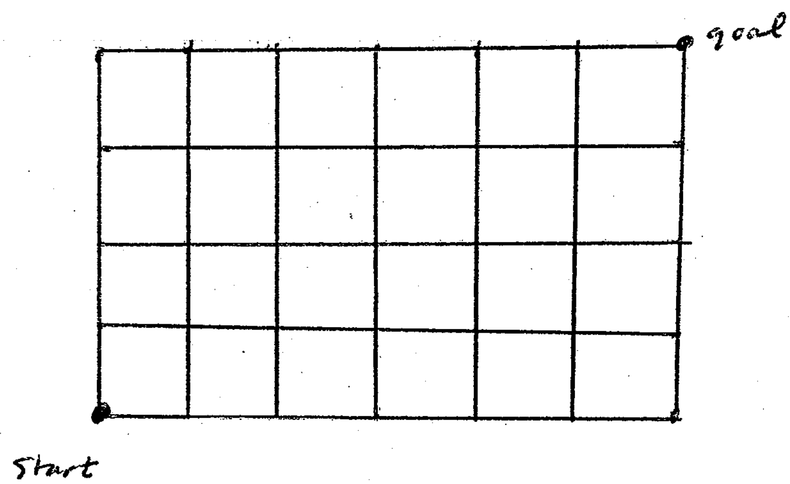
+ # 1-subsets of $S$

+ $\vdots$

+ # $n$-subsets of $S$

$= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ □

Ex Consider $4 \times 6$ grid of city streets!




goal is 6 blocks East
4 blocks North

Go from "start" to "goal" in 10 steps, each N or E

How many paths ?

Sol

Consider 10 steps of path

1 2 3 4 5 6 7 8 9 10

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

↑ ↑ ...

E or N E or N

Fill each blank with "E" or "N"

Require 6 E's and 4 N's.

For each path,

the set of blanks labelled N is a 4-subset of $\{1, 2, \dots, 10\}$

$$\text{set of paths} \overset{\text{bij}}{\Longleftrightarrow} \text{4-subsets of } \{1, 2, \dots, 10\}$$

$$\begin{aligned} \#\text{paths} &= \#\text{ 4-subsets of } \{1, 2, \dots, 10\} \\ &= \binom{10}{4} \end{aligned}$$

$\square$

Ex A committee of 5 people is chosen from a club that has a membership of 10 men and 12 women.

The Committee must have at least 2 women.

How many ways to form committee?

Sol

club

| |
|---|
| 10m |
| 12w |

"choose 5"

at least 2 w

We partition the committees according to the number of women

| # women | # ways to pick men | # ways to pick women | # committees |
|---|---|---|---|
| 2 | $\binom{10}{3}$ | $\binom{12}{2}$ | $\binom{10}{3}\binom{12}{2}$ |
| 3 | $\binom{10}{2}$ | $\binom{12}{3}$ | $\binom{10}{2}\binom{12}{3}$ |
| 4 | $\binom{10}{1}$ | $\binom{12}{4}$ | $\binom{10}{1}\binom{12}{4}$ |
| 5 | $\binom{10}{0} = 1$ | $\binom{12}{5}$ | $\binom{12}{5}$ |

$$\# \text{ comittees} = \binom{10}{3}\binom{12}{2} + \binom{10}{2}\binom{12}{3} + \binom{10}{1}\binom{12}{4} + \binom{12}{5}$$

$\square$

## 2.4 Permutations of multisets

We introduce the notion of a multiset with an example

Ex the letters in

MISSISSIPPI

form multiset

$$\{ I, I, I, I, M, P, P, S, S, S, S \}$$

which is abbreviated

$$\{ 4 \cdot I,\ 1 \cdot M,\ 2 \cdot P,\ 4 \cdot S \}$$

"repetition numbers"

A multiset is like a set except its elements need not be distinct.

EX letters in

ababab ...

form multiset

$\{\infty \cdot a, \infty \cdot b\}$

Def A <u>multiset</u> $M$ is a set $S$

together with a function

$R: S \to \{0,1,2,\dots\} \cup \infty$

$\forall x \in S$ $R(x)$ is the <u>repetition number</u> of $x$

By the <u>cardinality</u> (or <u>size</u>) of $M$ we mean

$$|M| = \sum_{x \in S} R(x)$$

Ex How many ways can we order the letters in

MISSISSIPPI ?

This problem asks for the number of <u>permutations</u>

of the multiset $\{4 \cdot I, 1 \cdot M, 2 \cdot P, 4 \cdot S\}$

Size is $11 = 4+1+2+4$

To Create an ordering

Fill in 11 blanks

__ __ __ __ __ __ __ __ __ __ __

with 4 I's
1 M
2 P's
4 S's.

We proceed in stages:

| stage | to do | # choices | |
|---|---|---|---|
| 1 | choose location for 4 I's | $\binom{11}{4}$ | |
| 2 | choose location of 1 M | $\binom{7}{1} = 7$ | $11-4=7$ |
| 3 | choose location of 2 P's | $\binom{6}{2}$ | $7-1=6$ |
| 4 | choose location of 4 S' | $\binom{4}{4} = 1$ | $6-2=4$ |

By the multiplication principle

$$\# \text{ perms} = \binom{11}{4}\binom{7}{1}\binom{6}{2} 1$$

$$= \frac{11!}{4!\,7!} \; \frac{7!}{1!\,6!} \; \frac{6!}{2!\,4!}$$

size

$$= \frac{11!}{4!\;1!\;2!\;4!}$$

repetition numbers

thm Given a multiset

$$M = \{ n_1 \cdot a_1,\ n_2 \cdot a_2,\ \dots\ n_k \cdot a_k \}$$

the number of permutations of $M$ is

$$\frac{n!}{n_1!\ n_2!\ \cdots\ n_k!}$$

where

$$n = |M| = n_1 + n_2 + \cdots + n_k$$

pf: just like above example

Def Given a multiset $M$.

For an integer $r \geq 0$

An <u>r-permutation</u> of $M$ is a sequence

$a_1 a_2 \cdots a_r$

where $a_1, a_2, \ldots a_r$ are from $M$

Ex For the multiset

$$M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$$

Find the number of r-perms for all $r \geq 0$

Sol. To create an r-perm fill in blanks

r blanks

$\underset{\substack{\uparrow \\ k \text{ choices}}}{\_\_\_} \quad \underset{\substack{\uparrow \\ k \text{ choices}}}{\_\_\_} \quad \_\_\_ \quad \cdots \quad \underset{\substack{\uparrow \\ k \text{ choices}}}{\_\_\_}$

# of r-perms is $k^r$

Ex For the multiset

$$M = \{5 \cdot x,\ 7 \cdot y,\ 8 \cdot z\}$$

Find the number of 18-permutations of M.

Sol. size of M is 20

To obtain an 18-perm, remove 2 elements of M and order the remaining elements.

We partition the set of 18-perms according to what is removed

| elements removed from M | # 18-perms |
|---|---|
| x, x | $\frac{18!}{3!\,7!\,8!}$ |
| y, y | $\frac{18!}{5!\,5!\,8!}$ |
| z, z | $\frac{18!}{5!\,7!\,6!}$ |
| x, y | $\frac{18!}{4!\,6!\,8!}$ |
| x, z | $\frac{18!}{4!\,7!\,7!}$ |
| y, z | $\frac{18!}{5!\,6!\,7!}$ |

the # of 18-perms of M is sum of

Ex Given finite set $S$ $n = |S|$

Given labelled boxes

Box 1, Box 2, ..., Box $k$.

Distribute all the elements of $S$ into boxes

as follows:

Box 1 gets exactly $n_1$ elements

.. 2 -- $n_2$ --

...

$k$ ... $n_k$ --

How many ways to distribute?

Sol. Fill the boxes in stages

| stage | to do | # choices |
|---|---|---|
| 1 | choose elements for Box 1 | $\binom{n}{n_1}$ |
| 2 | ... 2 | $\binom{n-n_1}{n_2}$ |
| 3 | .. 3 | $\binom{n-n_1-n_2}{n_3}$ |
| $\vdots$ | | |
| $k$ | ... $k$ | $\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ |

" $\binom{n_k}{n_k} = 1$

\# distributions =

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots$$

$$= \frac{n!}{n_1!\,(n-n_1)!}\;\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\;\frac{(n-n_1-n_2)!}{n_3!\,(n-n_1-n_2-n_3)!}\cdots$$

$$= \frac{n!}{n_1!\,n_2!\cdots n_k!}$$

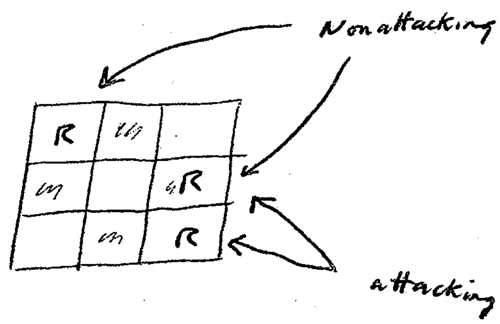
$\square$

Consider $n \times n$ chessboard

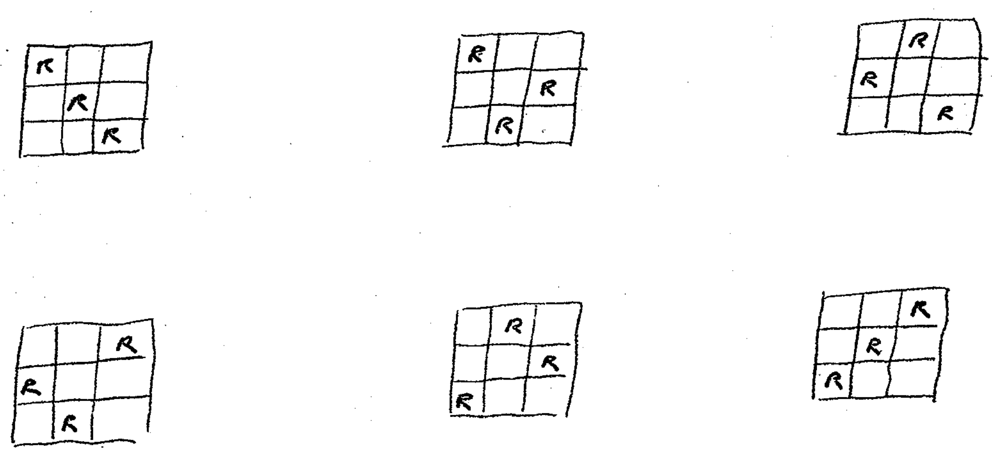
Recall in game of chess

rooks move horiz or vert

2 rooks are <u>nonattacking</u> if they are in distinct rows and distinct cols




Ex. In How many ways can we place 3 mutually nonattacking rooks on an $3 \times 3$ chessboard?



6 ways

Ex For $n \geq 1$ In how many ways can we place $n$ mutually nonattacking rooks on an $n \times n$ chessbd?

Sol. Obs

For each rook placement on chessbd

each row gets unique rook

each col gets unique rook

For $1 \leq i \leq n$

let $R_i$ = rook in row $i$

let $c_i$ = number of column that contains $R_i$

sequence $c_1 c_2 \cdots c_n$

is permutation of $\{1, 2, \ldots, n\}$

[ Ex $n=3$

| | | |
|---|---|---|
| | R | |
| R | | |
| | | R |

$c_1 = 2$
$c_2 = 1$
$c_3 = 3$ ]

this gives bijection

$$\left\{ \begin{array}{c} \text{set of} \\ \text{rook placements} \\ \text{on chessbd} \end{array} \right\} \overset{\text{bij}}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{set of} \\ \text{permutations of} \\ \{1, 2, \ldots, n\} \end{array} \right\}$$

$\#$ rook placements on chessbd $= \#$ perms of $\{1, 2, \ldots, n\}$

$= n!$

$\square$

Ex Referring to previous example

Suppose we now color the rooks with $k$ dist colors

Require:

| | | |
|---|---|---|
| $n_1$ | rooks get color | 1 |
| $n_2$ | ... | 2 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $n_k$ | ... | $k$ |

$$(n = n_1 + n_2 + \cdots + n_k)$$

How many colored-rook arrangements are possible?

Sol We create a colored-rook arrangement in stages

| stage | to do | # choices |
|---|---|---|
| 1 | put $n$ mut-nonattacking uncolored rooks on chessbd | $n!$ |
| 2 | color the rooks | $\frac{n!}{n_1! n_2! \cdots n_k!}$ |

$$\# \text{ colored-rook arrangements} = \frac{(n!)^2}{n_1! n_2! \cdots n_k!}$$

$\square$

2.5 Combinations of multisets

Given a multiset $M$

For $0 \le r \le |M|$

an <u>r-combination</u> of $M$ is an unorded selection

of $r$ elements of $M$

ex $M = \{a, a, b, c\}$

| $r$ | $r$-combinations of $M$ | | | |
|---|---|---|---|---|
| 4 | $\{a,a,b,c\}$ | | | |
| 3 | $\{a,a,b\}$ | $\{a,a,c\}$, | $\{a,b,c\}$ | |
| 2 | $\{a,a\}$ | $\{a,b\}$ | $\{a,c\}$ | $\{b,c\}$ |
| 1 | $\{a\}$ | $\{b\}$ | $\{c\}$ | |
| 0 | $\phi$ | | | |

Given multiset

$$M = \{ n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k \}$$

Given $0 \le r \le |M|$

How many $r$-combinations of $M$?

The $r$-combinations of $M$ are

$$\{ x_1 \cdot a_1, x_2 \cdot a_2, \ldots, x_k \cdot a_k \}$$

$$\left.\begin{array}{l} x_1, x_2, \ldots, x_k \in \mathbb{Z} \\ 0 \le x_1 \le n_1 \\ 0 \le x_2 \le n_2 \\ \vdots \\ 0 \le x_k \le n_k \\ x_1 + x_2 + \cdots + x_k = r \end{array}\right\} \quad *$$

The number of $r$-combinations of $M$ = # solutions to $(*)$.

[no simple general formula. We will find general method in ch 6]

Special Case Ref to (*). The condition

$$x_1 + x_2 + \cdots + x_k = r$$

implies

$$x_1 \le r$$
$$x_2 \le r$$
$$\vdots$$
$$x_k \le r$$

So if each of $n_1, n_2, \ldots, n_k$ is at least $r$ then the upper bounds

$$x_1 \le n_1$$
$$\vdots$$
$$x_k \le n_k$$

play no role. In this case the problem * becomes:

Ex. Given integers $k \ge 1$ and $r \ge 0$

Find the number of *integral* solutions to

$$\left.\begin{array}{l} x_1 \ge 0, \quad x_2 \ge 0, \quad \ldots \quad x_k \ge 0 \\ x_1 + x_2 + \cdots + x_k = r \end{array}\right\} \quad **$$

take $r = 2$

integral solutions:

| $x_1$ | $x_2$ | $x_3$ |
|---|---|---|
| 2 | 0 | 0 |
| 0 | 2 | 0 |
| 0 | 0 | 2 |
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |

# integral sols = 6

Sol Given an integral sol to ** :

$x_1, x_2, \ldots, x_k$

Consider $r$ dots in a row

Partition the set of dots into groups of size

$x_1, x_2, \ldots, x_k$

Separate adjacent groups with a marker |

Note $x_1, x_2, \ldots, x_k$ are determined by the location of the markers

$r = 8$ $k = 4$

$x_1 = 3$ $x_2 = 1$ $x_3 = 0$ $x_4 = 4$

$\bullet\ \bullet\ \bullet \mid \bullet \mid \mid \bullet\ \bullet\ \bullet\ \bullet$

| symbol | # used |
|---|---|
| $\bullet$ | $r$ |
| $\mid$ | $k-1$ |

# symbols used $= r+k-1$

# integral sols to ** $=$ # ways to fill $r+k-1$ blanks with $r$ dots and $k-1$ $|$'s

$= \#$ $r$-subsets of $r+k-1$ element set

$= \#$ $(k-1)$-subsets of $r+k-1$ ...

$= \binom{r+k-1}{r} = \binom{r+k-1}{k-1}$ □

In summary

Thm Given a multiset

$$M = \{ n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k \}$$

Given $0 \le r \le |M|$

The number of $r$-combinations of $M$ is

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

provided each of $n_1, n_2, \ldots, n_k$ is at least $r$. $\square$

Ex Find the number of integral solutions to

$$x_1 \geq 3, \quad x_2 \geq 0, \quad x_3 \geq 1$$

$$x_1 + x_2 + x_3 = 11$$

Sol. Change variables: define

$$y_1 = x_1 - 3$$

$$y_2 = x_2$$

$$y_3 = x_3 - 1$$

Requirement becomes

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0$$

$$y_1 + y_2 + y_3 = 7 \quad (= 11 - 3 - 1)$$

$$\text{ans} = \binom{7 + 3 - 1}{3 - 1} = \binom{9}{2} = 36 \qquad \square$$

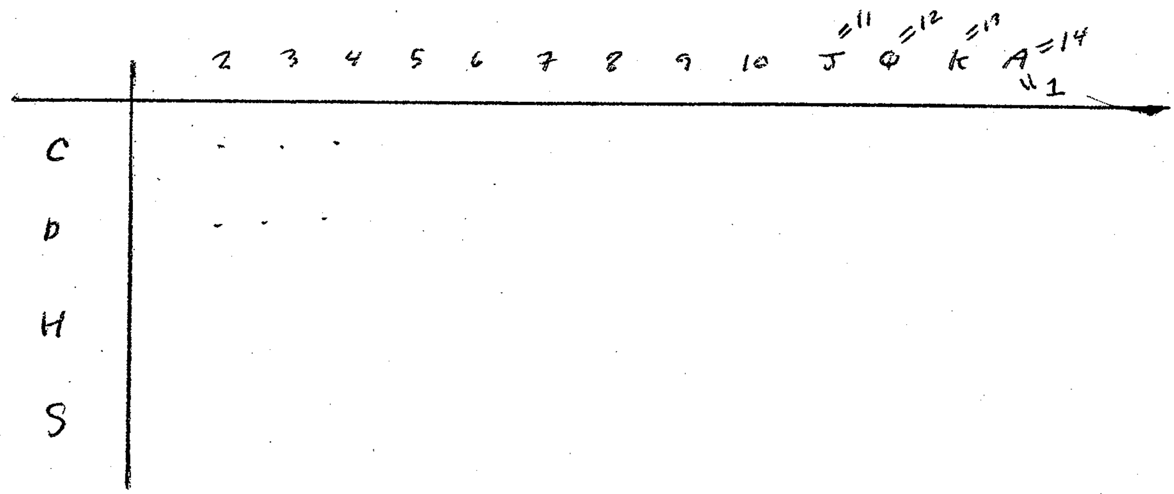
## 2.6 Probability

Recall game of Poker

52 card deck

13 ranks

4 suits




Poker hand is 5 cards ie 5-element subset of Deck

Each Poker hand equally Likely

Ex In a Poker hand, what is the probability of getting a "full house":

3 cards of one rank

2 cards of another rank

Sol. "Sample space" $S$ is set of all possible poker hands

Each element of $S$ equally Likely

$$|S| = \binom{52}{5}$$

"Event" $E$ is the set of full-house poker hands.

$$\text{Prob}(E) = \frac{|E|}{|S|}$$

Find $|E|$.

We construct a full-house in stages:

| stage | to do | # choices |
| --- | --- | --- |
| 1 | pick the 3-card rank | 13 |
| 2 | pick the suits for prev 3 cards | $\binom{4}{3} = 4$ |
| 3 | pick the 2-card rank | 12 |
| 4 | pick the suits for prev 2 cards | $\binom{4}{2}$ |

$$|E| = 13 \times 4 \times 12 \times \binom{4}{2}$$

$$\text{Prob}(E) = \frac{13 \times 4 \times 12 \times \binom{4}{2}}{\binom{52}{5}}$$

□

Ex In a poker hand, what is probability of getting "two pairs":

2 cards of one rank

2 cards of another rank

1 card of 3d rank

Sol Sample space $S$ consists of all poker hands

Event $E$ is set of all "two pair" poker hands

Find $|E|$

Construct "two pair" hand in stages:

| stage | to do | # choices |
| --- | --- | --- |
| 1 | pick the ranks of the 2 pairs | $\binom{13}{2}$ |
| 2 | pick suits for the pair with low rank | $\binom{4}{2}$ |
| 3 | ... high rank | $\binom{4}{2}$ |
| 4 | pick 5th card | $52-8=44$ |

$|E|$ is product of

$$\text{Prob}(E) = \frac{|E|}{|S|} = \frac{\binom{13}{2}\binom{4}{2}^2 44}{\binom{52}{5}}$$

$\square$

Ex In a Poker hand, what is probability of getting a "straight":

5 cards of consecutive ranks, view A as 14 or 1, ignore suits

Sol. Sample space $S$ consists of all poker hands

Event $E$ is set of "straight" poker hands

Find $|E|$.

Construct "straight" in stages:

| stage | to do | # choices |
|---|---|---|
| 1 | pick Lowest rank of straight | 10 |
| 2 | pick the five suits | $4^5$ |

poss ranks for straight are
1 – 5
2 – 6
⋮
10 – 14

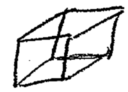
$$|E| = 10 \times 4^5$$

$$\text{Prob}(E) = \frac{10 \times 4^5}{\binom{52}{5}}$$

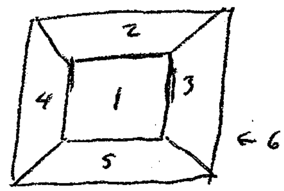
□

9/14/12 

Recall game of dice

One die is a cube 

faces are numbered




For a fair die each number equally likely to occur on top

Ex 3 dice are rolled.

What is probability that the sum of the top numbers is 8?

Sol Sample space $S$ consists of all 3-tuples

$(x, y, z)$ $\quad 1 \leq x, y, z \leq 6$

$|S| = 6^3$ Each outcome in $S$ equally likely

Event $E$ consists of the 3-tuples $(x, y, z)$ in $S$ such that

$x + y + z = 8$

$|E|$ = # integral sols to

$1 \leq x \leq 6, \quad 1 \leq y \leq 6, \quad 1 \leq z \leq 6$

$x + y + z = 8$

| ch vars |
|---|
| $X = x-1$ |
| $Y = y-1$ |
| $Z = z-1$ |

$=$ # integral sols to

$$0 \le X \le 5, \quad 0 \le Y \le 5, \quad 0 \le Z \le 5$$

$$X + Y + Z = 5$$

$=$ # integral sols to

$$0 \le X, \quad 0 \le Y, \quad 0 \le Z$$

$$X + Y + Z = 5$$

$$= \binom{5+3-1}{3-1} = \binom{7}{2} = 21$$

So

$$\text{Prob}(E) = \frac{|E|}{|S|}$$

$$= \frac{21}{6^3} \qquad \square$$