

# Math 475: Introduction to Combinatorics

## Lecture 1, MWF 12:05–12:55 p.m., B115 Van Vleck

### Syllabus for Semester I, 2012/2013

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Text: Introductory Combinatorics, 5th Edition, by Richard A. Brualdi.

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**Course Content:** We will cover roughly chapters 1–8 and 14 in the text. The main topics include permutations and combinations; pigeon-hole principle; partial orders; Dilworth's theorem; the inclusion-exclusion principle; recurrence relations and generating functions; difference sequences; Catalan numbers; Stirling numbers; partition numbers; counting equivalence classes in the presence of symmetries.

**Exams and Grades:** The course grade is based on two in-class exams and the final exam. Each of the in-class exams is worth 100 points, and the final exam is worth 150 points. In addition the graded homework is worth 50 points. Here is the exam schedule:

- Exam 1: Monday, October 15
- Exam 2: Monday, November 26
- Final exam: Thursday, Dec 20, at 7:45–9:45 a.m.

**Homework:** At the end of the syllabus there is a list of exercises to be handed in. These will be marked by a graduate student assigned to me. Your work on these exercises should be well presented, in good English. A clear explanation is just as important as the correct answer. It is suggested, but not required, to type your answers using for example Latex. It is fine to form a study group and discuss the exercises with your classmates; however the work you hand in should be your own and not copied from someone else. When you turn in your homework it must be a paper copy; please do not email your homework to me or the grader. Late homework will not be accepted.

**Calculator Policy:** During an exam no books, notes, calculators, cell phones, pagers, or any electronic devices will be allowed.

**How to prepare for the exams:** The list of exercises at the end of this syllabus is the minimal homework requirement; it is recommended that you do many more exercises on your own. For each exam including the final, the exam problems will be based on, and in many cases taken verbatim from, the exercises that appear at the end of the relevant chapters in the text. These exercises might not appear in the table at the end of the syllabus. Generally speaking, the more exercises from the text that you work out and understand, the easier the exam problems will seem.

**Rough Schedule:**

Week	M	W	F
Sept 3	Labor day	Ch 1	Ch 2
Sept 10	Ch 2	Ch 2	Ch 2
Sept 17	Ch 3	Ch 3	Ch 3
Sept 24	Ch 4	Ch 4	Ch 4
Oct 1	Ch 4	Ch 5	Ch 5
Oct 8	Ch 5	Ch 5	Ch 6
Oct 15	Exam I	Ch 6	Ch 6
Oct 22	Ch 6	Ch 6	Ch 7
Oct 29	Ch 7	Ch 7	Ch 7
Nov 5	Ch 7	Ch 8	Ch 8
Nov 12	Ch 8	Ch 8	Ch 8
Nov 19	Ch 8	Rev	Thanksgiving
Nov 26	Exam II	Ch 14	Ch 14
Dec 3	Ch 14	Ch 14	Ch 14
Dec 10	Ch 14	Ch 14	Ch 14

**Homework assignments:**

Chapter	Exercises	Tent. Due date
1	3, 4, 7, 14, 17, 30, 31, 36, 43	Not to hand in
2	4b, 5b, 6, 7, 9, 11, 13, 14, 15, 19b, 26, 27	Friday, Sept. 14
2	30, 38, 39, 41, 42, 45, 55b, 60, 61, 63	Friday, Sept. 21
3	4, 5, 9, 10, 14, 16, 18, 20, 27, 28	Friday, Sept. 28
4	7b, 8, 15a, 15c, 16a, 16c, 17, 23, 24, 29, 33	Friday, Oct. 5
5	5, 6, 7, 12, 13, 16, 18, 23, 24, 25, 27, 28, 29	Friday, Oct. 12
5	8, 9, 19, 30, 31, 32, 34, 39, 40, 43, 46, 48	Not to hand in
6	2, 9, 12, 13, 16, 17, 21, 23, 24a, 26, 27, 28, 31	Friday, Oct. 26
7	8, 9, 11, 14, 15, 17, 18, 19, 21, 29, 30	Friday Nov. 2
7	26, 27, 28, 32, 34, 35, 36, 40, 43, 47, 48, 50	Friday, Nov. 9
8	1, 2, 3, 6, 7, 8, 12, 13, 14, 15, 19, 20, 26	Friday, Nov. 16
14	1, 11, 12, 13, 14, 18, 20, 22, 23, 24, 25, 29	Friday Dec 7
14	40, 41, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53	Friday Dec 14

For the textbook a list of errata is located at <http://www.math.wisc.edu/~brualdi>

Errata for: Richard Brualdi, Introductory Combinatorics 5th Edition

Chapter 1

(i) page 24, problem 38.

An implicit assumption is that  $n$  points are colored red and  $n$  points are colored blue.

Chapter 2

(i) page 56, middle.

..the set  $S$  of consisting of..  
should be

..the set  $S$  consisting of..

Chapter 3

(i) page 83, problem 4.

An implicit assumption is that the chosen integers are mutually distinct.

Chapter 4

(i) page 112, first example.

In the display of the 3-permutations,  
the entry 312 at the bottom left should be 213.

(ii) page 122, problem 39.

The partially ordered set  $(X, \subseteq)$  should be  $(\mathcal{P}(X), \subseteq)$ .

Chapter 5

(i) page 142, middle.

..with  $|S|=k$  ..

should be

..with  $|A|=k$  ..

(ii) page 157, 158, problems 26, 27, 28.

There should be no reference to  $k$  in the opening sentence,  
since  $k$  is the index of summation.

Also, in problem 26, the expression  $2n+1$  should be  $2n+2$ .

(iii) page 159, problem 44.

In the displayed equation the right-hand side should be  $(-3)^n$ .

(iv) page 159, problem 45.

In the displayed equation the right-hand side should be  $(-4)^n$ .

Chapter 6

(i) page 201, problem 30.

..for each type of letter..

should be

..for each type of letter except  $d$ ..

Without this adjustment the answer is zero.

(ii) page 202, problem 33. The quantity

$a(n,k)$  is defined incorrectly.

In the correct definition,  $a(n,k)$  is the  
number of ways to place the  $k$  nonattacking rooks  
in the forbidden positions.

Chapter 7 OK

Chapter 8 OK

Chapter 14 OK

Math 475

Introduction to Combinatorics

Paul Terwilliger

- Hand out syllabus

Text: Introductory Combinatorics 5th Ed  
Richard Brualdi

- Will cover ch 1-9, 14
- Book is a classic - we will follow it closely
- ~weekly hw assignments will be graded  
(see syllabus)
- 1st assignment:

Read ch 1 and do hw listed in syl

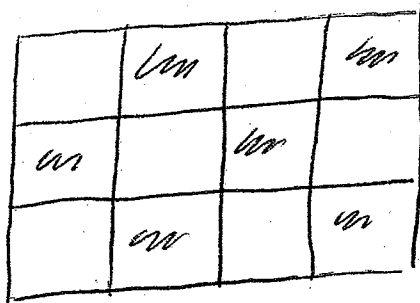
- Today: a few topics from ch 1
- Friday: start ch 2

# Perfect covers of chessboards

Given positive integers  $m, n$

Consider  $m \times n$  chessboard

ex  $m=3$   $n=4$

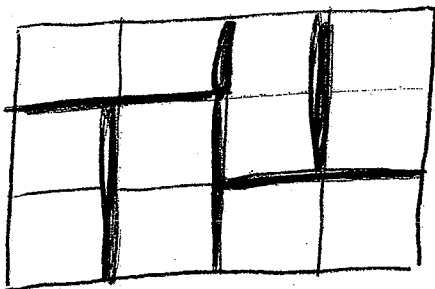


A domino consists of 2 squares



Consider a partition of the chessboard into dominoes  
"perfect cover"

ex



Assume our  $m \times n$  chess bd has a perfect cover by dominoes

let  $T = \#$  dominoes used

Obs  $2T = mn = \#$  squares in chessbd.

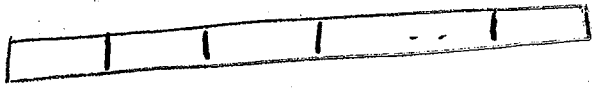
So  $mn$  is even

Assume  $mn$  is even.

How many perfect covers by dominoes are there?

ex  $m=1$   $n$  even


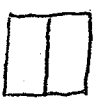
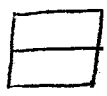
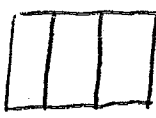

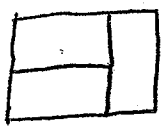
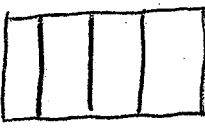

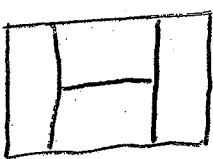
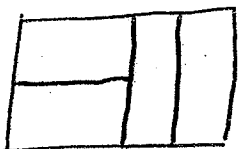
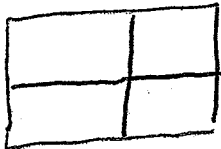
Unique perfect cover is



Try  $m=2$

For  $n=1, 2, \dots$  let  $h_n = \#$  perfect covers of  $2 \times n$  chessbd by dominoes

Find  $h_n$

$n$	examples	$h_n$
1		1
2	 	2
3	  	3
4	    	5

LEMMA 1 the sequence  $h_1, h_2, \dots$  is uniquely <sup>9/5/12</sup> determined by 5

$$h_n = h_{n-1} + h_{n-2} \quad n \geq 3$$

$$h_1 = 1, \quad h_2 = 2$$

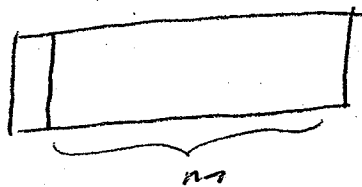
"Fibonacci" sequence "

pf Given a perfect cover of a  $2 \times n$  chessboard by dominoes

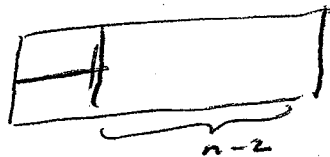
Consider dominoes that cover column 1

Either 1 vertical domino

or 2 horizontal dominoes



or



$h_{n-2}$

# choices

$h_{n-1}$

So  $h_n = h_{n-1} + h_{n-2}$

□



EX 2. A  $2 \times 10$  chessboard has how many perfect covers by dominoes?

Sol

$n$	1	2	3	4	5	6	7	8	9	10
$h_n$	1	2	3	5	8	13	21	34	55	89

ans = 89

Ex Given integer  $b \geq 2$

Suppose  $\exists$  perfect cover of  $m \times n$  chessbd using  $1 \times b$  tiles

let  $T = \#$  tiles used

Observe

$$Tb = mn = \# \text{ squares in chessbd}$$

So

$b$  is a factor of  $mn$

" $b$  divides  $mn$ "  
or " $b \mid mn$ "

LEM 3

In fact  $b \mid m$  or  $b \mid n$

pf:

Divide  $m$  by  $b$  and consider remainder  $r$   
..  $n$  ..  $b$  ..  $a$

So

$$m = pb + r$$

$$n = qb + a$$

$$p, r \in \mathbb{Z}, \quad 0 \leq r < b$$

$$q, a \in \mathbb{Z}, \quad 0 \leq a < b$$

Observe

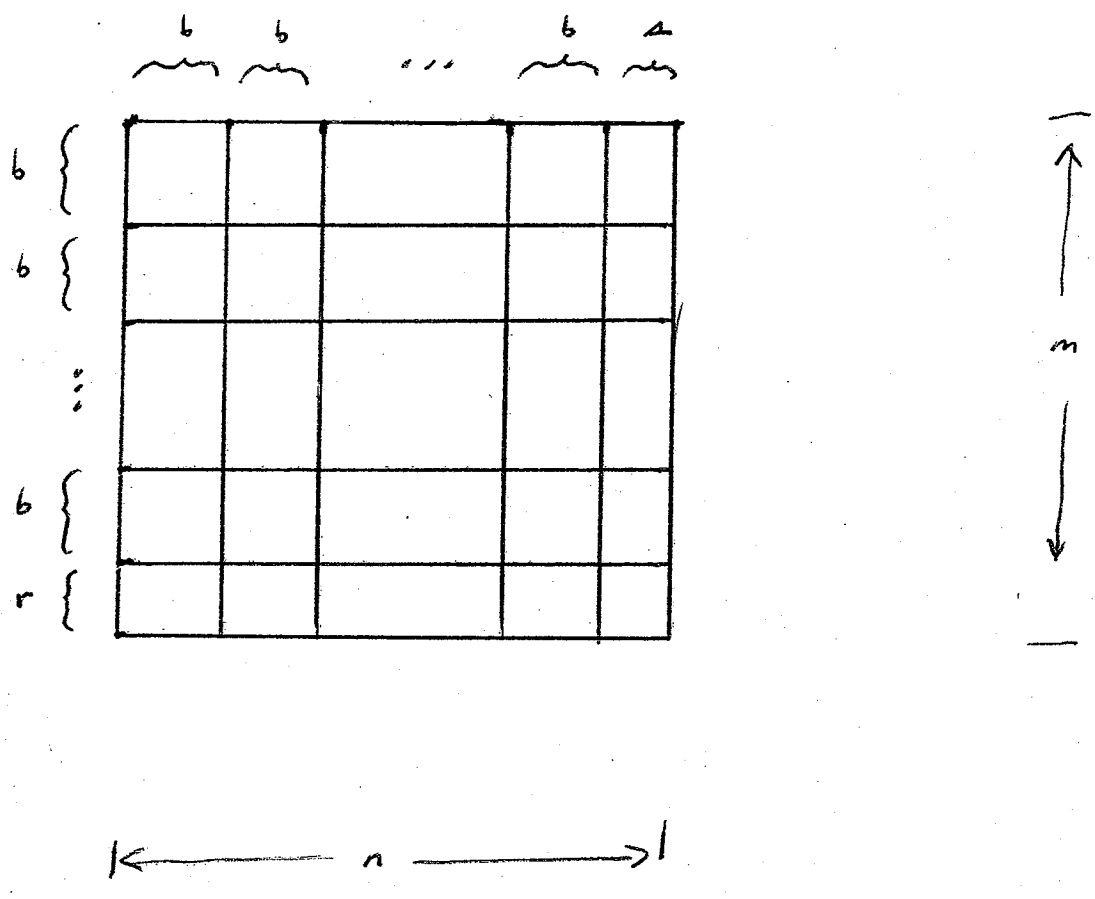
$$r=0 \quad \text{iff} \quad b \mid m$$

$$a=0 \quad \text{iff} \quad b \mid n$$

Suf to show  $r=0$  or  $a=0$

I will assume  $r \neq 0, a \neq 0$  and get contradiction.

Partition the squares of the chessboard as follows



In chessbd give each square a label from 1, 2, ..., 6  
as follows:

1 2 -- 6 2 3 -- 6 1 : : 6 1 -- 6	1 2 -- 6 2 3 -- 1 : : 6 -- 6	...	1 2 -- 6 2 3 -- 1 : : 6 -- 6	1 2 -- 6 2 3 -- 1 : : 6 -- 6
1 2 -- 6 2 3 : : 6 1 -- 6				
: : :				
1 2 6 2 3 : : 6 1 -- 6				
1 2 6 2 3 : : r		...		1 2 -- 6 2 3 -- : : r

Consider our perfect cover of chessbd by  $1 \times b$  tiles

Obs: Each  $1 \times b$  tile covers one each of  $1, 2, \dots, b$  (in some order)

let  $T = \# \text{ tiles}$

$$\begin{aligned}
 T &= \# \text{ of squares in chessbd labelled } 1 \\
 &= \dots && 2 \\
 &= \dots && \vdots \\
 &= \dots && b
 \end{aligned}$$

Consider our partition of the chessbd.

- For each  $b \times b$  block exactly  $b$  squares labelled  $1, 2, \dots, b$
- For each  $r \times b$  block exactly  $r$  squares labelled  $1, 2, \dots, b$
- For each  $b \times s$  block exactly  $s$  squares labelled  $1, 2, \dots, b$

For the unique  $r \times s$  block

must have

$$\begin{aligned}
 &= \# \text{ squares labelled } 1 \\
 &= \# \text{ squares labelled } 2 \\
 &= \dots \\
 &= \# \text{ squares labelled } b
 \end{aligned}$$



Given a perfect matching of an  $m \times n$  chessbd  
by  $1 \times b$  tiles

Call it trivial whenever all the tiles are  
oriented the same way:

- a all horizontal
- a all vertical.

Cor 4. Given an integer  $b \geq 2$  and a  $m \times n$   
chessboard TFAE:

- (i)  $\exists$  perfect cover of chessbd using  $1 \times b$  tiles
- (ii)  $\exists$  trivial perfect cover of chessbd using  $1 \times b$  tiles

pf (i)  $\rightarrow$  (ii) By Lem 3  
 $b \mid m$  or  $b \mid n$

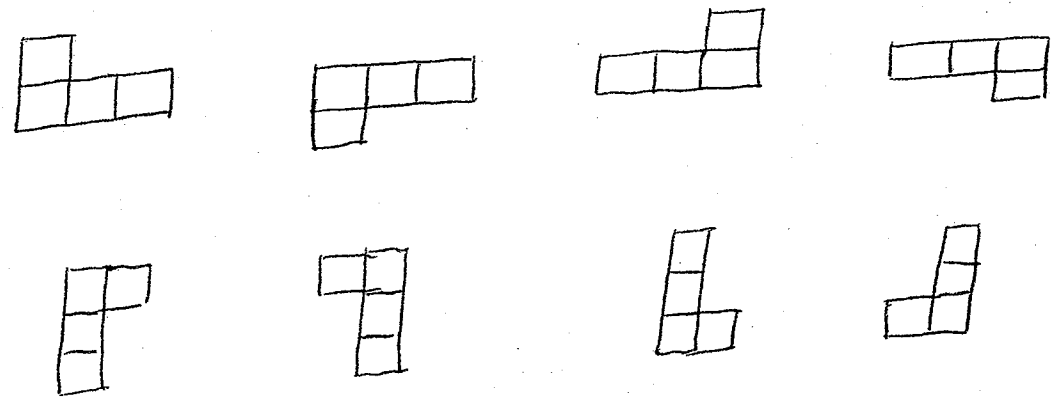
wlog  $b \mid m$   
obs  $\exists$  perfect cover of chessbd with all tiles  
vertical

(ii)  $\rightarrow$  (i) clear



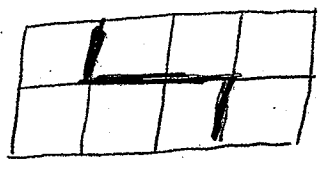
tetromino coverings of chessbds

A tetromino consists of an L-shape of 4 squares



Which  $m \times n$  chessbds have a perfect covering tetrominoes?

ex  $m=2$   $n=4$



Given  $m \times n$  chessbd suppose such a perfect cover exists

let  $T = \#$  tiles

$$4T = mn$$

So  $4 \mid mn$

This condition is nec but not suf, since  $m=2, n=2$  is counterex.



LEM For a  $m \times n$  chessbd suppose  $\exists$  perfect cover by tetrominoes. then  $8 | mn$

pf Let  $T = \#$  tiles used

$$4T = mn$$

wlog  $m$  even write  $m = 2r$

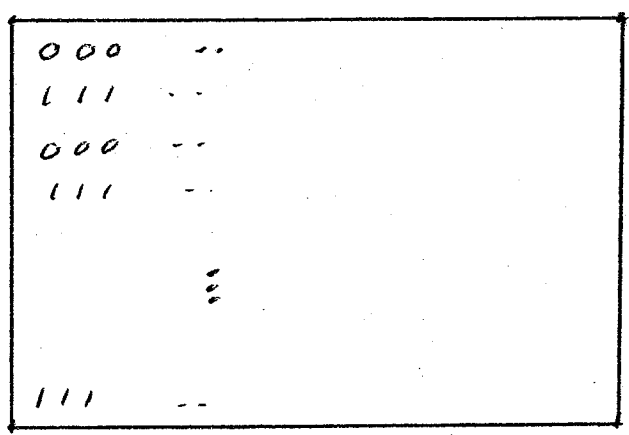
$$4T = 2rn$$

$$2T = rn$$

For  $m \times n$  chessbd label each square as follows:

$n$

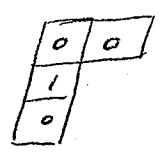
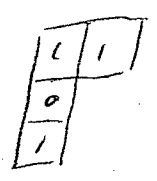
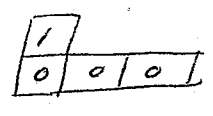
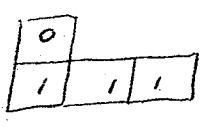
$m$



$$\begin{aligned} \text{Sum of Labels} &= rn \\ &= \text{even} \end{aligned}$$

Consider tetrominoes in the perfect cover

possible Labellings :



In all cases sum of labels is odd

So

even = sum of all labels on  $m \times n$  chessbd

$$= \sum_{i=1}^T \underbrace{\left( \text{sum of labels in } i^{\text{th}} \text{ tetra} \right)}_{\text{odd}}$$

So

T is even

We saw

$$2T = rn$$

So  $4 \mid rn$

$m=2r$

So  $8 \mid mn$

□

Matchings

Given a positive integer  $n$

Suppose  $2n$  people at a party pair off  
to form  $n$  teams of 2

In how many ways can this be done?

Call this number  $M_n$

Ex.  $n=2$  Label the persons 1, 2, 3, 4

1-2

3-4

1 2  
1 1  
3 4

1 2  
1  
3 4

$$M_2 = 3$$

LEM

$$M_n = (2n-1)(2n-3)(2n-5) \cdots 3 \cdot 1$$

$$n = 1, 2, \dots$$

pf

Label the persons  $1, 2, \dots, n$ .

Consider the partner for person 1: #choices =  $2n-1$

Remove person 1 and their partner

$2n-2$  persons left.

# ways to pair them off is  $M_{n-1}$

So

$$M_n = (2n-1) M_{n-1}$$

$$n=2, 3, \dots$$

obs

$$M_1 = 1$$

Result follows by induction on  $n$

□

## Ch 2 Permutations and combinations

## 2.1 4 basic counting principles

Given a set  $S$

A partition of  $S$  is a decomposition

of  $S$  into subsets such that

each element of  $S$  is contained in exactly one of the subsets.

Ex  $S = \{1, 2, 3, a, b\}$

$$A = \{1, a\}$$

$$B = \{2, b\}$$

$$C = \{3\}$$

$A, B, C$  partition  $S$

For a set  $S$  define

$$|S| = \text{number of elements in } S$$

"cardinality of  $S$ "

"size of  $S$ "

For above example

$$\begin{aligned} |S| &= 5 = 2+2+1 \\ &= |A| + |B| + |C| \end{aligned}$$

This illustrates

Addition Principle

Suppose a set  $S$  is partitioned

into subsets  $S_1, S_2, \dots, S_k$ .

Then

$$|S| = |S_1| + |S_2| + \dots + |S_k|$$

Given sets  $A, B$

their Cartesian product denoted  $A \times B$

is the set of all ordered pairs  $(a, b)$  with  $a \in A$   
and  $b \in B$

Ex  $A = \{1, 2, 4\}$

$B = \{r, s\}$

els of  $A \times B$  are in table:

		B	
		r	s
A	1	(1, r)	(1, s)
	2	(2, r)	(2, s)
	4	(4, r)	(4, s)

Obs

$$|A \times B| = 6 = 3 \times 2 = |A| |B|$$

In general for any finite sets  $A, B$

$$|A \times B| = |A| |B|$$

9/7/12 4

Ex For an integer  $n \geq 1$  Find the number of sequences

$$a_1, a_2, \dots, a_n$$

$$a_i \in \{0, 1\} \quad 1 \leq i \leq n$$

Sol Def  $A = \{0, 1\}$

View set of sequences as Cartesian product

$$\underbrace{A \times A \times \dots \times A}_{n \text{ copies}}$$

$$\begin{aligned} \# \text{ sequences} &= |A|^n \\ &= 2^n \end{aligned}$$



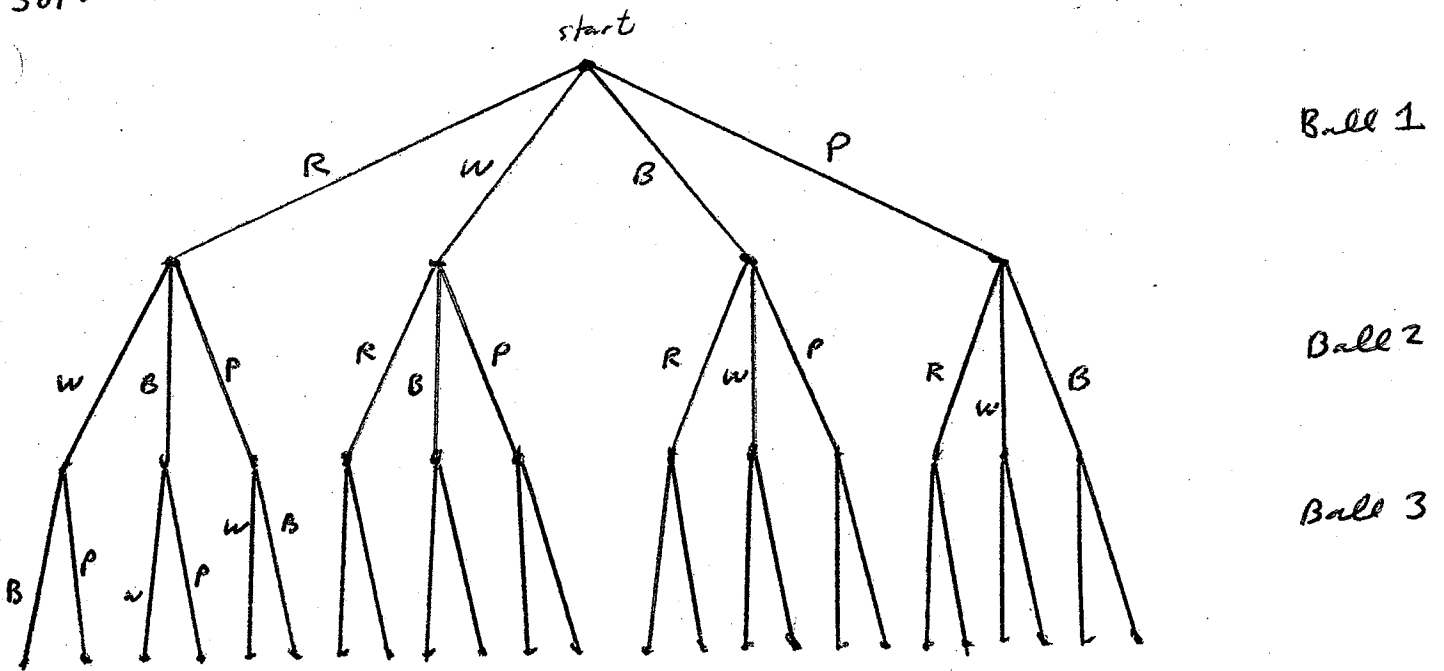
Prob: A box contains

- 1 Red ball
- 1 white ball
- 1 blue ball
- 1 purple ball

3 balls are withdrawn, one at a time, without replacement and colors noted.

How many outcomes are possible?

Sol: Illustrate with "tree diagram"



Each outcome is represented by a path in the tree diagram, from "start" to a "leaf"

$$\# \text{outcomes} = 4 \times 3 \times 2 = 24$$

Above examples illustrate the

9/7/12 6

## Multiplication principle

Consider a multistage experiment with  $k$  stages

Assume:

- stage 1 has  $n_1$  possible outcomes
- stage 2 has  $n_2$  possible outcomes  
(regardless of result of stage 1)
- stage 3 has  $n_3$  possible outcomes  
(regardless of results of stages 1, 2)
- $\vdots$
- stage  $k$  has  $n_k$  possible outcomes  
(regardless of results of prev stages)

then the total number of outcomes is

$$n_1 \times n_2 \times \dots \times n_k$$

Ex For an integer  $n \geq 0$

the set  $\{1, 2, \dots, n\}$  has how many subsets?

Sol. View the situation as an  $n$ -stage experiment in which we select a subset

stage	to do	# choices
1	decide is "1" in subset	2
2	decide is "2" in subset	2
⋮		⋮
$n$	decide is " $n$ " in subset	2

( $2^n$ )

By multi principle the # subsets is  $2^n$

Given sets  $S \subseteq U$

The complement of  $S$  in  $U$  is

$$\bar{S} = \{x : x \in U \text{ and } x \notin S\}$$

We have partition

$$U = S \cup \bar{S} \quad (\text{disjoint})$$

So  $|U| = |S| + |\bar{S}|$

In other words

Subtraction principle : For  $|U| < \infty$  and  $S \subseteq U$ ,

$$|\bar{S}| = |U| - |S|$$

Ex A product code consists of a string of 4 dist letters taken from the set  $\{A, H, Q, T, Z\}$

How many such codes contain the letter "H" ?

Sol View

$U$  = set of all product codes

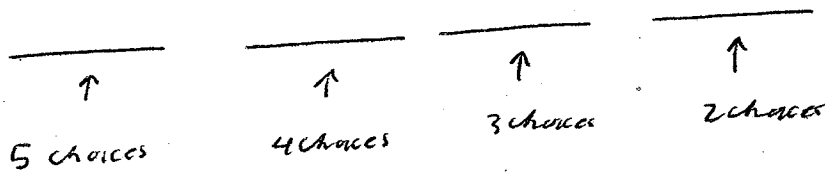
$S$  = set of product codes that do not contain H

Find  $|\bar{S}|$  using

$$|\bar{S}| = |U| - |S|$$

Find  $|U|$ :

Construct prod code by filling in blanks left to right

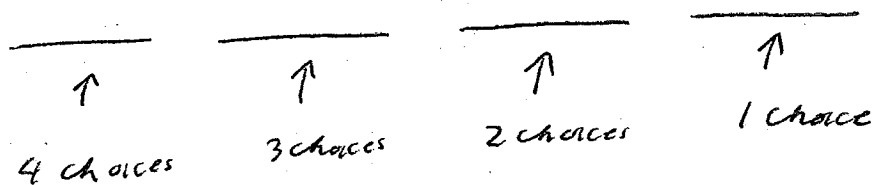


By mult principle

$$|U| = 5 \times 4 \times 3 \times 2 = 120$$

9/7/12 10

Find  $|S|$  :



$$|S| = 4 \times 3 \times 2 \times 1$$

$$= 24$$

$$|\bar{S}| = 120 - 24$$

$$= 96$$

D

Division principle

Given a finite set  $U$

partition  $U$  into subsets with equal cardinality.

Then

$$\text{the number of subsets} = \frac{|U|}{\text{cardinality of each subset}}$$

Ex Given integers  $3 \leq k \leq n$

We create a pearl necklace as follows.

Given  $n$  distinct pearls

Necklace uses  $k$  of these pearls.

How many necklaces are possible?

[two necklaces are same if they differ by rot / refl]

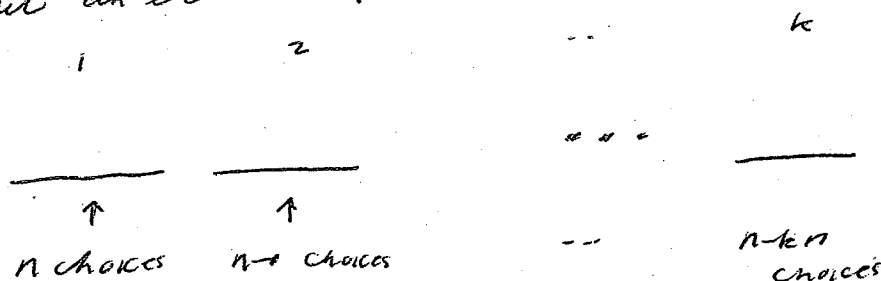
Sol: Call the pearls  $1, 2, \dots, n$ .

Define

$U =$  set of all sequences of  $k$  pearls taken from  $\{1, 2, \dots, n\}$

First find  $|U|$ :

To select an element of  $U$ , fill in blanks left to right.



By multiplication principle

$$|U| = n(n-1) \dots (n-k+1)$$

Define a binary relation  $\sim$  on  $U$  as follows.

$$\forall x, y \in U$$

$x \sim y$  whenever  $y$  is obtained from  $x$  by rot/refl.

$\sim$  is an equiv relation.

Each equiv class consists of exactly  $2k$  elements



Each necklace corresponds to an equiv class of  $\sim$

$$\begin{aligned} \# \text{necklaces} &= \# \text{equiv class for } \sim \\ &= \frac{|U|}{\text{card of each equiv class}} \\ &= \frac{n(n-1) \dots (n-k+1)}{2k} \end{aligned}$$



## 2.2 Permutations of sets

Given a set  $S$

Given integer  $r \geq 0$

An  $r$ -permutation of  $S$  is a sequence

$$a_1, a_2, \dots, a_r$$

where  $a_1, a_2, \dots, a_r$  are distinct elements of  $S$ .

By a permutation of  $S$  we mean an  $r$ -perm where  $r = |S|$

Ex  $S = \{a, b, c\}$

the permutations of  $S$  are

$abc$	$cba$
$bca$	$acb$
$cab$	$bac$

the 2-perms of  $S$  are

$ab$	$ba$	$ca$
$ac$	$bc$	$cb$

the 1 perms of  $S$  are

$a$                        $b$                        $c$

No  $r$ -perms of  $S$  for  $r > 3$

Ex Given finite set  $S$  write  $n = |S|$

For  $0 \leq r \leq n$  find the number of  $r$ -permutations of  $S$

Sol Select an  $r$ -perm  $a_1 a_2 \dots a_r$

in stages

stage	to do	# choices
1	pick $a_1$	$n$
2	pick $a_2$	$n-1$
$\vdots$	$\vdots$	$\vdots$
$r$	pick $a_r$	$n-r+1$

By multiplication principle

$$\# \text{ } r\text{-perms of } S = \underbrace{n(n-1)(n-2) \dots (n-r+1)}_{\text{call this } P(n,r)}$$

9/10/12 2

For an integer  $n \geq 0$  define

$$n! = n(n-1)(n-2) \dots 2 \cdot 1$$

By convention  $0! = 1$

Obs

$$P(n, r) = \frac{n!}{(n-r)!}$$

Obs

$$\begin{aligned} \# \text{ perms of } S &= P(n, n) \\ &= n! \end{aligned}$$

"n factorial"

Ex At a party there are 8 women and 11 men. How many ways are there to form 8 couples consisting of 1 man and 1 woman?

Sol. Label women 1, 2, ..., 8

Proceed in stages:

stage	to do	# choices
1	select partner for w1	11
2	...	10
...	...	...
8	...	4

By mult principle

$$\begin{aligned} \# \text{ ways} &= 11 \times 10 \times \dots \times 4 \\ &= P(11, 8) \end{aligned}$$

□

EX Given set  $S$  of people  $|S|=n$

For  $1 \leq r \leq n$  seat  $r$  people from  $S$  around circular

table. Only care about who sits to your left/right

"circular  $r$ -perm"

$$\left[ \begin{array}{ccc} r=4 & & \\ d & \begin{matrix} a \\ b \\ c \end{matrix} & = & \begin{matrix} d \\ c \\ b \end{matrix} & a & \neq & \begin{matrix} d \\ a \\ b \end{matrix} & c \end{array} \right]$$

How many arrangements possible?

Sol (Sim to necklace problem)

Let  $U$  = set of all  $r$ -perms of  $S$

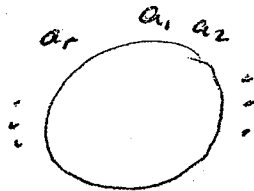
$$|U| = P(n,r)$$

Given  $r$ -perm  $a_1, a_2, \dots, a_r$  in  $U$

wrap  $a_1, a_2, \dots, a_r$  around table clockwise

$a_1, a_2, \dots, a_r$

$\Rightarrow$



The following  $r$ -perms get sent to the same table arrangement:

$$a_1 a_2 \dots a_r$$

$$a_2 a_3 \dots a_r a_1$$

$$a_3 \dots a_r a_1 a_2$$

$$\vdots$$

$$a_r a_1 \dots a_{r-1}$$

(\*)

Define a binary relation  $\sim$  on  $\mathcal{U}$  as follows:

$$\forall x, y \in \mathcal{U}$$

$x \sim y$  whenever  $x, y$  give same table arrangement

$\sim$  is equiv relation

Each equiv class has form (\*) with  $r$  elements

Each table arrangement corresponds to an equiv class  $\sim$

$$\begin{aligned} \# \text{ table arrangements} &= \# \text{ equiv classes of } \sim \\ &= \frac{|\mathcal{U}|}{\text{card of each equiv class}} \\ &= \frac{P(n, r)}{r} \end{aligned}$$

□

In summary

Then Given a finite set  $S$  with  $|S|=n$

For  $1 \leq r \leq n$  the number of circ  $r$ -permutations of  $S$

$$\text{is } \frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$$

In particular taking  $r=n$ , the number of

circular permutations of  $S$  is

$$\frac{n!}{n} = (n-1)!$$

### 2.3 Combinations of Sets

Given a set  $S$   $|S|=n$

For  $0 \leq r \leq n$

By an  $r$ -combination (or  $r$ -subset) of  $S$

we mean a subset of  $S$  with cardinality  $r$ .

View as selection of  $r$  objects from  $S$  where selection order does not matter.

Ex Find the number of  $r$ -combinations of  $S$ .

Sol. Call this number  $C(n,r)$

To find it, we count the number of  $r$ -perms of  $S$

as follows.

We construct an  $r$ -perm from  $S$  using this 2-stage procedure:



stage	to do	# choices
1	select an $r$ -combination $X$ from $S$	$C(n,r)$
2	order the elements of $X$	$r!$

By multiplication principle

$$\begin{aligned} \# \text{ } r\text{-perms of } S &= C(n,r) r! \\ \text{"} & \\ P(n,r) & \end{aligned}$$

$$\begin{aligned} \text{So } C(n,r) &= \frac{P(n,r)}{r!} \\ &= \frac{n!}{r!(n-r)!} \end{aligned}$$

Other notations for  $C(n,r)$ :

${}^n C_r$

$$\binom{n}{r}$$

"binomial coefficient"

↑

we will use

Convention

$\binom{n}{r} = 0$  if  $r < 0$  or  $r > n$

Note

$\binom{n}{0} = 1$

$\binom{n}{n} = 1$

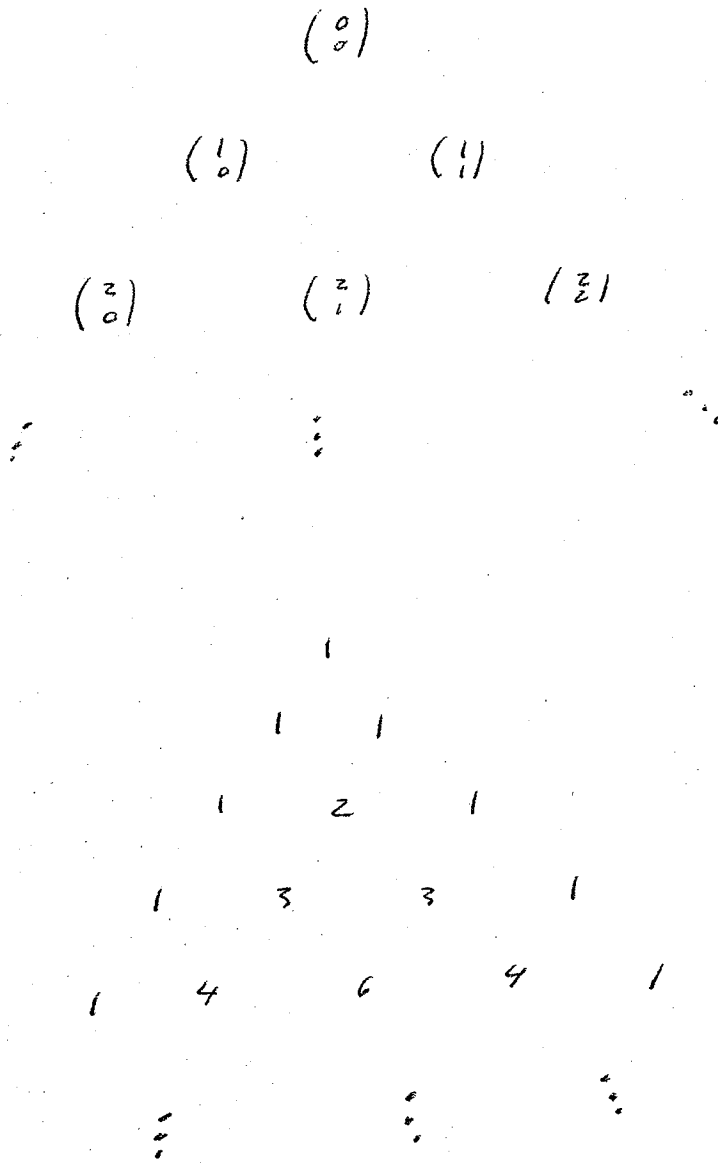
$\binom{n}{r} = \binom{n}{n-r}$

$0 \leq r \leq n$

Recall "Pascal triangle"

9/10/12 10

the numbers



are

then  $F_{n+1} = F_n + F_{n-1}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

pf 1 Use formula  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

pf 2

Consider set

$$S = \{1, 2, \dots, n\}$$

Partition the set of  $k$ -subsets of  $S$  into 2 types:

type of $k$ -subset of $S$	# choices
$k$ -subsets of $S$ that contain "1"	$\binom{n-1}{k-1}$ ← # ways to select remaining $k-1$ elements from $\{2, \dots, n\}$
$k$ -subsets of $S$ that do not contain "1"	$\binom{n-1}{k}$ ← # ways to select $k$ elements from $\{2, 3, \dots, n\}$

So by addition principle

$$\begin{aligned} \binom{n}{k} &= \# k\text{-subsets of } S \\ &= \binom{n-1}{k-1} + \binom{n-1}{k} \end{aligned}$$

□

Here is another formula for binom coefficients

thm For an integer  $n \geq 0$

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

pf. Consider the set

$$S = \{1, 2, \dots, n\}$$

$$2^n = \# \text{ subsets of } S$$

$$= \# \text{ 0-subsets of } S$$

$$+ \# \text{ 1-subsets of } S$$

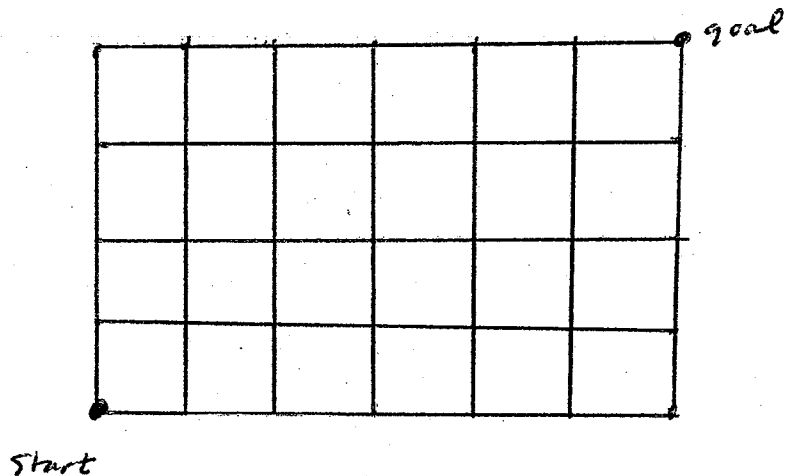
$$+ \vdots$$

$$+ \# \text{ n-subsets of } S$$

$$= \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

□

Ex Consider 4x6 grid of city streets!



goal is 6 blocks East  
4 blocks North

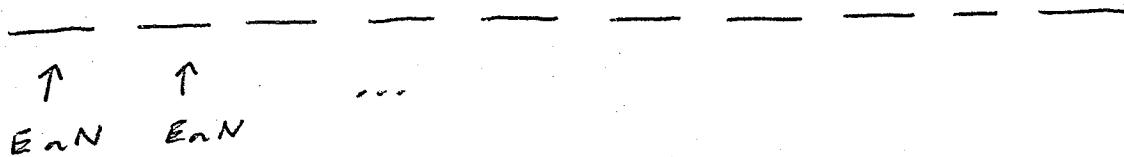
Go from "start" to "goal" in 10 steps, each N or E

How many paths?

Sol

Consider 10 steps of path

1    2    3    4    5    6    7    8    9    10



Fill each blank with "E" or "N"

Require 6 E's and 4 N's.

For each path,

the set of blanks labelled N is a 4-subset of  $\{1, 2, \dots, 10\}$

set of paths  $\overset{\text{bij}}{\iff}$  4-subsets of  $\{1, 2, \dots, 10\}$

$$\begin{aligned} \# \text{ paths} &= \# \text{ 4-subsets of } \{1, 2, \dots, 10\} \\ &= \binom{10}{4} \end{aligned}$$

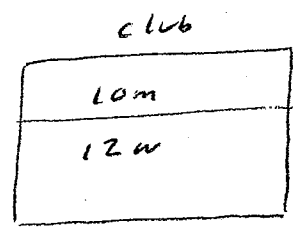
□

Ex A committee of 5 people is chosen from a club that has a membership of 10 men and 12 women.

The Committee must have at least 2 women.

How many ways to form committee?

Sol



"choose 5"  
at least 2 w

We partition the committees according to the number of women

# women	# ways to pick men	# ways to pick women	# committees
2	$\binom{10}{3}$	$\binom{12}{2}$	$\binom{10}{3} \binom{12}{2}$
3	$\binom{10}{2}$	$\binom{12}{3}$	$\binom{10}{2} \binom{12}{3}$
4	$\binom{10}{1}$	$\binom{12}{4}$	$\binom{10}{1} \binom{12}{4}$
5	$\binom{10}{0} = 1$	$\binom{12}{5}$	$\binom{12}{5}$

# committees =  $\binom{10}{3} \binom{12}{2} + \binom{10}{2} \binom{12}{3} + \binom{10}{1} \binom{12}{4} + \binom{12}{5}$



## 2.4 Permutations of multisets

We introduce the notion of a multiset with an example

Ex the letters in

MISSISSIPPI

form multiset

$$\{ I, I, I, I, M, P, P, S, S, S, S \}$$

which is abbreviated

$$\{ 4 \cdot I, 1 \cdot M, 2 \cdot P, 4 \cdot S \}$$

↑ ↑ ↑ ↑  
"repetition numbers"

A multiset is like a set except its elements need not be distinct.

Ex letters in

ababab ...

form multiset

$$\{\infty a, \infty b\}$$

Def A multiset  $M$  is a set  $S$

together with a function

$$R: S \rightarrow \{0, 1, 2, \dots\} \cup \infty$$

$\forall x \in S$

$R(x)$  is the repetition number of  $x$

By the cardinality (a size) of  $M$  we mean

$$|M| = \sum_{x \in S} R(x)$$