[Monday Nov 26 Exam II]

Lecture 33 Wed Nov 28

We will do ch 14 next, and return to ch 9 time permitting

Ch 14 Polya counting

14.1 Permutations and symmetry groups

Let X = nonempty finite set

Say $X = \{1, 2, \dots, n\}$

Consider perm of X:

$a_1 a_2 \cdots a_n$

View this as a bijection

$$X \longrightarrow X$$
$$i \longrightarrow a_i$$

To emphasize this view we often write

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

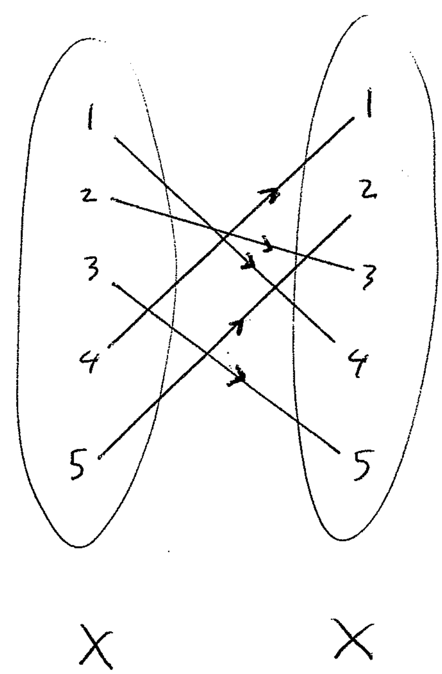
the bijection sends each number in top row to the number beneath it

Ex $n=5$

the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$$

satisfies



Def For $n \geq 1$

S_n = set of all perms of $\{1, 2, \ldots, n\}$

Composition of permutations

Given perms $f : X \to X$ $g : X \to X$

their <u>composition</u> $f \circ g : X \to X$ satisfies

$$(f \circ g)(x) = f(g(x)) \quad \forall x \in X$$

" First apply g and then apply f "

$$X \xrightarrow{g} X \xrightarrow{f} X$$

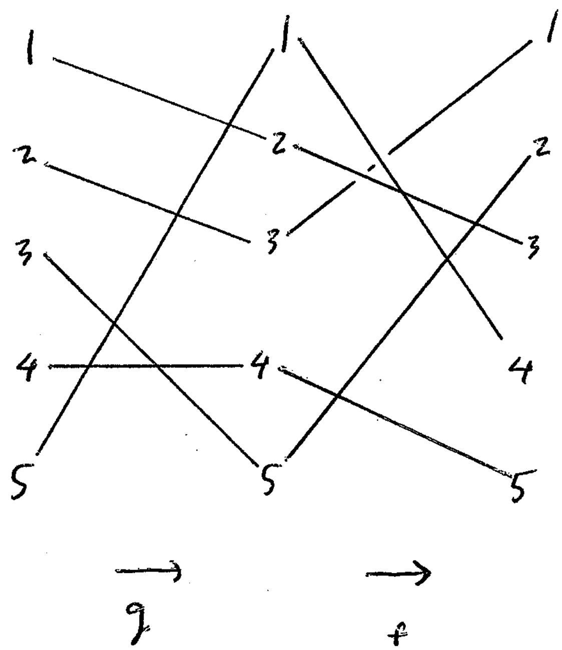
$f \circ g$: $x \longrightarrow g(x) \longrightarrow f(g(x))$

Ex fn perms

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$$

Find $f \circ g$

Sol

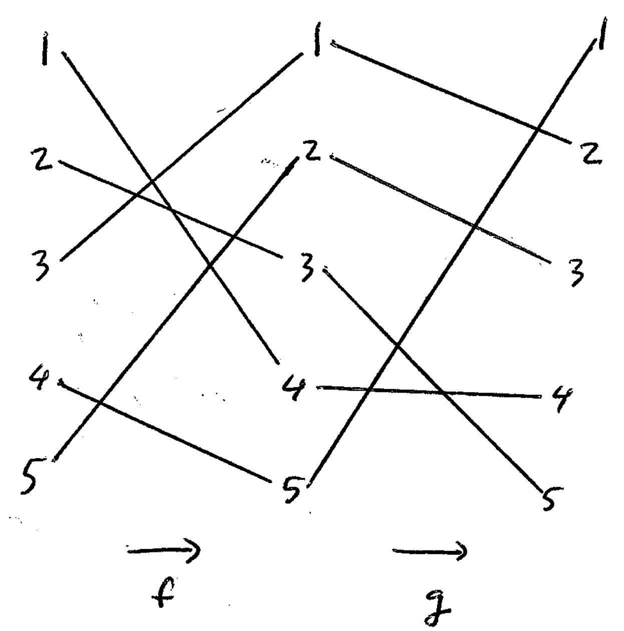


$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

$$\begin{aligned} (f \circ g)(1) &= f(g(1)) \\ &= f(2) \\ &= 3 \end{aligned}$$

etc.

Ex Referring to above f, g find g∘f



$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

Note $g \circ f \neq f \circ g$ in general

— o —

We view composition ∘ as a binary operation on S_n: given f and g in S_n their composition $f \circ g$ is an element of S_n

LEM For $f, g, h \in S_n$

$$(f \circ g) \circ h = f \circ (g \circ h) \qquad *$$

" composition is associative "

pf Each side of $*$ is a function $X \to X$

where $X = \{1, 2, \ldots, n\}$

Show each function sends each $x \in X$ to the same thing

$$(f \circ g) \circ h : \quad x \xrightarrow{h} h(x) \xrightarrow{f \circ g} (f \circ g)(h(x)) = f(g(h(x)))$$

$$\shortparallel$$

$$f \circ (g \circ h) : \quad x \xrightarrow{g \circ f} (g \circ h)(x) = g(h(x)) \xrightarrow{f} f(g(h(x)))$$

$\square$

From now on we drop parenthesis and write

$$f \circ g \circ h$$

We abbrev

$$f^2 = f \circ f, \qquad f^3 = f \circ f \circ f \qquad f^1 = f$$

etc

DEF The <u>identity</u> permutation

$$I : X \to X$$

satisfies

$$I(x) = x \qquad \forall x \in X$$

" I leaves everything alone "

So

$$I = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

LEM For a perm $f : X \to X$,

$$f \circ I = f$$

$$I \circ f = f$$

pf $\forall x \in X$

$$(f \circ I)(x) = f(\underbrace{I(x)}_{"x})$$
$$= f(x)$$

Also

$$(I \circ f)(x) = I(f(x))$$

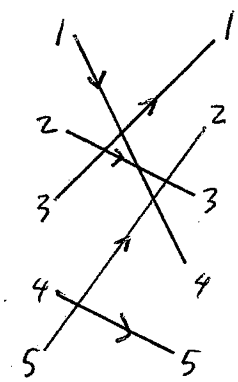
$$= f(x)$$

$\square$

Inverses

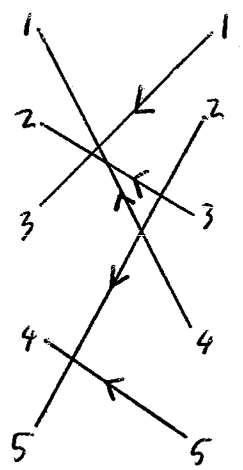
Given perm $f: X \to X$

say $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}$



f

Change direction of arrows to get a new perm f^{-1}:



$\longleftarrow$

f^{-1}

$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$

Call f^{-1} the inverse of f

" f^{-1} undoes f "

Given a perm $f: X \to X$

then f^{-1} is described as follows:

<u>View I</u> For all $x, y \in X$

$f(x) = y$ if and only if $f^{-1}(y) = x$

<u>View II</u> For all $x \in X$

$f^{-1}(f(x)) = x$ ie $f^{-1} \circ f = I$

<u>View III</u> For all $y \in X$

$f(f^{-1}(y)) = y$ ie $f \circ f^{-1} = I$

LEM Given perm $f: X \to X$

then for all perms $g: X \to X$

the following are equivalent:

(i) $g \circ f = I$

(ii) $f \circ g = I$

(iii) $g = f^{-1}$

pf (iii) → (i) From View II alone

(iii) → (ii) From View III alone

(i) → (iii)

$$g \circ f = I$$

$$(g \circ f) \circ f^{-1} = I \circ f^{-1}$$

$$\| \qquad\qquad\qquad \|$$

$$g \circ (f \circ f^{-1}) \qquad\qquad f^{-1}$$

$$\|$$

$$g \circ I$$

$$\|$$

$$g$$

(ii) → (iii) Similar

∎

EX: Given perms

$f: X \to X, \qquad g: X \to X$

find $(f \circ g)^{-1}$

Sol

$f \circ g: \quad X \xrightarrow{g} X \xrightarrow{f} X$

$f \circ g$ is undone by

$(f \circ g)^{-1} \quad X \xleftarrow{g^{-1}} X \xleftarrow{f^{-1}} X$

So

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

check

$$\left(g^{-1} \circ f^{-1}\right) \circ \left(f \circ g\right) \stackrel{?}{=} I$$

$$\| $$

$$\underbrace{g^{-1} \circ \underbrace{f^{-1} \circ f}_{= I} \circ g}_{= g^{-1} \circ g} = I \checkmark$$

More generally, given perms

$f_1 : X \to X,$
$f_2 : X \to X,$
$\dots$
$f_r : X \to X$

$$(f_1 \circ f_2 \circ \dots \circ f_r)^{-1} = f_r^{-1} \circ \dots \circ f_2^{-1} \circ f_1^{-1}$$

In particular, for any perm

$$f : X \to X$$

and $r \geq 1$

$$(f^r)^{-1} = (f^{-1})^r$$

Call this common value f^{-r}

Further define

$$f^0 = I$$

By construction

$$f^r \circ f^s = f^{r+s} \qquad \text{for all integers } r, s$$

$$(f^r)^s = f^{rs}$$

Lecture 34 Friday Nov 30

14.1 Cont.

X = nonempty finite set

Say
$$X = \{1, 2, \dots, n\}$$

Given perm
$$f : X \to X$$

Consider
$$I, f, f^2, f^3, \dots \qquad \ast$$

Finitely many perms $X \to X$

Must be duplication among $\ast$:
$$f^r = f^s \qquad r < s$$

so
$$f^{s-r} = I$$

So $\exists\ m \geq 1$ such that
$$f^m = I$$

Note
$$f^{-1} = f^{m-1}$$

since

$$f \circ f^{m-1} = f^m = I$$

— o —

DEF A <u>permutation group</u> on X is a set

G of perms $X \to X$ such that:

(1) For all $f, g \in G$ "G is closed under composition $\circ$"

$$f \circ g \in G$$

(2) $I \in G$

LEM Given a perm gp G on X.

Then for all $f \in G$

$$f^{-1} \in G$$

"G is closed under taking inverses"

pf Case $f = I$: ok since $I^{-1} = I$

Case $f \neq I$: $\exists m \geq 1$ such that $f^m = I$

$$f^{-1} = f^{m-1}$$

$$= \underbrace{f \circ f \circ \cdots \circ f}_{m-1}$$

$\in G$ since G closed under comp

$\square$

Examples of Permutation Groups

EX 1 Recall

S_n = set of all perms of $X = \{1, 2, \ldots, n\}$

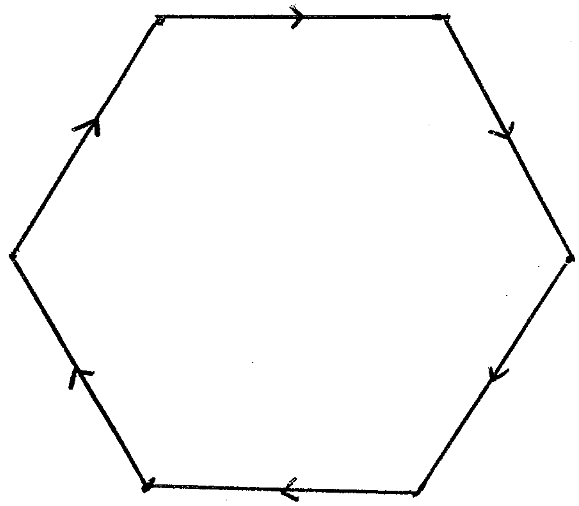
S_n = perm group on X

This group is called the symmetric group of order n

Ex 2 Consider an oriented regular n-gon P in the plane

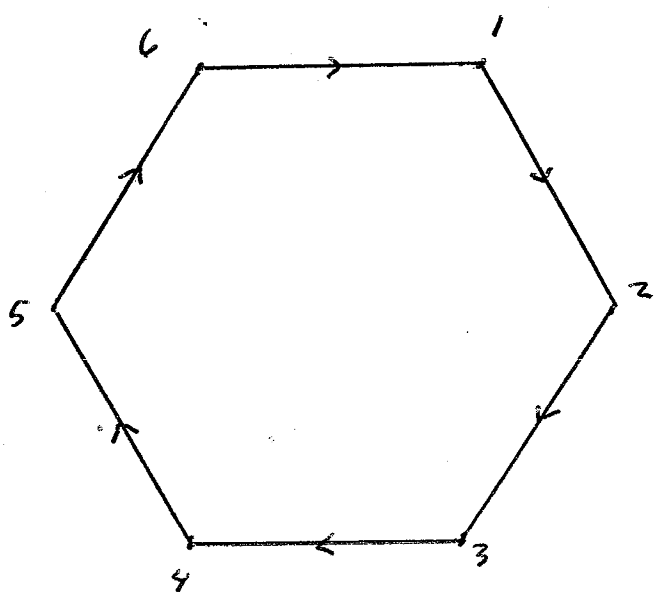
ex n=6

P:



View X = set of corners (vertices) of P

P:



$X = \{1, 2, 3, 4, 5, 6\}$

P has rotational symmetries:

If we rotate P clockwise by some multiple m of 60° $(0 \le m \le 5)$

then result coincides with P_0

Each rotation induces perm of X:

m	perm of X	name of perm
0	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$	I
1	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$	R
2	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$	$R^2 = R \circ R$
3	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$	R^3
4	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$	R^4
5	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	R^5

Note $R^6 = I$

Define

G = set of all rotational symmetries of P

$= \{ I, R, R^2, R^3, R^4, R^5 \}$

So $|G| = 6$

Then G is a perm group on X

"cyclic group of order 6"

inverses:

f	I	R	R^2	R^3	R^4	R^5
f^{-1}	I	R^5	R^4	R^3	R^2	R

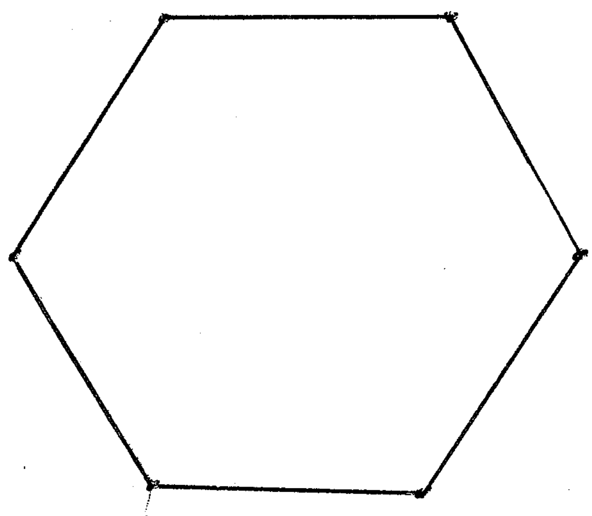
In general, for $n \geq 2$

the <u>cyclic group of order n</u> = group of rotational symmetries of an oriented regular n-gon

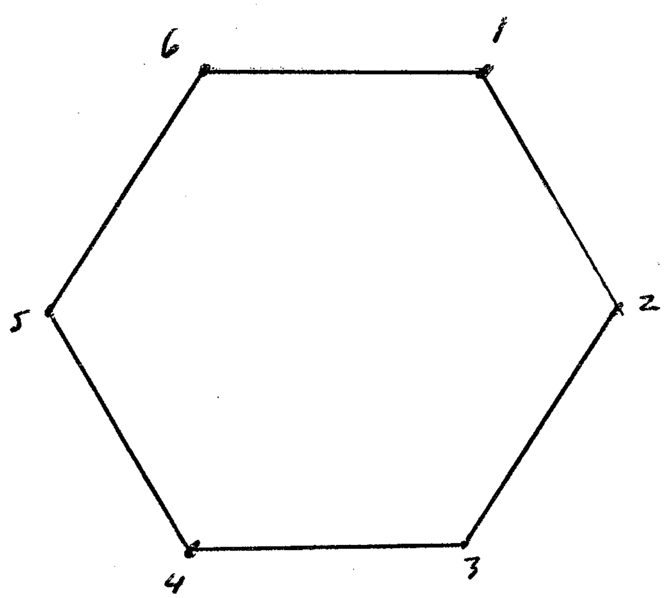
Ex 3 Consider a nonoriented regular n-gon in the plane

$n = 6$

P:



View X = set of vertices of P as before



$X = \{1, 2, 3, 4, 5, 6\}$

P has rotational and reflectional symmetries

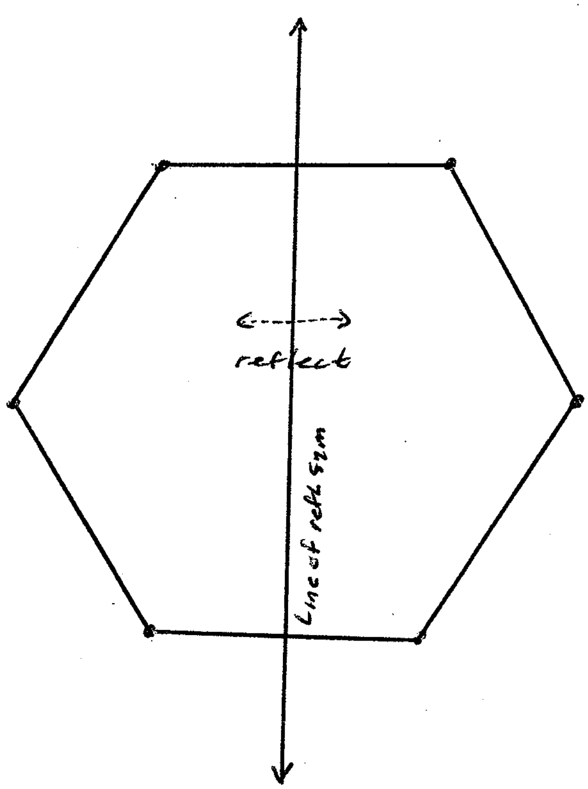
<u>rotational symmetries</u> : 6 of these just as in oriented case

<u>reflectional symmetries</u>

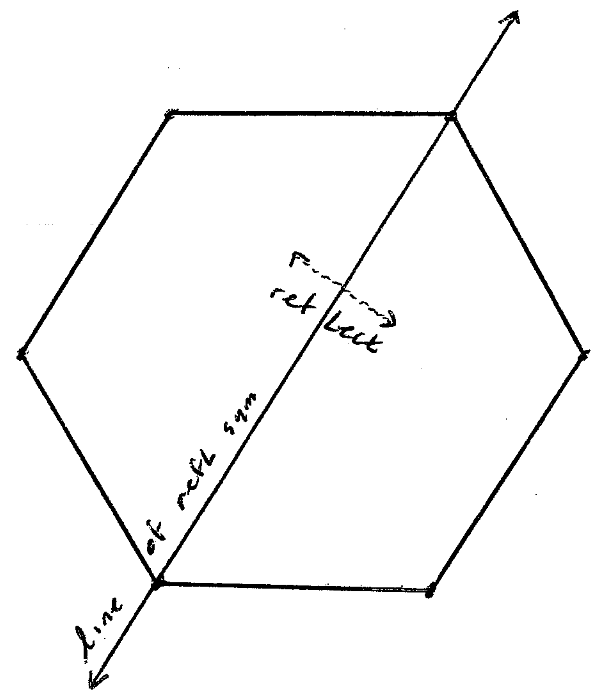
If we reflect P thru a line of reflectional symmetry, the result coincides with P

ex

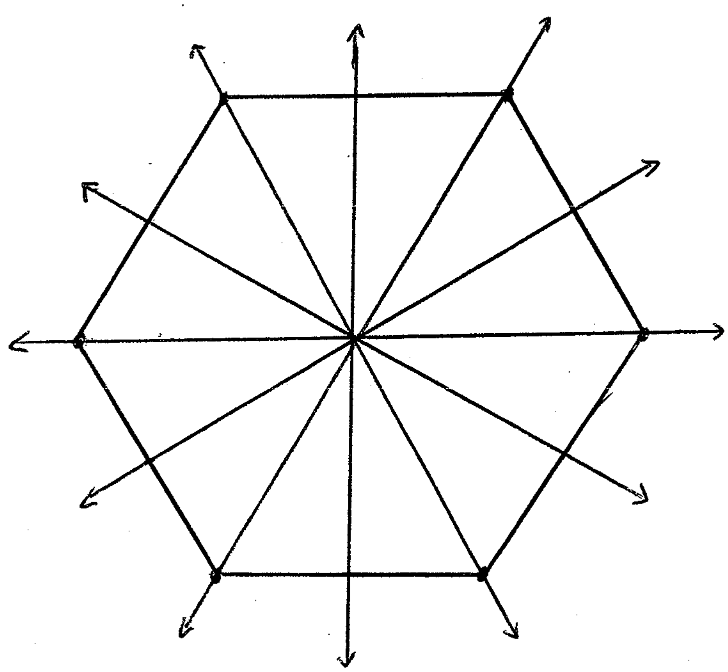
P:



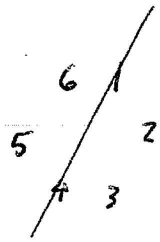
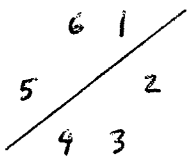
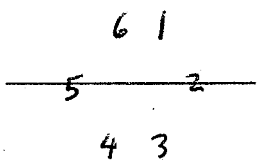
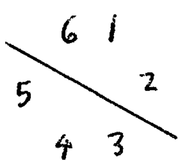
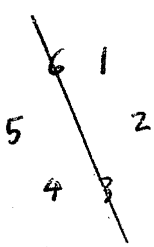
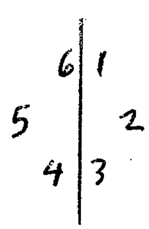
P:



P has 6 lines of reflectional symmetry:



Each reflectional symmetry of P induces perm of X:

line of sym	perm of X	name
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$	τ
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$	$R \circ \tau$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$	$R^2 \circ \tau$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}$	$R^3 \circ \tau$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$	$R^4 \circ \tau$
	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$	$R^5 \circ \tau$

Recall $R^6 = I$

For any reflection

$$(\text{reflection})^2 = I$$

so

each reflection is its own inverse

obs $\quad \tau^2 = I \qquad \tau^{-1} = \tau$

For $0 \le i \le 5$

$$(R^i \circ \tau)^2 = I$$

$$\begin{aligned} R^i \circ \tau &= (R^i \circ \tau)^{-1} \\ &= \tau^{-1} \circ R^{-i} \\ &= \tau \circ R^{-i} \\ &= \tau \circ R^{6-i} \end{aligned}$$

Let G = set of all symmetries of P_6 both rotational and reflectional

$$= \{R^i\}_{i=0}^{5} \cup \{R^i \circ \tau\}_{i=0}^{5}$$

$|G| = 12$

then G is a perm group on X

"dihedral group of order 12"

In general for $n \geq 3$

The <u>dihedral group of order $2n$</u> = group of symmetries of the regular n-gon

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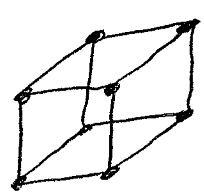
Any geometric figure of any dimension has a symmetry group

ex the 5 platonic solids in 3 dimensions

cube
tetrahedron
octahedron
dodecahedron
icosahedron

cube



in 3 dimensions

X = set of vertices $|X| = 8$.

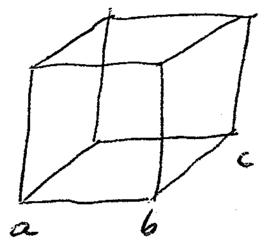
Each symmetry of the cube induces perm of X

Let G = set of resulting perms of X

G = perm group on X

<u>claim</u> $|G| = 48$

pf consider 3 vertices



To construct $f \in G$ we define $f(a)$, $f(b)$, $f(c)$

in stages

stage	to do	# choices
1	Pick $f(a)$	8
2	Pick $f(b)$ from among the 3 vertices adjacent a	3
3	Pick $f(c)$ from among 2 vertices adjacent $f(b)$ other than $f(a)$	2

$$\# \text{poss} = 8 \times 3 \times 2 = 48$$

$\square$

Lecture 35 Monday Dec 3 12/3/12

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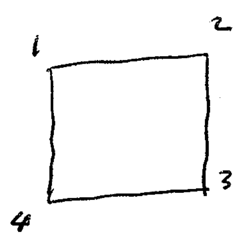
14.1 Cont.

let X = nonempty finite set

Say $X = \{1, 2, \ldots, n\}$

let G = permutation group on X

Running example (REX): X is set of vertices for the regular 4-gon

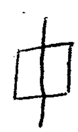


G = the group of symmetries

= dihedral group of order 8

Define

ρ: 90° clockwise rotation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

τ: reflection about  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$G = \{I, \rho, \rho^2, \rho^3, \tau, \rho\tau, \rho^2\tau, \rho^3\tau\}$

Def: A <u>coloring</u> of X is an assignment of a color to each element of X (distinct elements of X might get the same color)

REX Using colors Red (R) and Blue (B) there are $2^4 = 16$ possible colorings. They are:

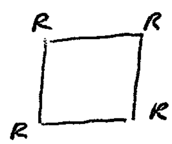
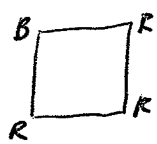
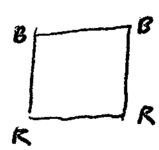
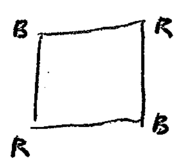
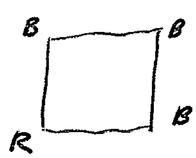
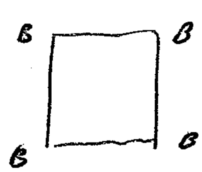
# B	desc		# colorings
0			1
1		+ cyclic perms	4
2		...	4
2		...	2
3		...	4
4			1

Table 1

View each coloring alone as a function

$$\begin{array}{ccc} X & \longrightarrow & \{R, B\} \\ i & \longrightarrow & \text{color assigned to } i \end{array}$$

Given $f \in G$

Given a coloring c of X

Permute X to get another coloring of X:

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "f"'] & \{R, B\} \\ X & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "c"] & \{R, B\} \\ X \arrow[u, "f^{-1}"] \arrow[ur, bend left] & \end{tikzcd}$$

composition $c \circ f^{-1}$ is a coloring of X

<u>Define</u>

$$f * c = c \circ f^{-1}$$

Thus

$f * c$ is the coloring of X

that assigns each element $i \in X$ the

color $c(f^{-1}(i))$

<u>Note</u> f induces a perm of the set of all colorings of X:

$$\{\text{coloring of } X\} \longrightarrow \{\text{coloring of } X\}$$

$$c \longrightarrow f * c$$

LEM For a coloring c of X

$$I * c = c \qquad I = \text{identity elmt of } G$$

pf

$$\begin{aligned} I * c &= c \circ I^{-1} \\ &= c \circ I \\ &= c \end{aligned}$$

LEM Given $f, g \in G$,

Given a coloring c of X,

then

$$f * (g * c) = (f \circ g) * c$$

pf

$$\begin{aligned} f * (g * c) &= (g * c) \circ f^{-1} \\ &= (c \circ g^{-1}) \circ f^{-1} \\ &= c \circ (g^{-1} \circ f^{-1}) \\ &= c \circ (f \circ g)^{-1} \\ &= (f \circ g) * c \end{aligned}$$

$\square$

LEM Given $f \in G$

Given colorings c_1 and c_2 of X

such that

$$c_2 = f * c_1$$

then

$$c_1 = f^{-1} * c_2$$

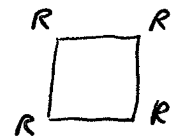
pf

$$\begin{aligned} f^{-1} * c_2 &= f^{-1} * (f * c_1) \\ &= (f^{-1} \circ f) * c_1 \\ &= I * c_1 \\ &= c_1 \end{aligned}$$

$\square$

<u>REX</u> (i) For the coloring

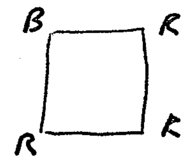
C :



$f * C = C$ for all $f \in G$

(ii) For

C :



✱

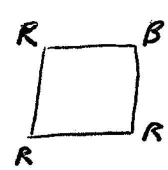
Find $\rho * C$ $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix} \xrightarrow{C} \begin{matrix} R & B \end{matrix}$$

$$\rho \downarrow \uparrow \rho^{-1} \qquad \overset{?}{\dashrightarrow}$$

$$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$$

$\rho * C$:



"apply ρ to picture ✱"

Find

$\tau * c$ $\qquad\qquad$ $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$$\begin{array}{cc} 1 & 2 \\ 4 & 3 \end{array} \xrightarrow{c} \begin{array}{cc} R & B \\ & \end{array}$$

$\tau \updownarrow$ $\qquad\qquad$ $\dashrightarrow$?

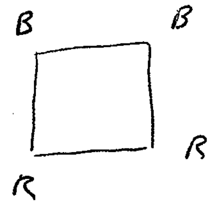
$$\begin{array}{cc} 1 & 2 \\ 4 & 3 \end{array}$$

$\tau * c$: $\quad \begin{array}{|cc|} \hline R & B \\ R & R \\ \hline \end{array}$

"apply τ to picture $*$"

We record a principle

Given a coloring c of X, say



$*$

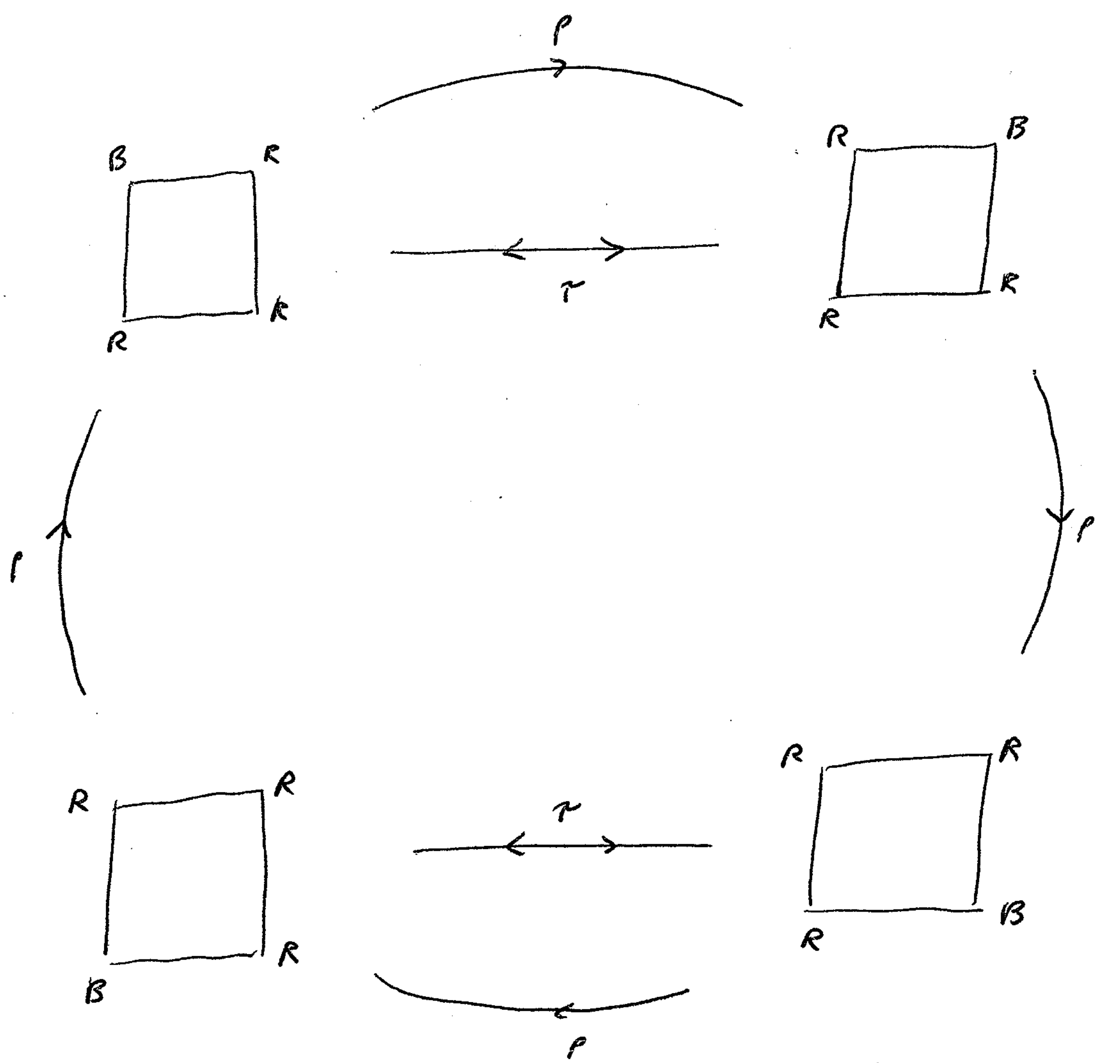
Then for all $f \in G$,

the coloring $f * c$ is obtained by applying f

to the picture $*$

Consider the set of colorings of X that have one B

G acts on this set as follows



key: For $f \in G$ and a coloring c of X,

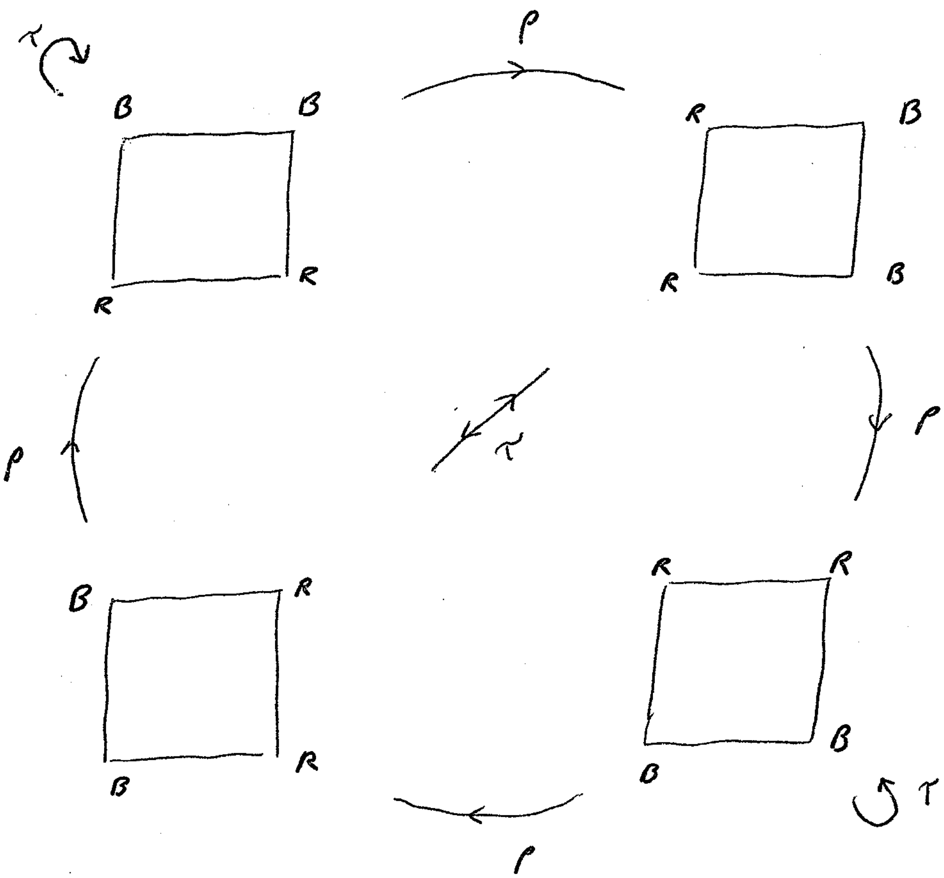
$$c \xrightarrow{f} f * c$$

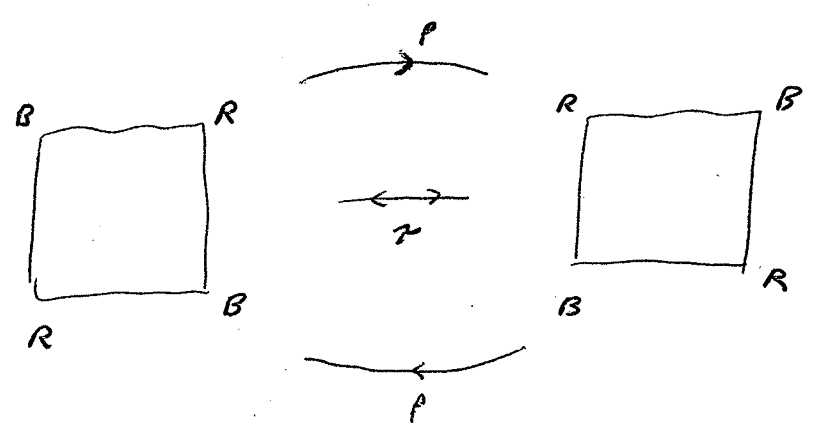
REX Consider the set of colorings of X that have two B's.

12/3/12

10

G acts on this set as follows:





Above diagram has 2 connected components.

Given colorings c_1, c_2 of X

call them G-<u>equivalent</u> and write $c_1 \overset{G}{\sim} c_2$

whenever they are in the same connected component of the diagram.

Thus

$$c_1 \overset{G}{\sim} c_2$$

means

there exists $f \in G$ such that

$$c_2 = f * c_1$$

The relation $\overset{G}{\sim}$ is an equivalence relation and the equivalence classes are the connected components of the diagram.

Formal verification that $\overset{G}{\sim}$ is an equiv rel

- For all colorings c of X show

$$c \overset{G}{\sim} c$$

pf $\qquad c = I * c \qquad\qquad I = \text{identity of } G$

- For all colorings c_1, c_2 of X such that $c_1 \overset{G}{\sim} c_2$ show

$$c_2 \overset{G}{\sim} c_1$$

pf Since $c_1 \overset{G}{\sim} c_2$ $\exists$ $f \in G$ s.t. $c_2 = f * c_1$

Now $c_1 = f^{-1} * c_2$ and $f^{-1} \in G$ so

$c_2 \overset{G}{\sim} c_1$

- For all colorings c_1, c_2, c_3 of X such that

$$c_1 \overset{G}{\sim} c_2 \qquad \text{and} \qquad c_2 \overset{G}{\sim} c_3$$

show

$$c_1 \overset{G}{\sim} c_3$$

pf $\exists\ f \in G$ s.t.

$$c_2 = f * c_1$$

$\exists\ g \in G$ s.t.

$$c_3 = g * c_2$$

So

$$\begin{aligned} c_3 &= g * c_2 \\ &= g * (f * c_1) \\ &= \underbrace{(g \circ f)}_{\in G} * c_1 \end{aligned}$$

So $c_1 \overset{G}{\sim} c_3$

$\square$

REX Consider the set of the colourings of X using colors R, B

The relation $\overset{G}{\sim}$ on this set has 6 equivalence classes, one for each row of table 1.

So up to G-equivalence, there are 6 colourings of X with R, B

— o —

Lecture 36 Wednesday Dec 5

14.2 Burnside's theorem

let

$X =$ nonempty finite set

$X = \{1, 2, \ldots, n\}$

G = a permutation group on X

Pick some colors and consider the set of all colorings of X with those colors.

We saw that G induces a perm gp on that set

let C = a nonempty subset of the above set of colorings of X

Call C <u>G-closed</u> whenever

$f * c \in C$ for all $f \in G$ and $c \in C$

In this case G induces perm gp on C

REX For each row in table I, the given set of colorings is G-**closed**.

From now on assume C is G-closed.

Recall that for colorings $c_1, c_2 \in C$

$c_1 \overset{G}{\sim} c_2$ means "G-equivalent"

$\exists f \in G$ s.t. $f * c_1 = c_2$

$\overset{G}{\sim}$ is an equiv relation on C

C = disjoint union of $\overset{G}{\sim}$ equivalence classes

Define

$N(G, C)$ = # equivalence classes of $\overset{G}{\sim}$ on C

= # of mutually G-inequivalent colorings in C

REX If we take

C = all 16 colorings of X

then $N(G, C) = 6$.

Since there is 1 $\overset{G}{\sim}$ equiv class for each of the 6 rows of table I.

— o —

Next goal: Find formula for $N(G, C)$

For $c \in C$ and $f \in G$ we say

<u>f stabilizes c</u>

or

<u>f fixes c</u>

whenever

$$f * c = c$$

For $c \in C$ define

$$G(c) = \{ f \in G / \ f * c = c \}$$

"the stabilizer of c in G"

LEM Given $c \in C$. Then

$G(c)$ is a permutation gp on C

pf • Given $f, g \in G(c)$ show $f \circ g \in G(c)$:

We have $f * c = c$, $g * c = c$

Obs
$$(f \circ g) * c = f * (g * c) = f * c = c$$

• Show $I \in G(c)$:

$I * c = c$ ✓

□

LEM Given $c \in C$.

Then for all $f, g \in G$ the following are equivalent:

(i) $f * c = g * c$

(ii) $f^{-1} \circ g \in G(c)$

pf

$$f * c = g * c$$

$\iff$

$$\begin{array}{ccc} f^{-1} * (f * c) & = & f^{-1} * (g * c) \\ \| & & \| \\ (f^{-1} \circ f) * c & & (f^{-1} \circ g) * c \\ \| & & \\ I * c & & \\ \| & & \\ c & & \end{array}$$

$\iff$ $f^{-1} \circ g \in G(c)$

$\square$

Given $c \in C$

Consider the equivalence class of $\overset{G}{\sim}$ containing c

this is

$$\{ f * c \,/\, f \in G \}$$

<u>thm</u> For $c \in C$

$$|\{ f * c \,/\, f \in G \}| = \frac{|G|}{|G(c)|}$$

pf Abbr

$$Y = \{ f * c \,/\, f \in G \}$$

For $y \in Y$ define

$$G^{(y)} = \{ f \in G \,/\, f * c = y \}$$

$\{ G^{(y)} \}_{y \in Y}$ partition G

so

$$|G| = \sum_{y \in Y} |G^{(y)}|$$

For $y \in Y$ show

$$|G^{(y)}| = |G(c)|$$

For $f \in G^{(y)}$, the map

$$\begin{aligned} G(c) &\to G^{(y)} \\ h &\to f \circ h \end{aligned}$$

is a bijection (by prev LEM)

So $$|G(c)| = |G^{(y)}|$$

Now

$$|G| = \sum_{y \in Y} \underbrace{|G^{(y)}|}_{= |G(c)|}$$

$$= |Y|\,|G(c)|$$

So $$|Y| = \frac{|G|}{|G(c)|}$$ $\square$

<u>Def</u> For $f \in G$ define

$$C(f) = \{ c \in C \,/\, f * c = c \}$$

"set of colorings in C that are fixed by f"

So for $f \in G$ and $c \in C$

$c \in C(f)$ iff $f * c = c$ iff $f \in G(c)$

thm (Burnside)

$$\underset{\substack{\uparrow \\ \#\text{ of equiv classes of } \overset{G}{\sim} \text{ on } C}}{\overset{N}{\overset{\shortparallel}{N(G,C)}}} = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

pf let

S = set of ordered pairs (f, c) such that
$f \in G$ and $c \in C$ and $f * c = c$

We compute $|S|$ in two ways

I $\quad |S| = \sum_{f \in G} |C(f)|$

II $\quad |S| = \sum_{c \in C} |G(c)|$

let $C_1, C_2, \ldots, C_N$ denote the equiv classes of $\overset{G}{\sim}$ on C

$$|S| = \sum_{c \in C} |G(c)|$$

$$= \sum_{i=1}^{N} \sum_{c \in C_i} \underbrace{|G(c)|}_{= \frac{|G|}{|C_i|} \text{ by prev thm}}$$

$$= \sum_{i=1}^{N} |C_i| \frac{|G|}{|C_i|}$$

$$= \sum_{i=1}^{N} |G|$$

$$= N\,|G|$$

So

$$N = \frac{|S|}{|G|}$$

$$= \frac{\sum_{f \in G} |C(f)|}{|G|}$$

$\square$

12/5/12

10

REX Take

$C =$ set of all 16 colorings of X using colors R, B

$G =$ dihedral gp of order 8

We saw $N(G, C) = 6$

Let us verify this using Burnside.

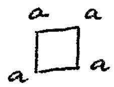
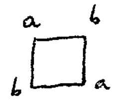
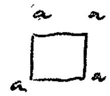
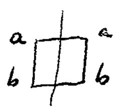
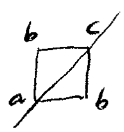
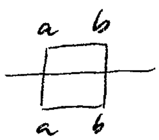
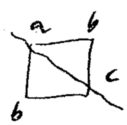
By Burnside

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

$|G| = 8$

$$G = \{1, \rho, \rho^2, \rho^3, \tau, \rho\tau, \rho^2\tau, \rho^3\tau\}$$

For all $f \in G$ find $|C(f)|$

$f \in G$	$C(f)$ desc		$\lvert C(f) \rvert$
1	C		$2^4 = 16$
ρ		$a \in \{R, B\}$	2
ρ^2		$a, b \in \{R, B\}$	$2^2 = 4$
ρ^3		$a \in \{R, B\}$	2
τ		$a, b \in \{R, B\}$	$2^2 = 4$
$\rho\tau$		$a, b, c \in \{R, B\}$	$2^3 = 8$
$\rho^2\tau$		$a, b \in \{R, B\}$	$2^2 = 4$
$\rho^3\tau$		$a, b, c \in \{R, B\}$	$2^3 = 8$
			48

$$N(G, C) = \frac{48}{8} = 6$$ ✓

□

Ex X, G as above

Given integer $p \geq 1$

Given p distinct colors

let C = set of colorings of X with these colors.

So $|C| = p^4$

Find $N(G, C)$

Sol prev ex is case $p = 2$

In prev ex replace 2 by p

f	$\lvert C(f)\rvert$
1	p^4
ρ	p
ρ^2	p^2
ρ^3	p
τ	p^2
$\rho\tau$	p^3
$\rho^2\tau$	p^2
$\rho^3\tau$	p^3
	$p^4 + 2p^3 + 3p^2 + 2p$

$$N(G, C) = \frac{p^4 + 2p^3 + 3p^2 + 2p}{8}$$

p	$N(G,C)$
1	1
2	6
3	21
4	55
⋮	⋮

14.2 Cont

Recall Burnside

Given:

X a nonempty finite set

G a perm gp on X

C a nonempty set of colourings of X

fixed pts of f

Then

$$N(G,C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

\# $\overset{G}{\sim}$-equiv classes in C

Ex Consider multiset

$$S = \{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$$

Fix integer $n \geq 1$.

Recall # of n-perms of S is 4^n.

How many n-perms of S if we declare each

$a_1 a_2 \cdots a_n$ and its mirror image $a_n \cdots a_2 a_1$

to be equivalent?

[so for $n=6$ abbcad equiv dacbba]

Sol Define

$$X = \{1, 2, \ldots, n\}$$

View a, b, c, d as colors

C = set of all colorings of X with a, b, c, d

$$|C| = 4^n$$

Define

$$G = \{I, \tau\}$$

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$$

$\tau^2 = I$. G is a perm gp on X

We seek

$$N(G, C)$$

Apply Burnside

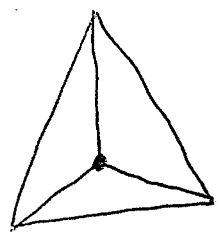
$f \in G$	$C(f)$ desc	$\lvert C(f) \rvert$
I	C	4^n
τ	Case $n = 2r$ is even $a_1 a_2 \cdots a_r \mid a_r \cdots a_2 a_1$ $a_1, \dots, a_r \in \{a, b, c, d\}$	4^r
	Case $n = 2r+1$ is odd $a_1 a_2 \cdots a_r\, a_{r+1} \mid a_r \cdots a_2 a_1$ $a_1, \dots, a_{r+1} \in \{a, b, c, d\}$	4^{r+1}

$$N(G, C) = \begin{cases} \dfrac{4^n + 4^{n/2}}{2} & \text{4 n even} \\[2ex] \dfrac{4^n + 4^{\frac{n+1}{2}}}{2} & \text{4 n odd} \end{cases}$$

$$= \frac{4^n + 4^{\lfloor \frac{n+1}{2} \rfloor}}{2}$$

$\square$

Ex Tetrahedron

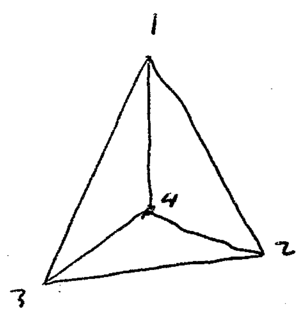


" in 3 dimensions "

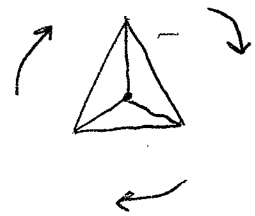
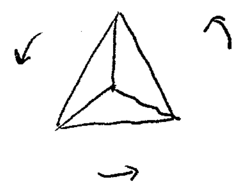
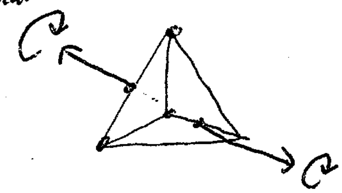
" top view "

X = set of vertices

Label X



the gp G of symmetries of the tetrahedron

types of symmetries on G		#
120° clockwise rot		4
120° c.c. rot		4
180° rot	halfturn 	3
identity I		1
		12

We allow only "physically possible" symmetries

$$|G| = 12$$

G is often called the <u>alternating group</u> A_4

For the above tetrahedron

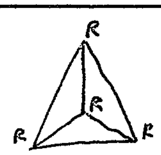
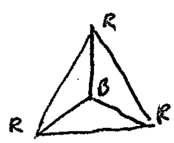
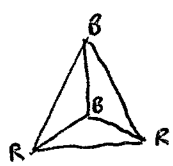
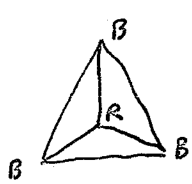
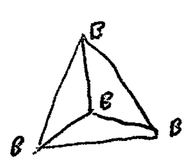
let

$C =$ set of colorings of X with colors R, B

So $|C| = 2^4 = 16$

Find $N(G, C)$

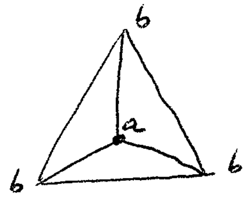
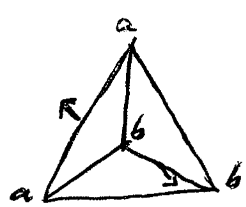
Sol 1: describe each $\overset{G}{\sim}$ equiv class

#B	desc	#colorings
0		1
1		4
2		6
3		4
4		1

Each row gives a $\overset{G}{\sim}$ equiv class

$N(G,C) = \#\text{rows} = 5$

Sol 2: Use Burnside

$f \in G$	$C(f)$ desc		$\lvert C(f)\rvert$
120° clockwise rot		$a,b \in \{R,B\}$	$2^2=4$
120° c.c. rot	"		"
180° rot		$a,b \in \{R,B\}$	$2^2=4$
I	C		$2^4=16$

$$N(G,C) = \frac{\sum_{f\in G} |C(f)|}{|G|}$$

$$= \frac{4\times4 + 4\times4 + 3\times4 + 1\times16}{12}$$

$$= \frac{60}{12}$$

$$= 5$$

□

Ex For the above tetrahedron

let

$C =$ set of colorings of X with R, W, B

So $|C| = 3^4 = 81$

Find $N(G, C)$

Sol 1 describe each $\overset{G}{\sim}$ equiv class

For a coloring $c \in C$

define

$r =$ # vertices colored R

$w =$ ... W

$b =$ ... B

up to $\overset{G}{\sim}$ c is det by r, w, b.

obs $r + w + b = 4$

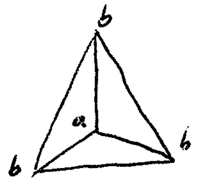
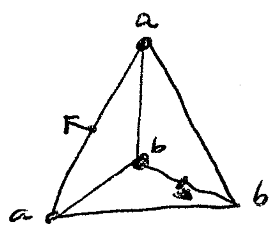
$r \geq 0, \quad w \geq 0, \quad b \geq 0$

#sols for r, w, b is

$$\binom{4+3-1}{3-1} = \binom{6}{2} = 15$$

$$N(G, C) = 15$$

Sol 2   Use Burnside

$f \in G$	$C(f)$ desc		$\lvert C(f)\rvert$
120° cl rot		$a, b \in \{R, W, B\}$	$3^2 = 9$
120° c.c. rot	"		"
180° rot		$a, b \in \{R, W, B\}$	$3^2 = 9$
I	C		3^4

$$N(G,C) = \frac{\sum_{f \in G} \lvert C(f) \rvert}{\lvert G \rvert}$$

$$= \frac{4 \times 3^2 + 4 \times 3^2 + 3 \times 3^2 + 1 \times 3^4}{12}$$

$$= 15$$

Ex For above tetrahedron

Given integer p

Given p dist colors

Let C = set of colorings of X with these colors

So $|C| = p^4$

Find $N(G, C)$

Use Sol 2: By Burnside

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

$$= \frac{4 \times p^2 + 4 \times p^2 + 3 \times p^2 + 1 \times p^4}{12}$$

$$= \frac{p^2(p^2 + 11)}{12}$$

Sol 1: gets complicated

$\square$

Ex For tetrahedron

Now take $G = S_4$ = set of all perms of X

$|G| = 4! = 24$

Given $p \geq 1$ Given p dist colors $c_1, c_2, \ldots, c_p$

C = set of colorings of X with these colors

Find $N(G, C)$

Sol 1 Desc each equiv class for $\overset{G}{\sim}$

For a coloring $c \in C$

For $1 \leq i \leq p$ let

n_i = # vertices colored c_i

c is det up to $\overset{G}{\sim}$ by

the sequence

$n_1, n_2, \ldots, n_p$

obs

$n_1 + n_2 + \cdots + n_p = 4$

$n_i \geq 0$ $1 \leq i \leq p$

sols are

$$\binom{4+p-1}{p-1} = \binom{p+3}{4}$$

$$N(G, C) = \binom{p+3}{4}$$

Sol 2 Use Burnside

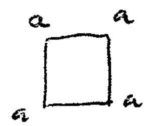
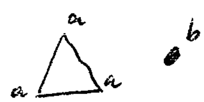
Describe the elements of G

ex $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

f: $1 \to 2 \to 3 \to 4$ $1 \to 4 \to 3$ 2

cycle type  

cycle type	# els of G
	$3! = 6$
	$4 \times 2 = 8$
$\leftrightarrow \quad \leftrightarrow$	3
$\leftrightarrow \quad \bullet \quad \bullet$	$\binom{4}{2} = 6$
$\bullet \quad \bullet \quad \bullet \quad \bullet$	1

$f \in G$	$C(f)$ desc		$\lvert C(f)\rvert$
		a arb	p
		a, b arb	p^2
$\leftrightarrow$ $\leftrightarrow$	$\underline{a\ a}$ $\underline{b\ b}$	a, b arb	p^2
$\leftrightarrow$ • •	$\underline{a\ a}$ b c	a, b, c arb	p^3
• • • •	a b c d	a, b, c, d arb	p^4

$$N(G,C) = \frac{\sum_{f\in G} \lvert C(f)\rvert}{\lvert G\rvert}$$

$$= \frac{6p + 8p^2 + 3p^2 + 6p^3 + p^4}{24}$$

$$= \frac{6p + 11p^2 + 6p^3 + p^4}{24}$$

$$= \frac{(p+3)(p+2)(p+1)p}{24}$$

$$= \binom{p+3}{4}$$

$\square$

Note

$$\frac{p^2(p^2+11)}{12} = \binom{p+3}{4}$$

for $p = 1, 2, 3$ but not in general

Lecture 38 Monday Dec 10

14.3 Polya Counting

the cycle factorization of a permutation

Given X = nonempty finite set

say $X = \{1, 2, \ldots, n\}$

Given integer r $(1 \le r \le n)$

Given mutually distinct $a_1, a_2, \ldots, a_r$ in X

Let

$$[a_1, a_2, \ldots, a_r]$$

"cycle notation"

denote the perm of X that sends

$$a_1 \to a_2 \to a_3 \to \cdots \to a_r$$

and fixes all other elements of X

Call this perm a cycle of order r

or

r-cycle

ex $n = 8$

let $f = [1\,3\,2\,6]$

Write f in 2-line notation

Sol

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 2 & 4 & 5 & 1 & 7 & 8 \end{pmatrix}$$

Note Each cycle has multiple cycle notations.

For instance

$$[1\,3\,2\,6] = [3\,2\,6\,1] = [2\,6\,1\,3] = [6\,1\,3\,2]$$

Note cycle of order 1 is just the identity:

$$[i] = I \qquad \text{for } 1 \le i \le n$$

Given two cycles:

$f = [a_1, a_2, \ldots, a_r]$ $\qquad g = [b_1, b_2, \ldots, b_s]$

Call these cycles <u>disjoint</u> whenever

$a_i \neq b_j \qquad 1 \le i \le r, \qquad 1 \le j \le s$

In this case f, g commute:

$$f \circ g = g \circ f$$

pf: obs

f fixes each of $b_1, b_2, \ldots, b_s$

g fixes each of $a_1, a_2, \ldots, a_r$

For $x \in X$ show

$$(f \circ g)(x) = (g \circ f)(x)$$

Case $x \in \{a_1, a_2, \ldots, a_r\}$

say $x = a_i$

$f \circ g$: $a_i \xrightarrow[g]{} a_i \xrightarrow[f]{} a_{i+1}$

" -

$g \circ f$: $a_i \xrightarrow[f]{} a_{i+1} \xrightarrow[g]{} a_{i+1}$

Case $x \in \{b_1, b_2, \ldots, b_s\}$:

say $x = b_j$

$$f \circ g: \quad b_j \xrightarrow[g]{} b_{j+1} \xrightarrow[f]{} b_{j+1}$$

$\shortparallel$ ✓

$$g \circ f: \quad b_j \xrightarrow[f]{} b_j \xrightarrow[g]{} b_{j+1}$$

Case $x \notin \{a_1, \ldots, a_r\}$ $\quad x \notin \{b_1, \ldots, b_s\}$:

$$f \circ g: \quad x \xrightarrow[g]{} x \xrightarrow[f]{} x$$

$\shortparallel$ ✓

$$g \circ f: \quad x \xrightarrow[f]{} x \xrightarrow[g]{} x$$

Ex Given any perm f of X

say $n = 8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 6 & 8 & 7 & 1 & 2 \end{pmatrix}$$

Express f as a product of disjoint cycles

↑

with respect to composition

f:

$$\begin{tikzcd} 1 \arrow[r] & 4 \arrow[r] & 6 \arrow[r] & 7 \arrow[lll, bend left] \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[r] & 5 \arrow[r] & 8 \arrow[ll, bend left] \end{tikzcd} \qquad \begin{tikzcd} 3 \arrow[loop right] \end{tikzcd}$$

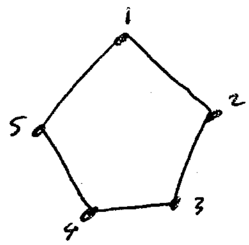
$$f = [1\,4\,6\,7] \circ [2\,5\,8] \circ [3]$$

"cycle factorization of f"

In general

LEM Each permutation of X is a product of disjoint cycles $\square$

Ex Consider regular 5-gon



$X = \{1, 2, 3, 4, 5\}$

Let $G =$ group of symmetries

$=$ dihedral gp of order 10

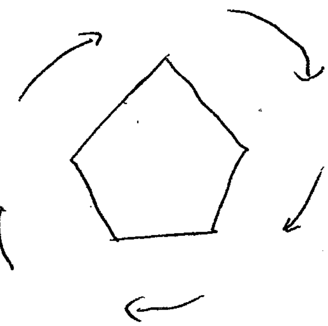
View as perm gp on X

$$G = \underbrace{\{\rho^i\}_{i=0}^{4}}_{\text{rot}} \cup \underbrace{\{\rho^i \tau\}_{i=0}^{4}}_{\text{refl}}$$

where

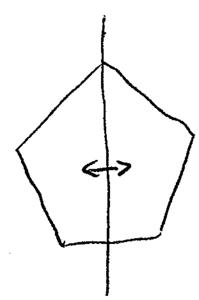
ρ is clockwise $72°$ rot

ρ:



τ is reflection

τ:



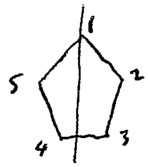
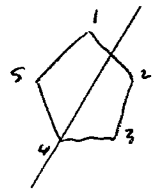
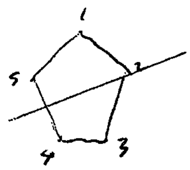
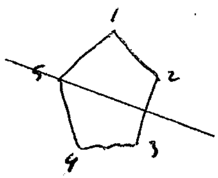
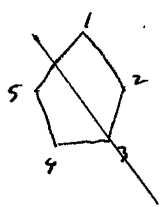
For each $f \in G$ find the cycle factorization the rotations in G

12/10/12

7

f	cycle factorization
I	$[1] \circ [2] \circ [3] \circ [4] \circ [5]$
ρ	$[1\ 2\ 3\ 4\ 5]$
ρ^2	$[1\ 3\ 5\ 2\ 4]$
ρ^3	$[1\ 4\ 2\ 5\ 3]$
ρ^4	$[1\ 5\ 4\ 3\ 2]$

the reflections in G 12/10/12

f	desc	cycle factorization of f
τ		$[1][25][34]$
$\rho\tau$		$[12][35][4]$
$\rho^2\tau$		$[13][2][45]$
$\rho^3\tau$		$[14][23][5]$
$\rho^4\tau$		$[15][24][3]$

Given $X = \{1, 2, \ldots, n\}$

Given perm f of X

Consider cycle factorization of f

Define $\#(f)$ = the number of cycles in this factorization.

Ex For above 5-gon

f	$\#(f)$
I	5
ρ	1
ρ^2	1
ρ^3	1
ρ^4	1
τ	3
$\rho\tau$	3
$\rho^2\tau$	3
$\rho^3\tau$	3
$\rho^4\tau$	3

Given $X = \{1, 2, \dots, n\}$

Given integer $p \geq 1$

Given p distinct colors $c_1, c_2, \dots, c_p$

Let $C =$ set of all colorings of X with these colors

Given a permutation f of X

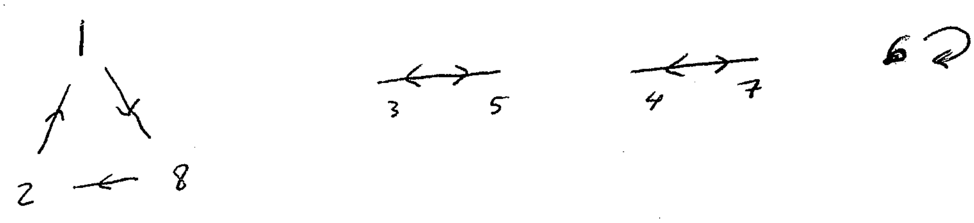
Find $|C(f)|$ in terms of $\#(f)$

pts in X fixed by f

Sol ex $n = 8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 5 & 7 & 3 & 6 & 4 & 2 \end{pmatrix}$$

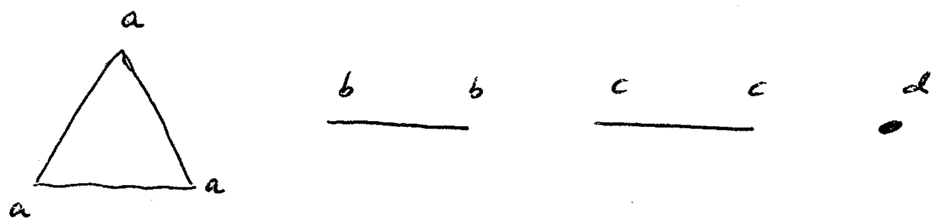
Find cycle factorization of f



$$f = [1\,8\,2] \circ [3\,5] \circ [4\,7] \circ [6]$$

$$\#(f) = 4$$

Describe $C(f)$



$a, b, c, d \in \{c_1, c_2, \dots, c_p\}$

choices for a, b, c, d is $p^4 = p^{\#(f)}$

So

$$|C(f)| = p^{\#(f)}$$

Cor. let X, C as above

Given G a perm gp on X

Then

$$N(G, C) = \frac{\sum_{f \in G} p^{\#(f)}}{|G|}$$

Given $X = \{1, 2, \dots, n\}$

Given a perm f of X

We now associate with f a polynomial in n variables

$z_1, z_2, \dots, z_n$

Consider cycle factorization of f

For $1 \le k \le n$ let

$$e_k = \#\ k\text{-cycles in this factorization}$$

So

$$\sum_{k=1}^{n} e_k k = n$$

Note $\sum_{k=1}^{n} e_k = \#$ of cycles in this factorization $= \#(f)$

Define

$$\mathrm{mon}(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$$ "the monomial of f"

ex $n = 8$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 5 & 7 & 3 & 6 & 4 & 2 \end{pmatrix}$$

$$f = [1\,8\,2] \circ [3\,5] \circ [4\,7] \circ [6]$$

k	1	2	3	4	5	6	7	8
e_k	1	2	1	0	0	0	0	0

$$\mathrm{mon}(f) = z_1 z_2^2 z_3$$

Def Given $X = \{1, 2, \ldots, n\}$

Given G = perm group on X

Define a polynomial P_G in n variables

$$z_1, z_2, \ldots, z_n$$

by

$$P_G(z_1, z_2, \ldots, z_n) = \frac{\sum_{f \in G} \text{mon}(f)}{|G|}$$

Thm Given $X = \{1, 2, \ldots, n\}$

Given $p \geq 1$

Given dist colors $c_1, c_2, \ldots, c_p$

Let C = set of all colorings of X with these colors

Let G = perm group on X.

Then

$$N(G, C) = P_G(p, p, \ldots, p)$$

pf Recall

$$N(G,C) = \frac{\sum_{f \in G} p^{\#(f)}}{|G|}$$

By def

$$P_G(z_1, z_2, \ldots, z_n) = \frac{\sum_{f \in G} mon(f)}{|G|}$$

For $f \in G$ write

$$mon(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$$

setting $z_i = p$ for $1 \le i \le n$, this becomes

$$p^{e_1} p^{e_2} \cdots p^{e_n}$$

$$= p^{e_1 + e_2 + \cdots + e_n}$$

$$= p^{\#(f)}$$

So

$$P_G(p, p, \ldots, p) = \frac{\sum_{f \in G} p^{\#(f)}}{|G|}$$

$$= N(G,C)$$

$\square$

14.3 Cont.

Recall

Fix $X = \{1, 2, \ldots, n\}$ $n \geq 1$

$p \geq 1$

Fix colors $c_1, c_2, \ldots, c_p$

C = set of all colorings of X with these colors

Fix

G = perm gp on X

Then

$$N(G, C) \overset{\#\ \overset{G}{\sim}\text{ equiv classes on } C}{=} \frac{\sum_{f \in G} |C(f)|}{|G|} \qquad \text{Burnside}$$

$$= P_G(p, p, \ldots, p)$$

where

$$P_G(z_1, z_2, \ldots, z_n) = \frac{\sum_{f \in G} \mathrm{mon}(f)}{|G|}$$

$$\mathrm{mon}(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$$

e_k = # k-cycles in cycle factorization of f $(1 \leq k \leq n)$

Function P_G called <u>cycle</u> <u>index</u> of G.

— o —

Given $c \in C$

For $1 \le i \le p$ define

n_i = # vertices in X colored c_i by c

So

$$\left.\begin{aligned} n_1 + n_2 + \cdots + n_p = n = |X| \\ n_i \ge 0 \qquad 1 \le i \le p \end{aligned}\right\} \qquad *$$

Given a sol $\{n_i\}_{i=1}^{p}$ of $*$, define

$C_{n_1, n_2, \ldots, n_p}$ = set of colorings of X that have exactly n_i vertices colored c_i for $1 \le i \le p$

Note that

$C_{n_1, n_2, \ldots, n_p}$ is G-invariant.

Problem Find

$$N(G, C_{n_1, n_2, \ldots, n_p}) \qquad **$$

Sol We give the generating function for ✩✩

Introduce variables

$u_1, u_2, \ldots, u_p$

(u_i corresponds to color c_i for $1 \le i \le p$)

Gen function for ✩✩ is

$$\sum_{n_1, n_2, \ldots, n_p} N(G, C_{n_1, n_2, \ldots, n_p})\, u_1^{n_1} u_2^{n_2} \cdots u_p^{n_p} \qquad ✩$$

where sum is over all sols $n_1, n_2, \ldots, n_p$ to ✱

thm (Polya counting formula)

With above notation

$$\bigstar \quad = \quad P_G\left(u_1+u_2+\cdots+u_p,\ u_1^2+u_2^2+\cdots+u_p^2,\ \ldots,\ u_1^n+u_2^n+\cdots+u_p^n\right)$$

where $P_G =$ cycle index of G

pf (for $p=2$)

2 colors c_1, c_2

show

$$\sum_{\substack{n_1\geq 0\\ n_2\geq 0\\ n_1+n_2=n}} N(G, C_{n_1,n_2})\, u_1^{n_1} u_2^{n_2} \quad = \quad P_G\left(u_1+u_2,\ u_1^2+u_2^2,\ \ldots,\ u_1^n+u_2^n\right)$$

Consider

$$P_G\left(u_1+u_2,\ u_1^2+u_2^2,\ \ldots,\ u_1^n+u_2^n\right).$$

as a poly in u_1, u_2.

For $n_1 \geq 0$ $n_2 \geq 0$

$$\underbrace{\text{coeff of } u_1^{n_1} u_2^{n_2}}_{\gamma}$$

Assume $n_1+n_2=n$, else $\gamma=0$ by constr.

Show

$$\gamma = N\left(G, C_{n_1,n_2}\right)$$

set of elements in C_{n_1,n_2} fixed by f

By Burnside

$$N\left(G, C_{n_1,n_2}\right) = \frac{\sum_f \left|C_{n_1,n_2}(f)\right|}{|G|}$$

Recall

$$P_G\left(z_1, z_2, \ldots, z_n\right) = \frac{\sum_{f\in G} \operatorname{mon}(f)}{|G|}$$

For $f \in G$ show

contrib of $\operatorname{mon}(f)$ to γ $=$ $\left|C_{n_1,n_2}(f)\right|$

$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$$

$e_k = \#$ k-cycles in cycle factorization of f

Replace

$$z_1 \to u_1 + u_2$$
$$z_2 \to u_1^2 + u_2^2$$
$$\vdots$$
$$z_n \to u_1^n + u_2^n$$

mult(f) becomes

$$(u_1+u_2)^{e_1}(u_1^2+u_2^2)^{e_2} \cdots (u_1^n+u_2^n)^{e_n}$$

In this polynomial show

coef of $u_1^{n_1} u_2^{n_2} = |C_{n_1,n_2}(f)|$

expand

$$(u_1+u_2)^{e_1}\,(u_1^2+u_2^2)^{e_2}\cdots(u_1^n+u_2^n)^{e_n}$$

$$=$$

$$\left(\sum_{l_1=0}^{e_1}\binom{e_1}{l_1}u_1^{l_1}u_2^{e_1-l_1}\right)$$

$$\times\left(\sum_{l_2=0}^{e_2}\binom{e_2}{l_2}u_1^{2l_2}u_2^{e_2-2l_2}\right)$$

$$\times$$

$$\cdots$$

$$\times\left(\sum_{l_n=0}^{e_n}\binom{e_n}{l_n}u_1^{nl_n}u_2^{e_n-nl_n}\right)$$

In the above polynomial the coef of $u_1^{n_1}u_2^{n_2}$ is

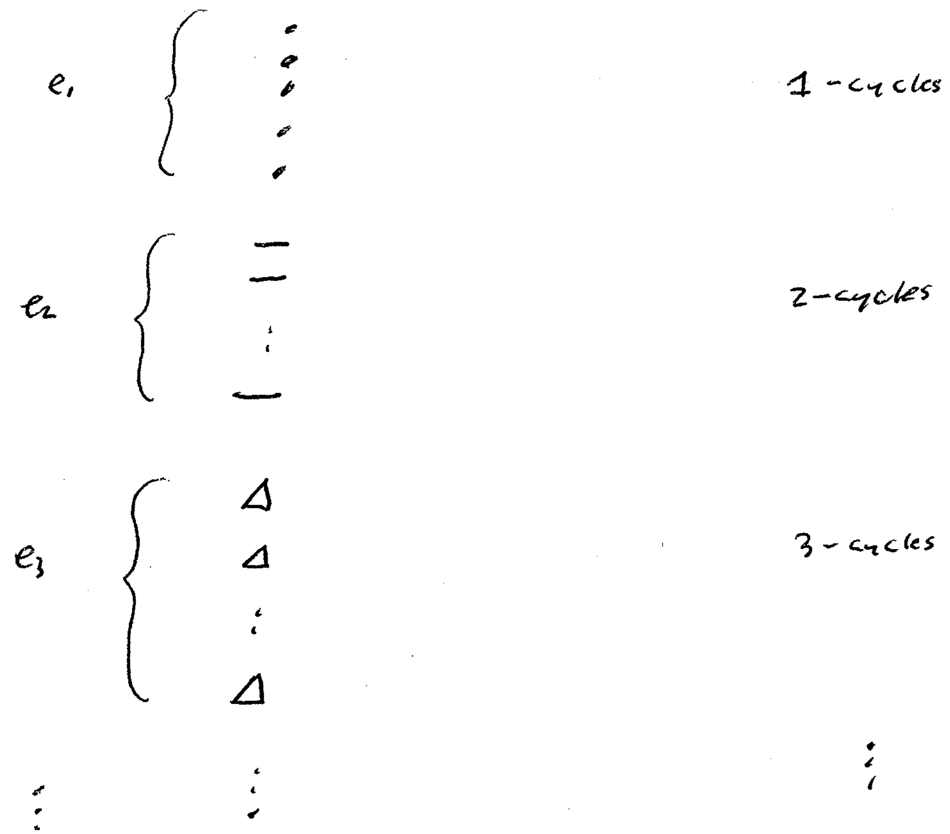
$$\underbrace{\sum_{\substack{l_1,l_2,\dots,l_n\\ 0\le l_1\le e_1\\ 0\le l_2\le e_2\\ \vdots\\ 0\le l_n\le e_n\\ l_1+2l_2+\cdots+nl_n=n_1}}\binom{e_1}{l_1}\binom{e_2}{l_2}\cdots\binom{e_n}{l_n}}$$

show this equals $|C_{n_1,n_2}(f)|$

12/12/12

Find $|C_{n_1,n_2}(f)|$

cycle factorization of f:



To find $|C_{n_1,n_2}(f)|$ we construct a coloring

$c \in C_{n_1,n_2}(f)$ in two steps

Step I decide how many

1 - cycles get colored c_1		(l_1)
2 - cycles ...	c_1	(l_2)
...		$\vdots$
n - cycles ...	c_1	(l_n)

One choice for each sol $\{l_i\}_{i=1}^n$ to

$$0 \le l_1 \le e_1$$
$$0 \le l_2 \le e_2$$
$$\vdots$$
$$0 \le l_n \le e_n$$

$$l_1 + 2l_2 + \dots + nl_n = n_1$$

For each choice $\{l_i\}_{i=1}^n$ we proceed to step II :

Step II

stage	to do	# choices
1	decide which 1-cycles to color c_1	$\binom{e_1}{l_1}$
2	-- 2-cycles ...	$\binom{e_2}{l_2}$
$\vdots$		
n	-- n-cycles ..	$\binom{e_n}{l_n}$

So

$$\left| C_{n_1, n_2}(f) \right| = \sum_{\substack{0 \le l_1 \le e_1 \\ 0 \le l_2 \le e_2 \\ \vdots \\ 0 \le l_n \le e_n \\ l_1 + 2l_2 + \cdots + nl_n = n_1}} \binom{e_1}{l_1}\binom{e_2}{l_2} \cdots \binom{e_n}{l_n}$$

this proves Polya thm for $p=2$

$\square$

Lecture 40 Friday Dec 14 12/14/12

[last day of class] 1

Recall

$X = \{1, 2, \dots, n\}$

G = permutation group on X

Cycle index of G is polynomial

$$P_G(z_1, z_2, \dots, z_n) = \frac{\sum_{f \in G} \text{mon}(f)}{|G|}$$

For $f \in G$

$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$$

e_k = # k-cycles in cycle factorization of f

Fix $p \geq 1$

dist colors $c_1, c_2, \dots c_p$

correp variables $u_1, u_2, \dots, u_p$

Polya thm Generating function for colorings of X up to G-equivalence

$$= P_G(u_1 + u_2 + \dots + u_p, u_1^2 + u_2^2 + \dots + u_p^2, \dots, u_1^n + u_2^n + \dots + u_p^n) \quad *$$

This means the following:

For each monomial

$$u_1^{n_1} u_2^{n_2} \cdots u_p^{n_p}$$

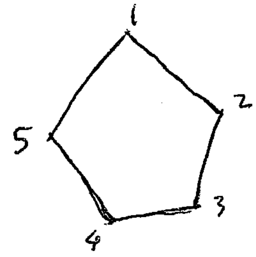
in $*$ the following are the same:

(i) the coefficient of $u_1^{n_1} u_2^{n_2} \cdots u_p^{n_p}$

(ii) Up to G-equivalence, the number of colorings of X with

n_1 vertices colored c_1
n_2 -- c_2
...
n_p -- c_p

Ex Consider regular 5-gon



$X = \{1,2,3,4,5\}$

G = gp of symmetries

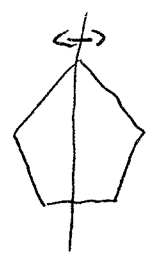
= dihedral group of order 10

$= \{\rho^i\}_{i=0}^{4} \cup \{\rho^i \circ \tau\}_{i=0}^{4}$

view as perm gp on X

ρ = clockwise 72° rotation

τ = reflection



Find P_G

f	cycle factorization of f	$mm(f)$
1	• • • • •	z_1^5
ρ		z_5
ρ^2		z_5
ρ^3		z_5
ρ^4		z_5
τ	• — —	$z_1 z_2^2$
$\rho\tau$	• — —	$z_1 z_2^2$
$\rho^2\tau$	• — —	$z_1 z_2^2$
$\rho^3\tau$	• — —	$z_1 z_2^2$
$\rho^4\tau$	• — —	$z_1 z_2^2$

$$P_G(z_1, z_2, z_3, z_4, z_5) = \frac{z_1^5 + 5 z_1 z_2^2 + 4 z_5}{10}$$

Take $p=3$ colors

c_i	Red	white	Blue
u_i	r	w	b

Find corresp Polya generating function

Sol:

$$P_G(r+w+b,\ r^2+w^2+b^2,\ r^3+w^3+b^3,\ r^4+w^4+b^4,\ r^5+w^5+b^5)$$

$$= \frac{(r+w+b)^5 + 5(r+w+b)(r^2+w^2+b^2)^2 + 4(r^5+w^5+b^5)}{10}$$

Up to G-equivalence, find the number of ways to color the vertices in X such that

2 vertices colored Red,
2 ... white
1 ... Blue

Sol: In Polya generating function

$$\text{coef of } r^2w^2b = \frac{\binom{5}{2\,2\,1} + 5\times 2}{10}$$

$$= 4$$

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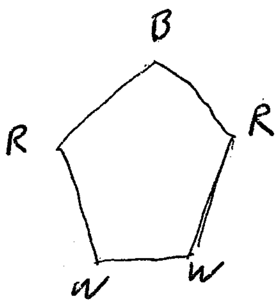
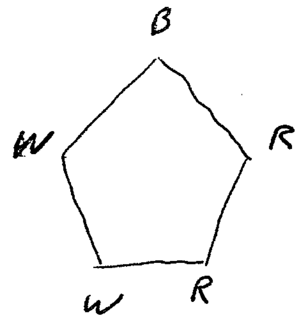
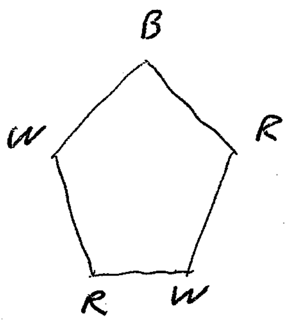
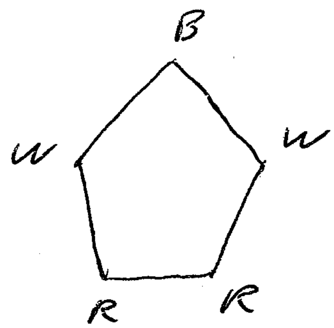
The colorings are

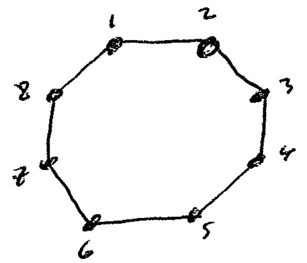








Consider regular 8-gon

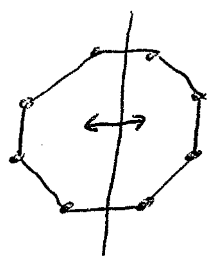


$X = \{1, 2, \dots, 8\}$

G = group of symmetries
= Dihedral group order 16
= $\{\rho^i\}_{i=0}^{7} \cup \{\rho^i \tau\}_{i=0}^{7}$

ρ = 45° clockwise rot

τ = reflection



Find P_G

f	cycle factorization of f	$\mathrm{mon}(f)$
1		z_1^8
ρ		z_8
ρ^2		z_4^2
ρ^3		z_8
ρ^4		z_2^4
ρ^5		z_8
ρ^6		z_4^2
ρ^7		z_8
τ		z_2^4
$\rho\tau$		$z_1^2 z_2^3$
$\rho^2\tau$		z_2^4
$\rho^3\tau$		$z_1^2 z_2^3$
$\rho^4\tau$		z_2^4
$\rho^5\tau$		$z_1^2 z_2^3$
$\rho^6\tau$		z_2^4
$\rho^7\tau$		$z_1^2 z_2^3$

$$P_G(z_1,\dots,z_8) = \frac{z_1^8 + 4z_1^2 z_2^3 + 5z_2^4 + 2z_4^2 + 4z_8}{16}$$

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9

Up to G-equiv, how many ways to color the vertices in X with colors

Red, white, Blue ?

Sol Set $z_i = 3$ for $1 \le i \le 8$

$$P_G(3,3,\ldots,3) = \frac{3^8 + 4\cdot3^5 + 5\cdot3^4 + 2\cdot3^2 + 4\cdot3}{16}$$

$$= 166 \times 3$$

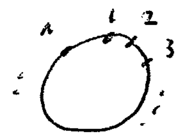
$$= 498$$

□

Ex let n = prime number

Fix $p \geq 1$, dist colors $c_1, \dots, c_p$

How many different necklaces can be made using n beads with above colors?



let X = set of bead locations $= \{1, 2, \dots, n\}$

Here symmetry gp is

$$G = \text{dihedral gp of order } 2n$$

$$= \{\rho^i\}_{i=0}^{n-1} \cup \{\rho^i \circ \tau\}_{i=0}^{n-1}$$

ρ = clockwise rotation thru $\frac{360}{n}$ degrees

τ = reflection

Find P_G

f	cycle factorization		mon (f)
I	$\cdot\ \cdot \cdots \cdot$		z_1^n
ρ^i, $1 \leq i \leq n-1$		n-cycle	z_n
$\rho^i \circ \tau$, $0 \leq i \leq n-1$	$\cdot\ \underbrace{\mid\ \mid\ \mid \cdots \mid}_{\frac{n-1}{2}}$		$z_1\ z_2^{\frac{n-1}{2}}$

$$P_G(z_1, \dots, z_n) = \frac{z_1^n + (n-1) z_n + n z_1 z_2^{\frac{n-1}{2}}}{2n}$$

necklaces

$= \#$ ways to color X with colors $c_1, c_2, \ldots, c_p$ (up to G-equivalence)

$$= P_G(p, p, \ldots, p)$$

$$= \frac{p^n + (n-1)p + n\, p^{\frac{n+1}{2}}}{2n}$$

Now for $p = 2$ colors Red, Blue

Find the Polya gen function

c_i	Red	Blue
w_i	r	b

Polya Gen function is

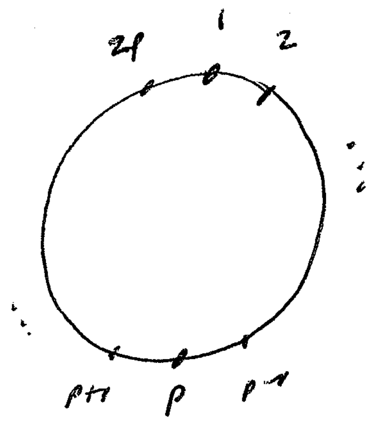
$$P_G(r+b, r^2+b^2, \ldots, r^n+b^n)$$

$$= \frac{(r+b)^n + (n-1)(r^n+b^n) + n(r+b)(r^2+b^2)^{\frac{n-1}{2}}}{2n}$$

Ex Find the cycle index for the dihedral group D_{2p} $\quad$ p prime $\quad$ ($D_{2p} = G$)

($9p$ has order $4p$)

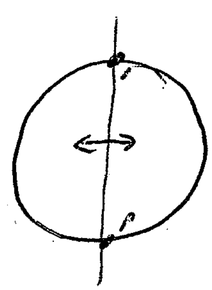
Sol View G as corner-symmetry group of regular $2p$-gon



$$G = \{\rho^i\}_{i=0}^{n-1} \cup \{\rho^i \tau\}_{i=0}^{n-1} \qquad n = 2p$$

ρ : clockwise rot thru $\frac{360}{n}$

τ : reflection



f	cycle factorization of f		$\mathrm{mon}(f)$
I	$\underbrace{\cdot\,\cdot\,\cdot\ \cdots\ \cdot}_{n}$		z_1^n
ρ^i i odd $1 \le i \le n-1$	$\bigcirc$	n-cycle	z_n
ρ^i i even $1 \le i \le n-1$	$\bigcirc\ \bigcirc$	two p-cycles	z_p^2
$\rho^i \gamma$ i even $0 \le i \le n-1$	$\cdot\ \cdot\ \underbrace{\mid\ \mid \cdots \mid}_{p-1}$		$z_1^2\, z_2^{p-1}$
$\rho^i \gamma$ i odd $0 \le i \le n-1$	$\underbrace{\mid\ \mid \cdots \mid}_{p}$		z_2^p

$$P_G(z_1, \ldots, z_n) = \frac{z_1^{2p} + p\, z_{2p} + (p-1)\, z_p^2 + p\, z_1^2 z_2^{p-1} + p\, z_2^p}{4p}$$

Ex Find the Polya gen function for the number of different necklaces that can be made with $2p$ beads and two colors Red, Blue

$p =$ prime

Sol. In prev problem replace

$$z_i = r^i + b^i \qquad 1 \le i \le 2p$$

Polya Gen Function is

$$\frac{(r+b)^{2p} + p(r^{2p}+b^{2p}) + (p-1)(r^p+b^p)^2 + p(r+b)^2(r^2+b^2)^{p-1} + p(r^2+b^2)^p}{4p}$$

The End