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[Monday Nov 26 Exam II]

Lecture 33 Wed Nov 28

We will do ch 14 next, and return to ch 9 time permitting

Ch 14 Polya counting

14.1 Permutations and symmetry groups

Let $X =$ nonempty finite set

Say $X = \{1, 2, \dots, n\}$

Consider perm of X :

$a_1 a_2 \dots a_n$

View this as a bijection

$X \longrightarrow X$

$i \longrightarrow a_i$

To emphasize this view we often write

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

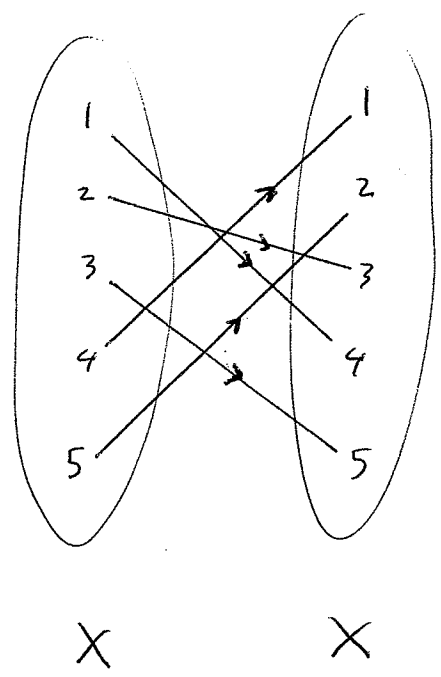
the bijection sends each number in top row to the number beneath it

Ex $n=5$

the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$$

satisfies



Def For $n \geq 1$ $S_n =$ set of all perms of $\{1, 2, \dots, n\}$ Composition of permutationsGiven perms $f: X \rightarrow X$ $g: X \rightarrow X$ their composition $f \circ g: X \rightarrow X$ satisfies

$$(f \circ g)(x) = f(g(x)) \quad \forall x \in X$$

" First apply g and then apply f "

$$X \xrightarrow{g} X \xrightarrow{f} X$$

$$f \circ g: \quad x \rightarrow g(x) \rightarrow f(g(x))$$

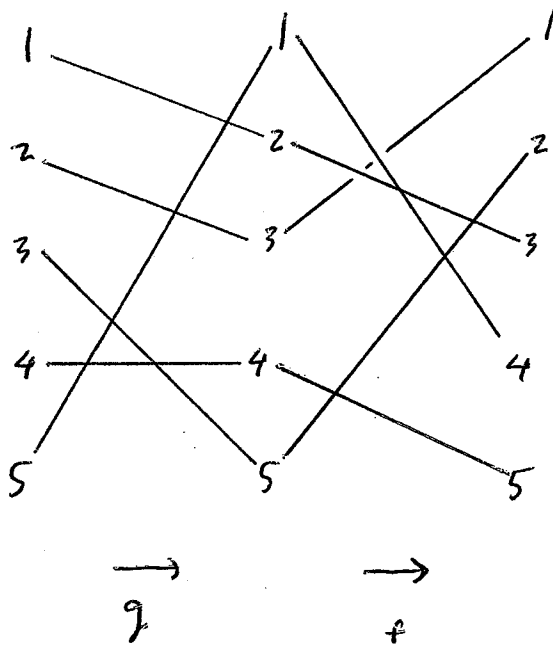
Ex for perms

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}$$

Find $f \circ g$

Sol

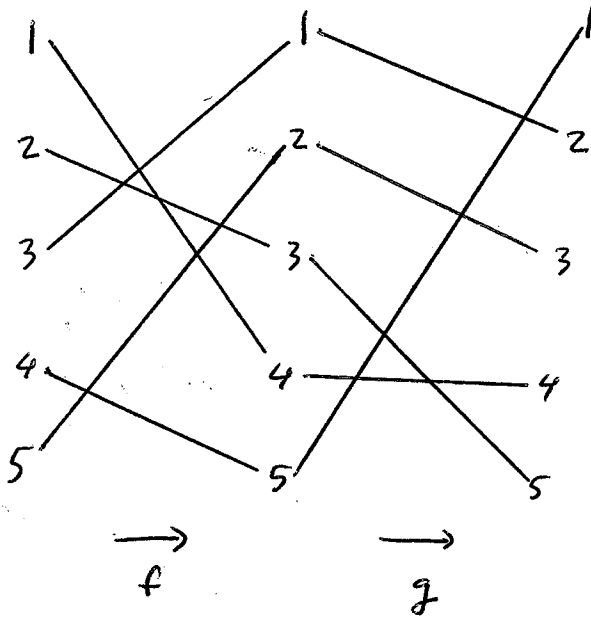


$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

$$\begin{aligned} (f \circ g)(1) &= f(g(1)) \\ &= f(2) \\ &= 3 \end{aligned}$$

etc.

Ex Referring to above f, g find $g \circ f$



$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

Note $g \circ f \neq f \circ g$ in general

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We view composition \circ as a binary operation

on S_n : given f and g in S_n their composition

$f \circ g$ is an element of S_n

LEM For $f, g, h \in S_n$

$$(f \circ g) \circ h = f \circ (g \circ h) \quad *$$

"composition is associative"

pf Each side of $*$ is a function $X \rightarrow X$

where $X = \{1, 2, \dots, n\}$

Show each function sends each $x \in X$ to the same thing

$$(f \circ g) \circ h : \quad x \xrightarrow{h} h(x) \xrightarrow{f \circ g} (f \circ g)(h(x)) = f(g(h(x)))$$

$$f \circ (g \circ h) : \quad x \xrightarrow{g \circ h} (g \circ h)(x) = g(h(x)) \xrightarrow{f} f(g(h(x)))$$

□

From now on we drop parenthesis and write

$f \circ g \circ h$

We abbrev

$$f^2 = f \circ f, \quad f^3 = f \circ f \circ f, \quad f^1 = f$$

etc

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DEF The identity permutation

$$I: X \rightarrow X$$

satisfies

$$I(x) = x$$

$$\forall x \in X$$

" I leaves everything alone "

So

$$I = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

LEM For a perm $f: X \rightarrow X$,

$$f \circ I = f$$

$$I \circ f = f$$

pf $\forall x \in X$

$$\begin{aligned}(f \circ I)(x) &= f(\underbrace{I(x)}_x) \\ &= f(x)\end{aligned}$$

Also

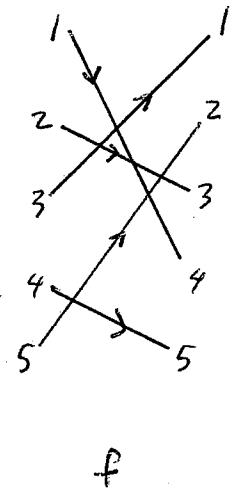
$$\begin{aligned}(I \circ f)(x) &= I(f(x)) \\ &= f(x)\end{aligned}$$

□

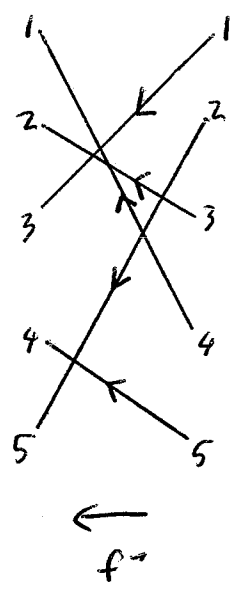
Inverses

Given perm $f: X \rightarrow X$

say $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 5 & 2 \end{pmatrix}$



Change direction of arrows to get a new perm f^{-1}



$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$$

Call f^{-1} the inverse of f

" f^{-1} undoes f "

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Given a perm $f: X \rightarrow X$

then f^{-1} is described as follows:

View I

For all $x, y \in X$

$f(x) = y$ if and only if $f^{-1}(y) = x$

View II

For all $x \in X$

$f^{-1}(f(x)) = x$

ie $f^{-1} \circ f = I$

View III

For all $y \in X$

$f(f^{-1}(y)) = y$

ie $f \circ f^{-1} = I$

LEM Given perm $f: X \rightarrow X$

then for all perms $g: X \rightarrow X$

the following are equivalent:

$$(i) \quad g \circ f = I$$

$$(ii) \quad f \circ g = I$$

$$(iii) \quad g = f^{-1}$$

pf (iii) \rightarrow (i) From View II alone

(iii) \rightarrow (ii) From View III alone

(i) \rightarrow (iii)

$$g \circ f = I$$

$$(g \circ f) \circ f^{-1} = I \circ f^{-1}$$

$$g \circ (f \circ f^{-1})$$

$$g \circ I$$

"

$$g$$

(ii) \rightarrow (iii) similar

□

EX: Given perms

$$f: X \rightarrow X, \quad g: X \rightarrow X$$

find $(f \circ g)^{-1}$

Sol

$$f \circ g: \quad X \xrightarrow{g} X \xrightarrow{f} X$$

$f \circ g$ is undone by

$$(f \circ g)^{-1} \quad X \xleftarrow{g^{-1}} X \xleftarrow{f^{-1}} X$$

So

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

check

$$(g^{-1} \circ f^{-1}) \circ (f \circ g) \stackrel{?}{=} I$$

$$\underbrace{g^{-1} \circ \underbrace{f^{-1} \circ f}_{I} \circ g}_{= I} = I$$

More generally, given perms

$$f_1: X \rightarrow X,$$

$$f_2: X \rightarrow X,$$

...

$$f_r: X \rightarrow X$$

$$(f_1 \circ f_2 \circ \dots \circ f_r)^{-1} = f_r^{-1} \circ \dots \circ f_2^{-1} \circ f_1^{-1}$$

In particular, for any perm

$$f: X \rightarrow X$$

and $r \geq 1$

$$(f^r)^{-1} = (f^{-1})^r$$

Call this commutative f^{-r}

Formally define

$$f^0 = I$$

By construction

$$f^r \circ f^s = f^{r+s}$$

for all integers r, s

$$(f^r)^s = f^{rs}$$

14.1 Cont.

$X =$ nonempty finite set

Say

$$X = \{1, 2, \dots, n\}$$

Given perm

$$f: X \rightarrow X$$

Consider

$$I, f, f^2, f^3, \dots$$

*

Finitely many perms $X \rightarrow X$

Must be duplication among * :

$$f^r = f^s \quad r < s$$

so

$$f^{s-r} = I$$

so

$$\exists m \geq 1 \text{ such that}$$

$$f^m = I$$

Note

$$f^{-1} = f^{m-1}$$

SINCE

$$f \circ f^{m-1} = f^m = I$$

— o —

DEF A permutation group on X is a set

G of perms $X \rightarrow X$ such that:

(1) For all $f, g \in G$

$$f \circ g \in G$$

" G is closed under composition "

(2) $I \in G$

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LEM Given a perm gp G on X .

Then for all $f \in G$

$$f^{-1} \in G$$

" G is closed under taking inverses"

pf

Case $f = I$: ok since $I^{-1} = I$

Case $f \neq I$: $\exists m \geq 1$ such that $f^m = I$

$$\begin{aligned} f^{-1} &= f^{m-1} \\ &= \underbrace{f \circ f \circ \dots \circ f}_{m-1} \end{aligned}$$

$\in G$ since G closed under comp

□

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Examples of Permutation Groups

EX 1

Recall

$S_n =$ set of all perms of $X = \{1, 2, \dots, n\}$

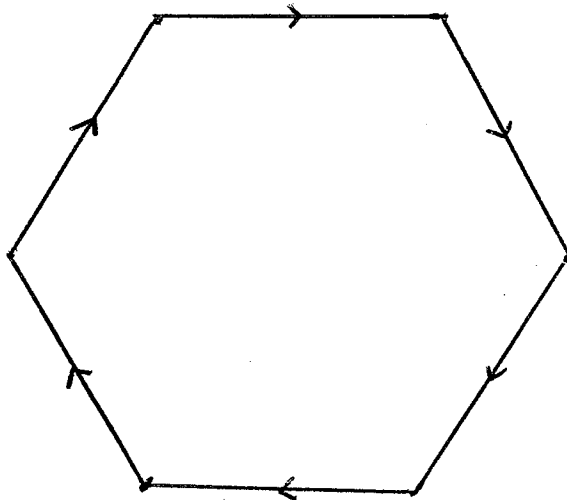
$S_n =$ perm group on X

This group is called the symmetric group of order n

EX 2 Consider an oriented regular n -gon P
in the plane

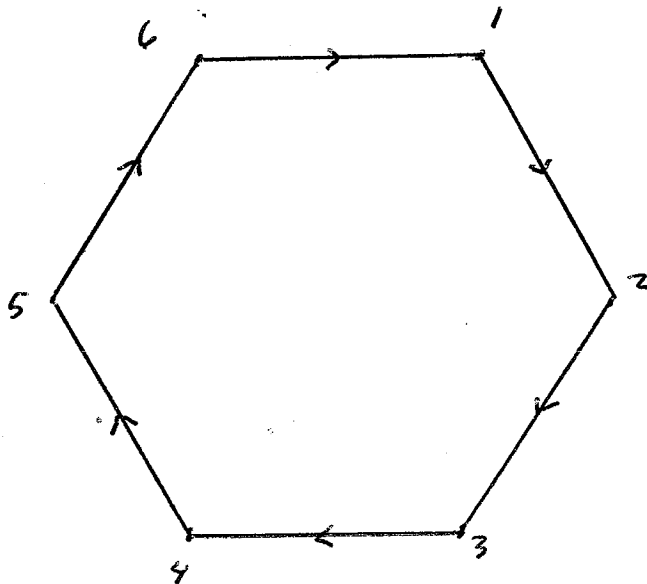
ex $n=6$

P :



View $X =$ set of corners (vertices) of P

P :



$$X = \{1, 2, 3, 4, 5, 6\}$$

P has rotational symmetries:

If we rotate P clockwise by some multiple m

of 60° (or $360/6$)

then result coincides with P_0

Each rotation induces perm of X:

m	perm of X	name of perm
0	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$	I
1	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$	R
2	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$	$R^2 = R \circ R$
3	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$	R^3
4	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$	R^4
5	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	R^5

Note $R^6 = I$

Define

$$G = \text{set of all rotational symmetries of } P \\ = \{ I, R, R^2, R^3, R^4, R^5 \}$$

$$\text{So } |G| = 6$$

Then G is a perm group on X

"cyclic group of order 6"

Inverses:

f	I	R	R^2	R^3	R^4	R^5
f^{-1}	I	R^5	R^4	R^3	R^2	R

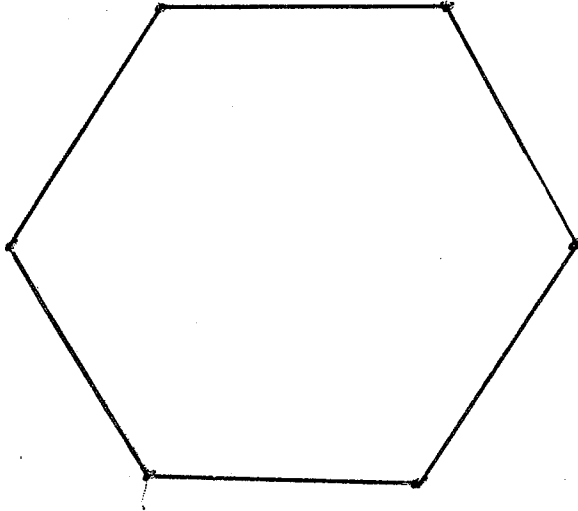
In general, for $n \geq 2$

the cyclic group of order n = group of rotational symmetries
of an oriented regular n -gon

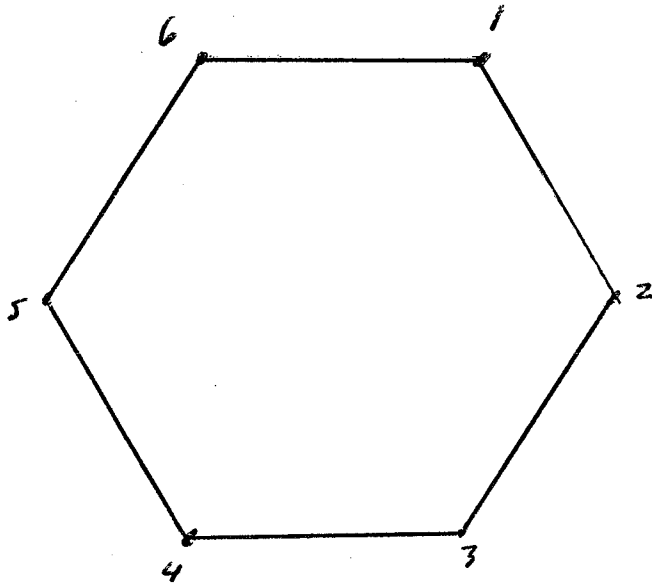
Ex 3 Consider a nonoriented regular n -gon
in the plane

$$n=6$$

$P =$



View $X =$ set of vertices of P as before



$$X = \{1, 2, 3, 4, 5, 6\}$$

P has rotational and reflectonal symmetries

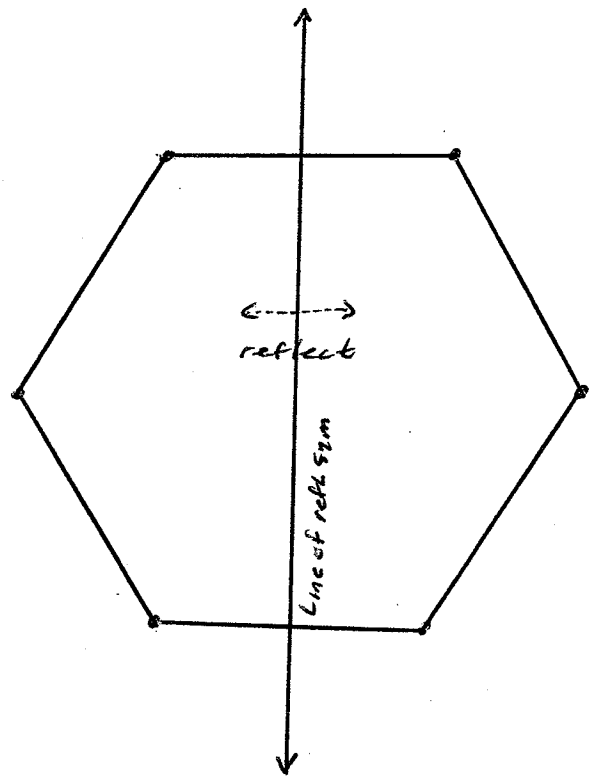
rotational symmetries : 6 of these just as in oriented case

reflectonal symmetries

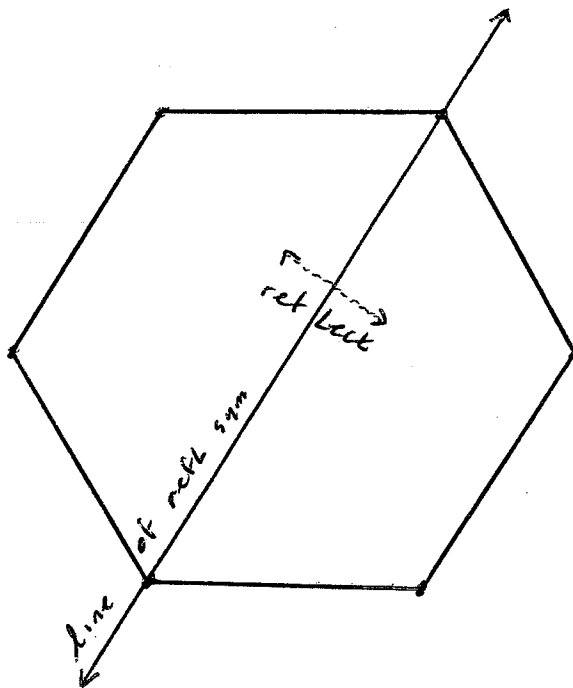
If we reflect P thru a line of reflectonal symmetry,
the result coincides with P

ex

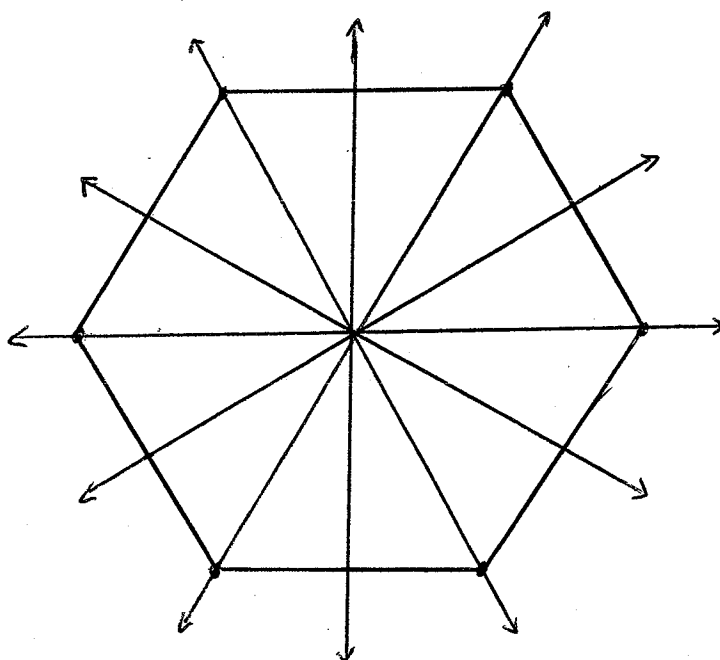
P:



P:



P has 6 lines of reflectonal symmetry:



Each reflectonal symmetry of P induces perm of $X =$

line of sym	perm of X	name
$\begin{array}{c} 6 \ 1 \\ \diagdown \quad / \\ 5 \quad 2 \\ \diagup \quad \diagdown \\ 4 \ 3 \end{array}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$	T
$\begin{array}{c} 6 \ 1 \\ \diagdown \quad / \\ 5 \quad 2 \\ \diagup \quad \diagdown \\ 4 \ 3 \end{array}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$	R ₀ T
$\begin{array}{c} 6 \ 1 \\ \hline 5 \quad 2 \\ 4 \ 3 \end{array}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$	R ² ₀ T
$\begin{array}{c} 6 \ 1 \\ \diagdown \quad / \\ 5 \quad 2 \\ \diagup \quad \diagdown \\ 4 \ 3 \end{array}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}$	R ³ ₀ T
$\begin{array}{c} 6 \ 1 \\ \diagdown \quad / \\ 5 \quad 2 \\ \diagup \quad \diagdown \\ 4 \ 3 \end{array}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$	R ⁴ ₀ T
$\begin{array}{c} 6 \ 1 \\ \quad \\ 5 \quad 2 \\ \quad \\ 4 \ 3 \end{array}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$	R ⁵ ₀ T

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Recall $R^6 = I$

For any reflection

$$(\text{reflection})^2 = I$$

So

each reflection is its own inverse

obs

$$\tau^2 = I$$

$$\tau^{-1} = \tau$$

For $0 \leq i \leq 5$

$$(R^i \circ \tau)^2 = I$$

$$R^i \circ \tau = (R^i \circ \tau)^{-1}$$

$$= \tau^{-1} \circ R^{-i}$$

$$= \tau \circ R^{-i}$$

$$= \tau \circ R^{6-i}$$

Let $G =$ set of all symmetries of P , both rotational and reflectonal

$$= \{R^i\}_{i=0}^5 \cup \{R^i \circ \tau\}_{i=0}^5$$

$$|G| = 12$$

then G is a perm group on X

"dihedral group of order 12"

In general for $n \geq 3$

The dihedral group of order $2n$ = group of symmetries of
the regular n -gon

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Any geometric figure of any dimension has a symmetry group

ex the 5 platonic solids in 3 dimensions

cube

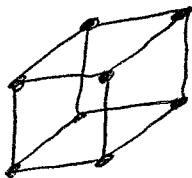
tetrahedron

octahedron

dodecahedron

icosahedron

cube



in 3 dimensions

$X =$ set of vertices $|X| = 8.$

Each symmetry of the cube induces perm of X

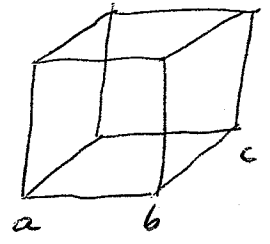
Let $G =$ set of resulting perms of X

$G =$ perm group on X

claim $|G| = 48$

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pf consider 3 vertices



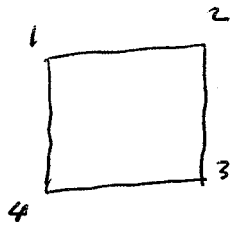
To construct $f \in G$ we define $f(a)$, $f(b)$, $f(c)$
in stages

stage	to do	#choices
1	Pick $f(a)$	8
2	Pick $f(b)$ from among the 3 vertices adjacent to a	3
3	Pick $f(c)$ from among 2 vertices adjacent $f(b)$ other than $f(a)$	2

$$\begin{aligned} \# \text{pos} &= 8 \times 3 \times 2 \\ &= 48 \end{aligned}$$

□

14.1 Cont.

Let $X =$ nonempty finite setSay $X = \{1, 2, \dots, n\}$ Let $G =$ permutation group on X Running example (REX): X is set of vertices for the regular 4-gon

$G =$ the group of symmetries
 $=$ dihedral group of order 8

define

 p : 90° clockwise rotation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

 τ : reflection about

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$G = \{ I, p, p^2, p^3, \tau, p\tau, p^2\tau, p^3\tau \}$$

Def: A coloring of X is an assignment of a color to each element of X
 (distinct elements of X might get the same color)

REX Using colors Red (R) and Blue (B) there are $2^4 = 16$ possible colorings. They are:

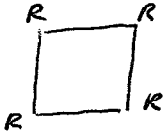
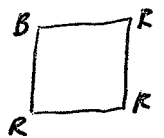
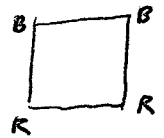


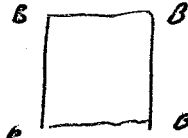
# B	desc	#colorings
0		1
1		4
2		4
2		2
3		4
4		1

Table 1

View each coloring alone as a function

$$\begin{array}{l}
 X \longrightarrow \{R, B\} \\
 i \longrightarrow \text{color assigned} \\
 \quad \quad \quad \text{to } i
 \end{array}$$

Given $f \in G$

Given a coloring c of X

Permute X to get another coloring of X :

$$\begin{array}{ccc}
 X & \xrightarrow{c} & \{R, B\} \\
 f \downarrow & & \\
 X & & \\
 & & \\
 X & \xrightarrow{c} & \{R, B\} \\
 f^{-1} \uparrow & \nearrow & \\
 X & & \text{composition } c \circ f^{-1} \text{ is a} \\
 & & \text{coloring of } X
 \end{array}$$

Define

$$f * c = c \circ f^{-1}$$

Thus

 $f * c$ is a coloring of X

that assigns each element $x \in X$ the

$$\text{color } c(f^{-1}(x))$$

Note f induces a perm of the set of all colorings of X :

$$\{\text{coloring of } X\} \rightarrow \{\text{coloring of } X\}$$

$$c \rightarrow f * c$$

LEM

For a coloring c of X

$$I * c = c$$

 $I = \text{identity elmt of } G$

pf

$$I * c = c \circ I^{-1}$$

$$= c \circ I$$

$$= c$$

LEM Given $f, g \in G$,

Given a coloring c of X ,

then

$$f * (g * c) = (f \circ g) * c$$

pf

$$\begin{aligned} f * (g * c) &= (g * c) \circ f^{-1} \\ &= (c \circ g^{-1}) \circ f^{-1} \\ &= c \circ (g^{-1} \circ f^{-1}) \\ &= c \circ (f \circ g)^{-1} \\ &= (f \circ g) * c \end{aligned}$$

□

LEM Given $f \in G$

Given colorings c_1 and c_2 of X

such that

$$c_2 = f * c_1$$

then

$$c_1 = f^{-1} * c_2$$

pf

$$\begin{aligned} f^{-1} * c_2 &= f^{-1} * (f * c_1) \\ &= (f^{-1} \circ f) * c_1 \\ &= I * c_1 \\ &= c_1 \end{aligned}$$

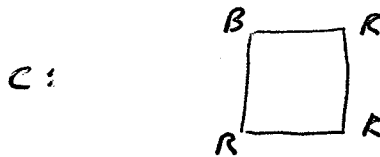
□

REX (i) For the coloring



$t * c = c$ for all $t \in G$

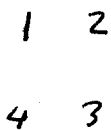
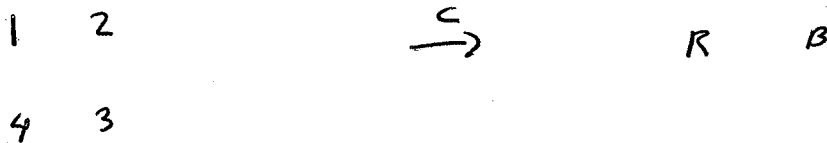
(ii) For



*

Find $p * c$

$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$



"apply p to picture *"

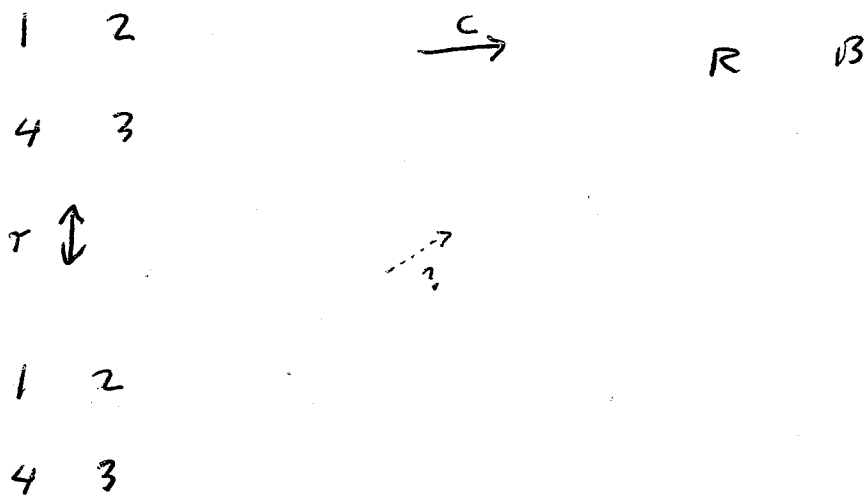
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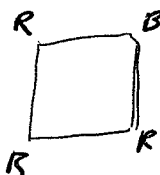
Find

$\gamma * C$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$



$\gamma * C$



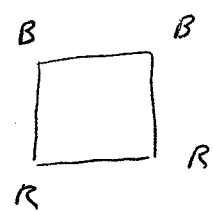
"applies γ to picture $*$ "

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We record a principle

Given a coloring c of X , say



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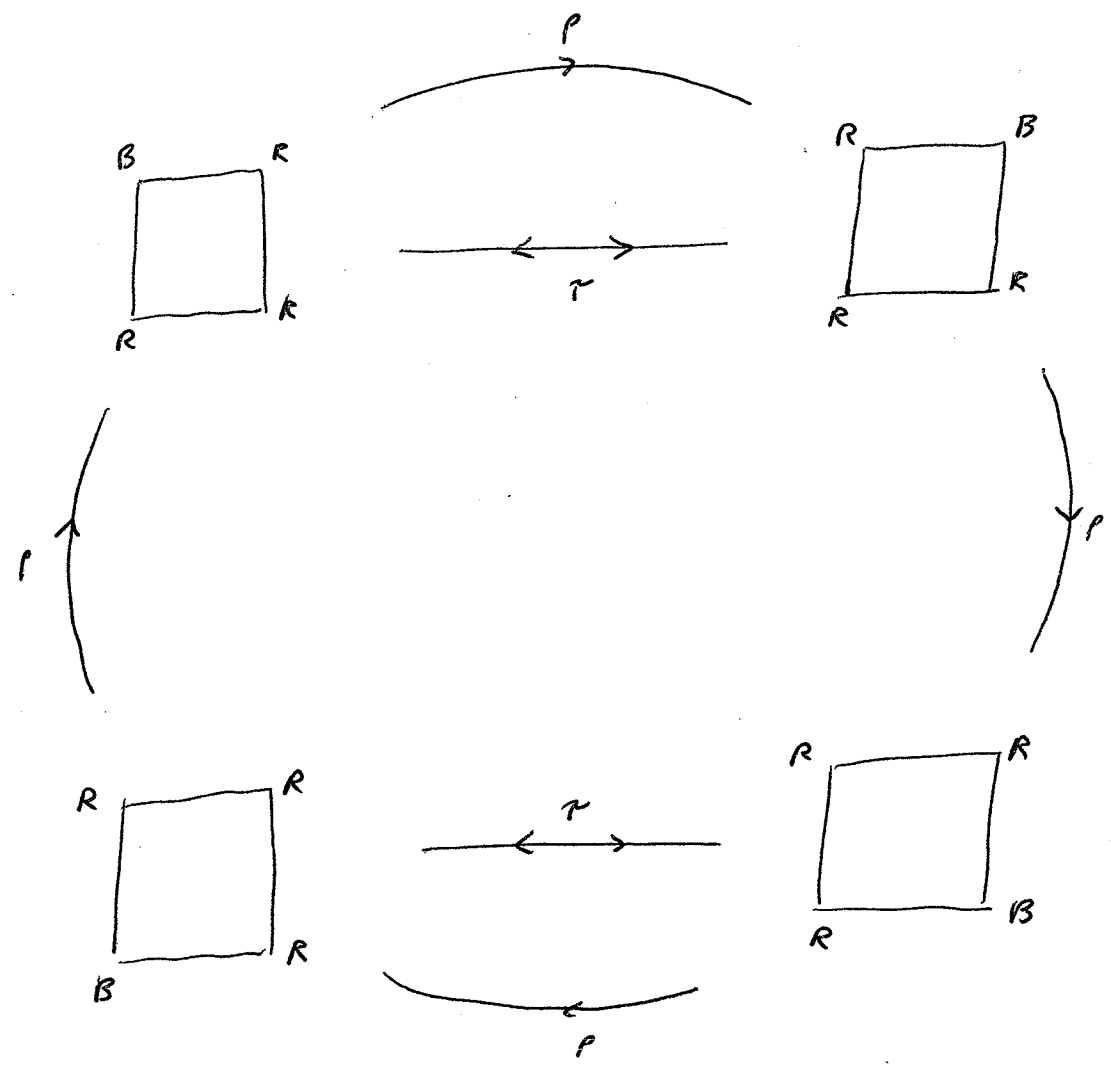
Then for all $f \in G$,

the coloring $f * c$ is obtained by applying f
to the picture *

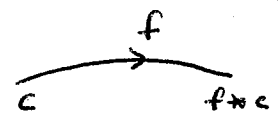
REX

Consider the set of colorings of X that have one B

G acts on this set as follows



key: For $f \in G$ and a coloring c of X ,



REX

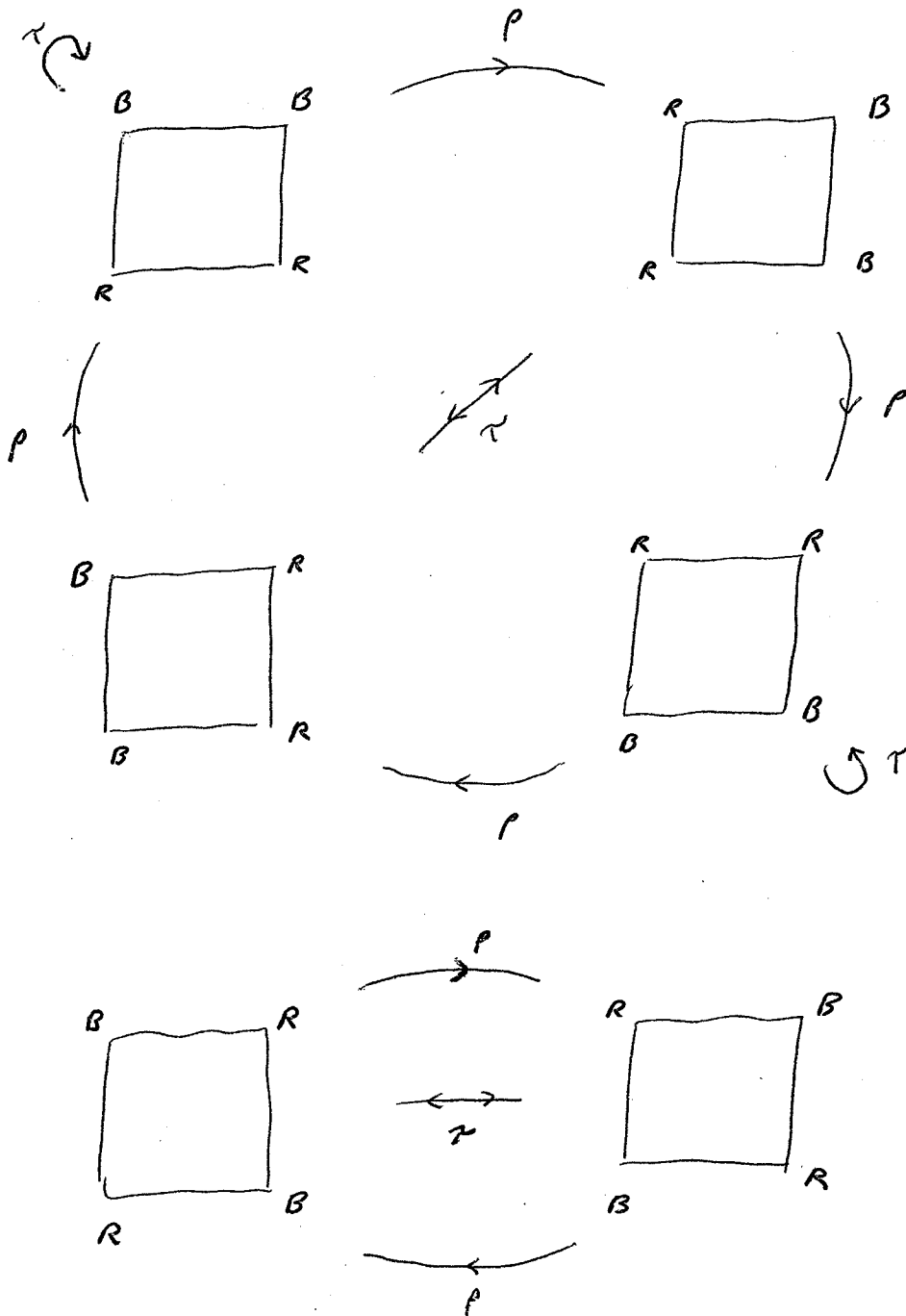
Consider the set of colorings of X

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that have two B's.

G acts on this set as follows:



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Above diagram has 2 connected components.

Given colourings c_1, c_2 of X

call them G -equivalent and write $c_1 \overset{G}{\sim} c_2$

whenever they are in the same connected component of the diagram.

then

$$c_1 \overset{G}{\sim} c_2$$

means

there exists $f \in G$ such that

$$c_2 = f * c_1$$

The relation $\overset{G}{\sim}$ is an equivalence relation and the equivalence classes are the connected components of the diagram.

Formal verification that \sim^G is an equivalence

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- For all colorings c of X show

$$c \sim^G c$$

pf

$$c = I * c$$

$$I = \text{identity} \in G$$

- For all colorings c_1, c_2 of X such that $c_1 \sim^G c_2$ show

$$c_2 \sim^G c_1$$

pf Since $c_1 \sim^G c_2 \exists f \in G$ st. $c_2 = f * c_1$

Now $c_1 = f^{-1} * c_2$ and $f^{-1} \in G$ so

$$c_2 \sim^G c_1$$

- For all colorings c_1, c_2, c_3 of X such that

$$c_1 \sim^G c_2 \quad \text{and} \quad c_2 \sim^G c_3$$

show

$$c_1 \sim^G c_3$$

pf

 $\exists f \in G$ s.t.

$$c_2 = f * c_1$$

 $\exists g \in G$ s.t.

$$c_3 = g * c_2$$

So

$$\begin{aligned}
 c_3 &= g * c_2 \\
 &= g * (f * c_1) \\
 &= \underbrace{(g \circ f)}_{\in G} * c_1
 \end{aligned}$$

So

$$c_1 \stackrel{G}{\sim} c_3$$

□

12/3/12

14

REX Consider the set of the colourings
of X using colors R, B

The relation \sim_G on this set has 6 equivalence
classes, one for each row of table 1.

So up to G -equivalence, there are 6
colourings of X with R, B

— 0 —

14.2 Burnside's Theorem

Let

 $X =$ nonempty finite set

$$X = \{1, 2, \dots, n\}$$

 $G =$ a permutation group on X

Pick some colors and consider the set of all colorings of X with those colors.

We saw that G induces a perm gp on that set.

Let $C =$ a nonempty subset of the above set of colorings of X .

Call C G -closed whenever

$$\forall c \in C \quad \forall g \in G \quad gc \in C$$

In this case G induces perm gp on C .

REX For each row in table I, the given set of colorings is G -closed.

12/5/12

From now on assume C is G -closed.

2

Recall that for colorings $c_1, c_2 \in C$

$$c_1 \stackrel{G}{\sim} c_2 \quad \text{means}$$

" G -equivalent"

$$\exists f \in G \text{ s.t. } f * c_1 = c_2$$

$\stackrel{G}{\sim}$ is an equiv relation on C

$C =$ disjoint union of $\stackrel{G}{\sim}$ equivalence classes

Define

$$N(G, C) = \# \text{ equivalence classes of } \stackrel{G}{\sim} \text{ on } C$$

$$= \# \text{ of mutually } G\text{-inequivalent}$$

colorings in C

REX If we take

$$C = \text{all 16 colorings of } X$$

$$\text{then } N(G, C) = 6.$$

Since there is 1 $\stackrel{G}{\sim}$ equiv class for each of the

6 rows of table I.

next goal: Find formula for $N(G, C)$

$\forall c \in C$ and $f \in G$

we say

f stabilizes c

or

f fixes c

whenever

$$f * c = c$$

$\forall c \in C$ define

$$G(c) = \{ f \in G \mid f * c = c \}$$

"the stabilizer of c in G "

LEM Given $c \in C$, then

$G(c)$ is a permutation gp on C

pf • Given $f, g \in G(c)$ show $f \circ g \in G(c)$:

We have $f \# c = c$, $g \# c = c$

Obs

$$\begin{aligned}(f \circ g) \# c &= f \# (g \# c) \\ &= f \# c \\ &= c\end{aligned}$$

• Show $I \in G(c)$:

$$I \# c = c \quad \checkmark$$

□

LEM Given $c \in C$.

Then for all $f, g \in G$ the following are equivalent:

(i) $f * c = g * c$

(ii) $f^{-1} \circ g \in G(c)$

pf

$$f * c = g * c$$

\Leftrightarrow

$$f^{-1} * (f * c) = f^{-1} * (g * c)$$

"

$$(f^{-1} \circ f) * c$$

"

$$I * c$$

"

$$c$$

"

$$(f^{-1} \circ g) * c$$

\Leftrightarrow

$$f^{-1} \circ g \in G(c)$$

□

Given $c \in C$

Consider the equivalence class \sim_G containing c

This is

$$\{ f * c \mid f \in G \}$$

thm For $c \in C$

$$|\{ f * c \mid f \in G \}| = \frac{|G|}{|G(c)|}$$

pf Abbr

$$Y = \{ f * c \mid f \in G \}$$

For $y \in Y$ define

$$G^{(y)} = \{ f \in G \mid f * c = y \}$$

$\{ G^{(y)} \}_{y \in Y}$ partition G

so

$$|G| = \sum_{y \in Y} |G^{(y)}|$$

For $y \in Y$ show

$$|G^{(y)}| = |G(c)|$$

For $f \in G^{(y)}$, the map

$$G(c) \rightarrow G^{(y)}$$

$$h \rightarrow f \circ h$$

is a bijection (by prev LEM)

So

$$|G(c)| = |G^{(y)}|$$

Now

$$|G| = \sum_{y \in Y} |G^{(y)}|$$

" $|G(c)|$

$$= |Y| |G(c)|$$

So

$$|Y| = \frac{|G|}{|G(c)|}$$

□

Def For $f \in G$ define

$$C(f) = \{c \in C \mid f * c = c\}$$

"set of colorings in C that are fixed by f "

So for $f \in G$ and $c \in C$

$$c \in C(f) \iff f * c = c \iff f \in G(c)$$

thm (Burnside)

$$N(G, C) = \frac{\sum_{f \in G} |C(f)|}{|G|}$$

↑

of equivalence classes of \sim on C

pf let

$S =$ set of ordered pairs (f, c) such that
 $f \in G$ and $c \in C$ and $f * c = c$

We compute $|S|$ in two ways

$$\text{I} \quad |S| = \sum_{f \in G} |C(f)|$$

$$\text{II} \quad |S| = \sum_{c \in C} |G(c)|$$

