MATH 846 Final Presentation:
Quantum Isomorphism of Graphs from Association Scheme

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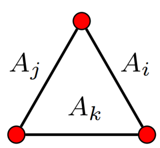
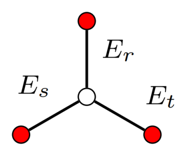
1 Delta-Wye parameters

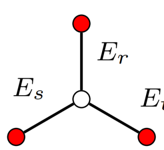
Theorem 1.1. *Let $(X, \mathcal{R})$ be an exactly triply regular d-class association scheme with Bose-Mesner algebra having an ordered basis $A_0, \ldots, A_d$ of adjacency matrices and an ordered basis $E_0, \ldots, E_d$ of primitive idempotents. Then there exist unique constants $\{\sigma_{r,s,t}^{i,j,k} : p_{ij}^k > 0, q_{rs}^t > 0\}$ and $\{\tau_{i,j,k}^{r,s,t} : p_{ij}^k > 0, q_{rs}^t > 0\}$ such that*



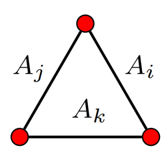
$$= \sum_{q_{rs}^t > 0} \sigma_{r,s,t}^{i,j,k}$$



(1)



$$= \sum_{p_{ij}^k > 0} \tau_{i,j,k}^{r,s,t}$$



(2)

A proof of Theorem 1.1 can be found in Williams[3]. These values are called the Delta-Wye parameters of $(X, \mathcal{R})$.

Definition 1.2. *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be exactly triply regular d-class association schemes. We say that they have the same Delta-Wye parameters, if there exists an ordering $A_0, \ldots, A_d$ and $A_0', \ldots, A_d'$ of the respective adjacency matrices and an ordering $E_0, \ldots, E_d$ and $E_0', \ldots, E_d'$ of their respective primitive idempotents such that $\sigma_{r,s,t}^{i,j,k} = (\sigma')_{r,s,t}^{i,j,k}$, and $\tau_{i,j,k}^{r,s,t} = (\tau')_{i,j,k}^{r,s,t}$.*

Lemma 1.3. *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be exactly triply regular d-class association schemes with the same Delta-Wye parameters. Then, the association schemes $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ have the same intersection numbers, Krein parameters, eigenvalues, and dual eigenvalues. Moreover, the bijective map $A_i \mapsto A_i'$ extends to a linear isomorphism $\kappa : \mathbb{A} \to \mathbb{A}'$, such that $\kappa(MN) = \kappa(M)\kappa(N)$ and $\kappa(M \circ N) = \kappa(M) \circ \kappa(N)$.*

Proof. We consider Eq. (1), for a fixed i,j,k. We sum up all coefficients of all the tensors on both the LHS and the RHS. We will first consider the RHS. We see that

$$\text{RHS} = \sum_{q^t_{rs}>0} \sigma^{i,j,k}_{r,s,t} \sum_{x,y,z,u\in X} (E_r)_{xu}(E_s)_{yu}(E_t)_{zu} = \sum_{q^t_{rs}>0} \sigma^{i,j,k}_{r,s,t} \sum_{y\in X}\sum_{x,z\in X} (E_r A^*_s E_t)_{xz}$$

where each A^*_s in the sum is with respect to y [4].

It is known that $\{E_r A^*_r E_t : 0 \le r,s,t \le d, q^t_{rs} > 0\}$ is an orthogonal basis of $\mathbb{A}\mathbb{A}^*\mathbb{A}$ [5]. We note that when $r = s = t = 0$, $E_0 A^*_0 E_0 = E_0 = \frac{1}{|X|}J$, where J is the all-ones matrix. Since every other matrix of the form $E_r A^*_s E_t$ is orthogonal to this matrix, it follows that for all $(r,s,t) \neq (0,0,0)$,

$$\sum_{x,z\in X} (E_r A^*_s E_t)_{xz} = 0.$$

Therefore, we see that

$$\text{RHS} = \sigma^{i,j,k}_{0,0,0}(|X|).$$

Now, when $i = j$ and $k = 0$, we see that the sum of coefficients on the LHS is

$$\text{LHS} = \sum_{x,y,z\in X} (A_i)_{x,y}(A_i)_{y,z}(A_0)_{z,x} = \sum_{x,y\in X} (A_i)_{x,y} = |X|p^0_{ii}.$$

This proves that $p^0_{ii} = \sigma^{i,i,0}_{0,0,0}$. For any other choice of i,j,k, we see that the sum of coefficients on the LHS is $|X|p^k_{ij}p^0_{ii}$.

This proves that the Delta-Wye parameters determine the intersection numbers. The remaining association scheme parameters are then recoverable from the intersection numbers.

Since $\kappa(A_i) = A'_i$ is a $0-1$ matrix and both $A_i \circ A_j = \delta_{ij}A_i$ and $A'_i \circ A'_j = \delta_{ij}A'_i$ holds for this pair of bases, we have $\kappa(M \circ N) = \kappa(M) \circ \kappa(N)$ by linearity. Also,

$$\kappa(A_iA_j) = \kappa\left(\sum_{k=0}^{d} p^k_{ij}A_k\right) = \sum_{k=0}^{d} p^k_{ij}A'_k = A'_iA'_j = \kappa(A_i)\kappa(A_j).$$

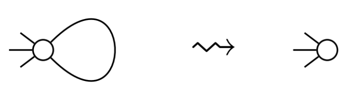
□

2 Main Theorem

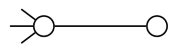
Theorem 2.1 (Main Theorem)**.** *Let $(X,\mathcal{R})$ and $(Y,\mathcal{S})$ be exactly triply regular symmetric association schemes that have the same Delta-Wye parameters. Let G be a graph whose adjacency matrix M lies in the Bose-Mesner algebra of $(X,\mathcal{R})$. Then, $\kappa(M)$ is the adjacency matrix of a graph G', and the two graphs G and G' are quantum isomorphic.*

2.1 Epifanov's theorem

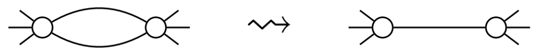
Given an embedding of a planar graph, we consider the following local operations:



Local Transformation 1: **loop**

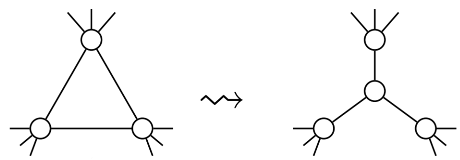


Local Transformation 2: **pendent**

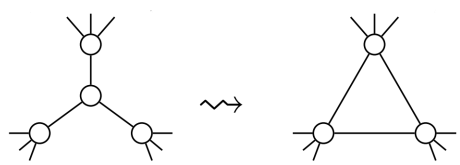


Local Transformation 3: **parallel**

Local Transformation 4: **series**



Local Transformation 5: **Delta**



Local Transformation 6: **Wye**

Theorem 2.2 (Epifanov's Theorem)**.** *Let F be a connected planar graph. There exist a sequence of planar graphs $(F_0, \ldots, F_l)$ such that*

1. *$F_0 = F$ and F_l is a graph with a single vertex and no edges.*
2. *F_{h+1} is obtained from F_h using one of the local transformations above, for $0 \leq h < l$.*

Proofs of Epifanov's Theorem can be found in Feo and Provan[2] and Truemper[6].

2.2 Main Lemma

We consider the problem of computing $\mathrm{S}(F, \emptyset; w)$ for planar F and $w : E(F) \to \mathbb{A}$. Recall that

$$\mathrm{S}(F, \emptyset; w) = \sum_{\phi: V(F) \to X} \left(\prod_{\substack{e \in E(F) \\ e=(u,v)}} w(e)_{\phi(u), \varphi(v)} \right).$$

If F has connected components $F^1, \ldots, F^k$, then it can be seen that

$$\mathrm{S}(F, \emptyset; w) = \prod_{i=1}^{k} \mathrm{S}(F^i, \emptyset; w|_{F_i}). \tag{3}$$

Therefore, we can restrict our attention to connected, planar F. We know from Epifanov's Theorem, that there exists a sequence of planar graphs $(F_0, \ldots, F_l)$.

The main technical lemma used in the proof of the Main Theorem is as follows:

Lemma 2.3 (Main Lemma)**.** *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be an exactly triply regular symmetric association schemes with the same Delta-Wye parameters. Let κ be the isomorphism from Lemma 1.3, and let $\kappa \bullet w$ represent the function composition. For any $0 \leq h < l$, there exists a positive integer m_h, constants $\{\alpha_{h,m}\}_{m=1}^{m_h}$, and weight functions $\{w_{h,m}\}_{m=1}^{m_h}$ where each $w_{h,m} : E(F_{h+1}) \to \mathbb{A}$, such that*

$$\mathrm{S}(F_h, \emptyset; w) = \sum_{m=1}^{m_h} \alpha_{h,m} \mathrm{S}(F_{h+1}, \emptyset; w_{h,m}); \qquad \mathrm{S}(F_h, \emptyset; \kappa \bullet w) = \sum_{m=1}^{m_h} \alpha_{h,m} \mathrm{S}(F_{h+1}, \emptyset; \kappa \bullet w_{h,m}). \tag{4}$$

Before proving Lemma 2.3, we will quickly see how it implies the Main Theorem. By repeated application of Lemma 2.3, we see that there exists a positive integer M, constants $\{\beta_m\}_{m=1}^{M}$ and weight functions $\{w_m\}_{m=1}^{M}$ such that

$$\mathrm{S}(F_0, \emptyset; w) = \sum_{m=1}^{M} \beta_m \mathrm{S}(F_l, \emptyset; w_m); \qquad \mathrm{S}(F_0, \emptyset; \kappa \bullet w) = \sum_{m=1}^{M} \beta_m \mathrm{S}(F_l, \emptyset; \kappa \bullet w_m).$$

Then, we note that F_l is the graph with just one vertex and no edges. Therefore, for any weight function $w : \emptyset \to \mathrm{Mat}_X(\mathbb{C})$, we have that $\mathrm{S}(F_l, \emptyset; w) = |X|$. This implies that $\mathrm{S}(F, \emptyset; w) = \mathrm{S}(F, \emptyset; \kappa \bullet w)$. If we let w be the function such that $w(e) = M$ for all $e \in E(F)$, then we see that for any planar F,

$$\hom(F, G) = \mathrm{S}(F, \emptyset; w) = \mathrm{S}(F, \emptyset; \kappa \bullet w) = \hom(F, G').$$

This implies that G and G' are quantum isomorphic.

2.3 Proof of the Main Lemma

We will focus on the weight function w. Consider an edge $e_1 = (u_1, v_1) \in E(F)$. We can represent w as $w = a_0 w_0 + \cdots + a_d w_d$, where $\{a_i\}_{i=0}^{d}$ are constants, and each w_i is defined as:

$$w_i(e) = \begin{cases} A_i & \text{if } e = e_1 \\ w(e) & \text{if } e \neq e_1 \end{cases}$$

Lemma 2.4.

$$\mathrm{S}(F, \emptyset; w) = \sum_{i=0}^{d} a_i \cdot \mathrm{S}(F, \emptyset; w_i).$$

Proof. By definition, we see that

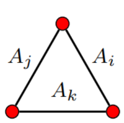
$$\begin{aligned}
\mathrm{S}(F,\emptyset;w) &= \sum_{\phi:V(F)\to X}\left((a_0w_0+\cdots+a_dw_d)(e_1)_{\phi(u_1),\phi(v_1)}\cdot\prod_{\substack{e\in E(F)\setminus\{e_1\}\\ e=(u,v)}} w(e)_{\phi(u),\varphi(v)}\right)\\
&= \sum_{i=0}^{d} a_i \sum_{\phi:V(F)\to X}\left(w_i(e_1)_{\phi(u_1),\phi(v_1)}\cdot\prod_{\substack{e\in E(F)\setminus\{e_1\}\\ e=(u,v)}} w(e)_{\phi(u),\varphi(v)}\right)\\
&= \sum_{i=0}^{d} a_i\cdot \mathrm{S}(F,\emptyset;w_i).
\end{aligned}$$

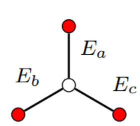
□

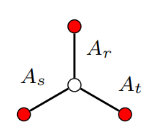
By repeated application of Lemma 2.4 on each edge of $E(F)$, we can assume that the range of the weight function w is $\{A_0,\dots,A_d\}$.

Now, we consider the local transformation $F_h \mapsto F_{h+1}$. We will consider each of the local transformations possible.

1. **loop**: Let e be the loop that is deleted. We may assume that $w(e)=A_r$. In this case, $m_h=1$, $\alpha_{h,1}=\delta_{r,0}$, and $w_{h,1}=w|_{E(F)\setminus\{e\}}$.
2. **pendent**: Let e be the edge that is deleted. We may assume that $w(e)=A_r$. In this case, $m_h=1$, $\alpha_{h,1}=p^0_{r,r}$, and $w_{h,1}=w|_{E(F)\setminus\{e\}}$.
3. **parallel**: Let e be the edge that is deleted, and let e' be its parallel edge. We may assume that $w(e)=A_r$ and $w(e')=A_s$. In this case, $m_h=1$, $\alpha_{h,1}=\delta_{s,r}$, and $w_{h,1}=w|_{E(F)\setminus\{e\}}$.
4. **series**: Let e and e' be the edges in series, that are replaced with the single edge e'' in F_{h+1}. We may assume that $w(e)=A_r$ and $w(e')=A_s$. In this case, $m_h=d+1$, $\alpha_{h,m}=p^m_{r,s}$, and $w_{h,m}$ is defined as
$$w_{h,m}(e)=\begin{cases}A_m & \text{if } e=e''\\ w(e) & \text{if } e\neq e''\end{cases}$$
5. **Delta**: Let e_1,e_2,e_3 be the edges that are replaced with the edges e'_1,e'_2,e'_3 in F_{h+1} (with e'_i being the edge that is not incident on any of the endpoints of the edge e_i). We may assume that $w(e_1)=A_i, w(e_2)=A_j$, and $w(e_3)=A_k$. Now, we note that from Eq. (1), we have that



$$=\sum_{q^c_{ab}>0}\sigma^{i,j,k}_{a,b,c}$$


$$=\frac{1}{|X|^3}\sum_{q^c_{ab}>0}\sigma^{i,j,k}_{a,b,c}\sum_{r,s,t}Q_{ra}Q_{sb}Q_{tc}$$


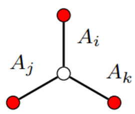
(5)

Consequently, we may let $m_h=(d+1)^3$, and for each $m=(r,s,t)\in[d]^3$, we let
$$\alpha_{h,(r,s,t)}=\frac{1}{|X|^3}\sum_{a,b,c:\ q^c_{ab}>0}\sigma^{i,j,k}_{a,b,c}Q_{ra}Q_{sb}Q_{tc},$$

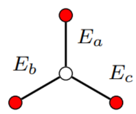
and we define $w_{h,(r,s,t)}$ as

$$w_{h,(r,s,t)}(e) = \begin{cases} A_r & \text{if } e = e'_1 \\ A_s & \text{if } e = e'_2 \\ A_t & \text{if } e = e'_3 \\ w(e) & \text{if } e \notin \{e'_1, e'_2, e'_3\} \end{cases}$$

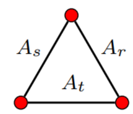
6. **Wye**: Let e_1, e_2, e_3 be the edges that are replaced with the edges e'_1, e'_2, e'_3 in F_{h+1} (with e'_i being the edge that is incident on the two vertices that e_i is not incident on). We may assume that $w(e_1) = A_i, w(e_2) = A_j$, and $w(e_3) = A_k$. Now, we note that from Eq. (2), we have that



$$= \sum_{a,b,c} P_{ai} P_{bj} P_{ck}$$



$$= \sum_{a,b,c} P_{ai} P_{bj} P_{ck} \sum_{p^t_{rs} > 0} \tau^{a,b,c}_{r,s,t}$$



(6)

Consequently, we may let $m_h = (d+1)^3$, and for each $m = (r,s,t) \in [d]^3$, we let

$$\alpha_{h,(r,s,t)} = \sum_{a,b,c=0}^{d} \tau^{a,b,c}_{r,s,t} P_{ai} P_{bj} P_{ck},$$

and we define $w_{h,(r,s,t)}$ as

$$w_{h,(r,s,t)}(e) = \begin{cases} A_r & \text{if } e = e'_1 \\ A_s & \text{if } e = e'_2 \\ A_t & \text{if } e = e'_3 \\ w(e) & \text{if } e \notin \{e'_1, e'_2, e'_3\} \end{cases}$$

We see that Eq. (4) holds true for each of the transformations above.

References

[1] Ada Chan and William J Martin. Quantum isomorphism of graphs from association schemes. *arXiv preprint arXiv:2209.04581*, 2022.

[2] Thomas A Feo and J Scott Provan. Delta-wye transformations and the efficient reduction of two-terminal planar graphs. *Operations Research*, 41(3):572–582, 1993.

[3] William J Martin. Scaffolds: a graph-theoretic tool for tensor computations related to bose-mesner algebras. *Linear Algebra and its Applications*, 619:50–106, 2021.

[4] Paul Terwilliger. Course lecture notes, Math 846 Algebraic Graph Theory - Lecture 7, Spring term 2023. https://people.math.wisc.edu/~terwilli/Htmlfiles/as7.pdf [Accessed: April 24, 2023].

[5] Paul Terwilliger. Course lecture notes, Math 846 Algebraic Graph Theory - Lecture 8, Spring term 2023. https://people.math.wisc.edu/~terwilli/Htmlfiles/as8.pdf [Accessed: April 24, 2023].

[6] Klaus Truemper. On the delta-wye reduction for planar graphs. *Journal of graph theory*, 13(2):141–148, 1989.