Spring 2023 Math 846

Algebraic Combinatorics: Association Schemes
Paul Terwilliger

Lecture 1

Our textbook is:

Bannai, Bannai, Ito, Tanaka. Algebraic Combinatorics. De Gruyter Series in Discrete Mathematics and Applications 5 (2021).

We will begin with Chapter 2. Chapter 1 is an elementary introduction, and mostly discusses special cases of the material in later chapters. Hopefully, we can cover Chapters 2–5.

In addition to the text, the following publications are handy references:

E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes.* Benjamin/Cummings, London, 1984.

A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs.* Springer-Verlag, Berlin, 1989.

W. Martin, H. Tanaka. Commutative association schemes. European J. Combin. 30 (2009) 1497–1525.

P. Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Research Reports Suppl.* 10 (1973).

1 The definition of an association scheme

Let X denote a nonempty finite set. We will speak of the Cartesian product $X \times X = \{(x, y)|x, y \in X\}$.

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Definition 1.1. For $d \in \mathbb{N}$, an *association scheme of class* d is a sequence $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$, where X is a nonempty finite set, and $\{R_i\}_{i=0}^d$ are nonempty subsets of $X \times X$ such that:

(i) $R_0 = \{(x, x)|x \in X\}$;

(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ (disjoint union);

(iii) for $0 \le i \le d$ there exists $i' \in \{0, 1, \ldots, d\}$ such that $R_i^t = R_{i'}$, where

$$R_i^t = \{(y, x)|(x, y) \in R_i\};$$

(iv) for $0 \le i, j, k \le d$ there exists a natural number $p_{i,j}^k$ such that for all $(x, y) \in R_k$,

$$p_{i,j}^k = |\{z \in X|(x, z) \in R_i \text{ and } (z, y) \in R_j\}|.$$

The elements of X are called the *vertices* of $\mathcal{X}$. We call R_i the i^{th} *relation* of $\mathcal{X}$. The relation R_0 is called *trivial.* We call $p_{i,j}^k$ an *intersection number* of $\mathcal{X}$.

We mention two special cases of association schemes.

Definition 1.2. Referring to the association scheme $\mathcal{X}$ in Definition 1.1,

(i) $\mathcal{X}$ is *commutative* whenever

$$p^k_{i,j} = p^k_{j,i} \qquad (0 \le i, j, k \le d).$$

(ii) $\mathcal{X}$ is *symmetric* whenever

$$i' = i \qquad (0 \le i \le d).$$

Lemma 1.3. *A symmetric association scheme is commutative.*

Proof. Referring to Definition 1.1, assume that $\mathcal{X}$ is symmetric. For $0 \le i, j, k \le d$ we show that $p^k_{i,j} = p^k_{j,i}$. Pick $x, y \in X$ with $(x, y) \in R_k$. Then $(y, x) \in R^t_k = R_{k'} = R_k$.
Since $(x, y) \in R_k$,

$$p^k_{j,i} = |\{z \in X | (x, z) \in R_j \text{ and } (z, y) \in R_i\}|.$$

Since $(y, x) \in R_k$,

$$p^k_{i,j} = |\{z \in X | (y, z) \in R_i \text{ and } (z, x) \in R_j\}|.$$

For $z \in X$,

$$(x, z) \in R_j \text{ iff } (z, x) \in R_j \qquad (z, y) \in R_i \text{ iff } (y, z) \in R_i.$$

By these comments $p^k_{i,j} = p^k_{j,i}$. □

We give some examples of association schemes.

Consider a finite group G acting on a set X. This action is called *transitive* whenever for all $x, y \in X$ there exists $g \in G$ such that $x^g = y$.

Example 1.4. Let G denote a finite group acting transitively on a set X. Consider the action of G on $X \times X$ such that

$$(x, y)^g = (x^g, y^g) \qquad g \in G, \quad x, y \in X.$$

Let $\{R_i\}_{i=0}^d$ denote the orbits of G on $X \times X$, ordered such that $R_0 = \{(x, x) | x \in X\}$. Then $(X, \{R_i\}_{i=0}^d)$ is an association scheme (not commutative in general).

Proof. We check the axioms in Definition 1.1.
(i), (ii) Clear.
(iii) For $0 \le i \le d$, R^t_i is an orbit of G on $X \times X$.
(iv) Let $0 \le i, j, k \le d$ and $(x, y) \in R_k$. We show that for $g \in G$, the following sets have the same size:

$$\{z \in X | (x, z) \in R_i \text{ and } (z, y) \in R_j\}, \tag{1}$$
$$\{w \in X | (x^g, w) \in R_i \text{ and } (w, y^g) \in R_j\}. \tag{2}$$

This holds because the map $z \mapsto z^g$ gives a bijection from (1) to (2). □

Consider a finite group G acting on a set X. This action is called *generously transitive* whenever for all $x, y \in X$ there exists $g \in G$ such that $x^g = y$ and $y^g = x$.

Example 1.5. Referring to Example 1.4, the following are equivalent:

(i) the action of G on X is generously transitive;

(ii) for all $x, y \in X$ the ordered pairs (x, y) and (y, x) are in the same orbit of G on $X \times X$;

(iii) the association scheme $(X, \{R_i\}_{i=0}^d)$ is symmetric.

Proof. Routine. □

Example 1.6. (Hamming association scheme $H(d, q)$). Fix integers $d, q \geq 1$. Fix a set F with $|F| = q$. Define a set

$$X = F \times F \times \cdots \times F \qquad\qquad (d \text{ copies}).$$

For $x = (x_1, x_2, \ldots, x_d) \in X$ and $y = (y_1, y_2, \ldots, y_d) \in X$, define their *Hamming distance*

$$\partial(x, y) = |\{i | 1 \leq i \leq d, \;\; x_i \neq y_i\}|.$$

For $0 \leq i \leq d$ define

$$R_i = \{(x, y) | x, y \in X, \;\; \partial(x, y) = i\}.$$

Then $(X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme, denoted $H(d, q)$.

Proof. This is a special case of Example 1.5, with G defined as follows. Let S_q denote the symmetric group on F, which consists of the permutations of F. Consider the direct sum $S = S_q \oplus S_q \oplus \cdots \oplus S_q$ (d copies). Then S acts on X by permuting each copy of F. Next consider the symmetric group S_d. This group acts on X by permuting the coordinates $1, 2, \ldots, d$. The group G consists of the permutations of X obtained by applying an element of S followed by an element of S_d. The group G is generously transitive on X. It is routine to check that $\{R_i\}_{i=0}^d$ are the orbits of G on $X \times X$. □

Lecture 2

Example 1.7. (Johnson association scheme $J(v, d)$). Fix integers $v, d \geq 1$ with $d \leq v/2$. Fix a set V with $|V| = v$. Let the set X consist of the subsets of V that have cardinality d. For $0 \leq i \leq d$ define

$$R_i = \{(x, y) | x, y \in X, \;\; |x \cap y| = d - i\}.$$

Then $(X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme, denoted $J(v, d)$.

Proof. This is a special case of Example 1.5, where $G = S_v$ is the symmetric group on V. The action of G on V induces an action of G on X. The action of G on X is generously transitive. It is routine to check that $\{R_i\}_{i=0}^d$ are the orbits of G on $X \times X$. □

Example 1.8. (Conjugacy classes of a finite group G). Let G denote a finite group. Elements $x, y \in G$ are called *conjugate* whenever there exists $g \in G$ such that $gxg^{-1} = y$. Conjugacy is an equivalence relation, and the equivalence classes are called conjugacy classes. let $\{C_i\}_{i=0}^d$ denote the conjugacy classes, ordered such that $C_0 = \{\mathbf{1}\}$ (the identity element of G). Define $X = G$. For $0 \le i \le d$ define

$$R_i = \{(x, y) | x, y \in G, \;\; y^{-1}x \in C_i\}.$$

Then $(X, \{R_i\}_{i=0}^d)$ is a commutative association scheme.

Proof. We apply Example 1.4. The group G acts on $X = G$ by left and right multiplication. The left action is $G \times X \to X, (g, x) \mapsto gx$. The right action is $G \times X \to X, (h, x) \mapsto xh^{-1}$. The two actions commute. Combining the two actions, we get an action of $G \oplus G$ on X such that (g, h) sends $x \mapsto gxh^{-1}$ for all $(g, h) \in G \oplus G$ and $x \in X$. The action of $G \oplus G$ on X is transitive. Next we check that the orbits of $G \oplus G$ on $X \times X$ are $\{R_i\}_{i=0}^d$. Pick any $(x, y) \in X \times X$ and $(u, v) \in X \times X$ that are in the same orbit of $G \oplus G$. We show that $y^{-1}x$ and $v^{-1}u$ are conjugate. By assumption, there exists $(g, h) \in G \oplus G$ such that $gxh^{-1} = u$ and $gyh^{-1} = v$. We have

$$v^{-1}u = \big(gyh^{-1}\big)^{-1}\big(gxh^{-1}\big) = hy^{-1}g^{-1}gxh^{-1} = hy^{-1}xh^{-1}$$

so $y^{-1}x$ and $v^{-1}u$ are conjugate. Conversely, pick any $(x, y) \in X \times X$ and $(u, v) \in X \times X$ such that $y^{-1}x$ and $v^{-1}u$ are conjugate. We show that (x, y) and (u, v) are in the same orbit of $G \oplus G$ on $X \times X$. By assumption there exists $h \in G$ such that $v^{-1}u = hy^{-1}xh^{-1}$. Rearranging this equation, we obtain $uhx^{-1} = vhy^{-1}$; denote this common value by g. We have $gxh^{-1} = u$ and $gyh^{-1} = v$. Therefore $(g, h) \in G \oplus G$ sends (x, y) to (u, v). Consequently (x, y) and (u, v) are in the same orbit of $G \oplus G$ on $X \times X$. We have shown that $(X, \{R_i\}_{i=0}^d)$ is an association scheme. Next, we check that this association scheme is commutative. For $0 \le i, j, k \le d$ we show that $p^k_{i,j} = p^k_{j,i}$. Pick $(x, y) \in R_k$. Note that xy^{-1} is conjugate to $y^{-1}x$, so $(y^{-1}, x^{-1}) \in R_k$. Consider the following sets:

$$\{z \in X | (x, z) \in R_i \text{ and } (z, y) \in R_j\}, \tag{3}$$

$$\{w \in X | (y^{-1}, w) \in R_j \text{ and } (w, x^{-1}) \in R_i\}. \tag{4}$$

The sets (3) and (4) have cardinality $p^k_{i,j}$ and $p^k_{j,i}$ respectively. These cardinalities are the same, because the map $z \mapsto z^{-1}$ gives a bijection from (3) to (4). We have shown that $(X, \{R_i\}_{i=0}^d)$ is commutative. □

Problem 1.9. Referring to the association scheme in Example 1.8, assume that G is the symmetric group S_n. For small $n = 2, 3, 4, ..$ describe the conjugacy classes and find the intersection numbers.

Problem 1.10. Find the intersection numbers of the Hamming association scheme $H(2, 4)$. Show that $H(2, 4)$ contains four vertices that are mutually in relation one (4-clique). Construct an association scheme that has the same intersection numbers as $H(2, 4)$ and has no 4-clique. This association scheme is called the Shrikhande scheme.

Problem 1.11. (Cyclotomic association schemes, I). Let $\mathrm{GF}(q)$ denote a finite field with q elements. Let $\mathrm{GF}(q)^*$ denote the multiplicative group. This group consists of the nonzero elements of $\mathrm{GF}(q)$, and the group operation is multiplication. It is known that $\mathrm{GF}(q)^*$ is cyclic; let ω denote a generator of $\mathrm{GF}(q)^*$. Define $X = \mathrm{GF}(q)$. Define $R_0 = \{(x, x)|x \in X\}$. For $1 \le i \le q-1$ define $R_i = \{(x, y)|x, y \in X, \;\; x - y = \omega^{i-1}\}$. Show that $(X, \{R_i\}_{i=0}^{q-1})$ is a commutative association scheme.

Problem 1.12. (Cyclotomic association schemes, II). We refer to Problem 1.11. Let d denote a divisor of $q-1$ and define $r = (q-1)/d$. Let $H_r = \langle \omega^d \rangle$ denote the subgroup of $\mathrm{GF}(q)^*$ generated by ω^d. Note that $|H_r| = r$. For $1 \le i \le d$ define the coset $C_i = \omega^{i-1} H_r$. For notational convenience, define the set $C_0 = \{0\}$. Define $X = \mathrm{GF}(q)$. Define $R_0 = \{(x, x)|x \in X\}$. For $1 \le i \le d$ define $R_i = \{(x, y)|x, y \in X, \;\; x - y \in C_i\}$. Show that $(X, \{R_i\}_{i=0}^{d})$ is a commutative association scheme.

2 The Bose-Mesner algebra

In this section we consider association schemes using linear algebra. We start with some notation.

Let $\mathbb{C}$ denote the field of complex numbers. Let X denote a nonempty finite set. Let $M_X(\mathbb{C})$ denote the algebra over $\mathbb{C}$ consisting of the matrices that have rows and columns indexed by X and all entries in $\mathbb{C}$. Let $I \in M_X(\mathbb{C})$ denote the identity matrix. The matrix $J \in M_X(\mathbb{C})$ has all entries 1. Let $A \in M_X(\mathbb{C})$. For $x, y \in X$ the (x, y)-entry of A is denoted $A_{x,y}$ or $A(x, y)$. The transpose of A is denoted A^t or ${}^tA$. For $A, B \in M_X(\mathbb{C})$ define a matrix $A \circ B \in M_X(\mathbb{C})$ with (x, y)-entry $A_{x,y} B_{x,y}$ for $x, y \in X$. We call $A \circ B$ the *entrywise product* or *Hadamard product* of A and B. We have

$$A \circ B = B \circ A, \qquad\qquad A \circ J = A. \tag{5}$$

Let $\mathcal{X} = (X, \{R_i\}_{i=0}^{d})$ denote an association scheme. For $0 \le i \le d$ define $A_i \in M_X(\mathbb{C})$ that has entries

$$A_i(x, y) = \begin{cases} 1 & \text{if } \;(x, y) \in R_i; \\ 0 & \text{if } \;(x, y) \notin R_i \end{cases} \qquad\qquad x, y \in X.$$

We call A_i the i^{th} *associate matrix for* $\mathcal{X}$, or the *adjacency matrix of* $\mathcal{X}$ *for* R_i. In terms of these matrices, the conditions (i)–(iv) in Definition 1.1 become:

(i) $A_0 = I$;

(ii) $J = \sum_{i=0}^{d} A_i$;

(iii) for $0 \le i \le d$ there exists $i' \in \{0, 1, \ldots, d\}$ such that $A_i^t = A_{i'}$;

(iv) for $0 \le i, j \le d$ there exist natural numbers $p_{i,j}^k$ $(0 \le k \le d)$ such that

$$A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k.$$

The scheme $\mathcal{X}$ is commutative if and only if

$$A_i A_j = A_j A_i \qquad (0 \le i \le d).$$

The scheme $\mathcal{X}$ is symmetric if and only if

$$A_i^t = A_i \qquad (0 \le i \le d).$$

By the above conditions (i)–(iv), the matrices $\{A_i\}_{i=0}^d$ form a basis for a subalgebra $\mathcal{M}$ of $M_X(\mathbb{C})$ that contains J and is closed under transpose. Note that $\mathcal{M}$ is closed under Hadamard multiplication, because

$$A_i \circ A_j = \delta_{i,j} A_i \qquad (0 \le i, j \le d).$$

We call $\mathcal{M}$ the *adjacency algebra* of $\mathcal{X}$. If $\mathcal{X}$ is commutative, then we call $\mathcal{M}$ the *Bose-Mesner algebra* of $\mathcal{X}$.

Lecture 3

Our next goal is to define adjacency algebras in a more abstract way.

Lemma 2.1. *Let $\mathcal{M}$ denote a nonzero subspace of the vector space $M_X(\mathbb{C})$. Assume that $\mathcal{M}$ is closed under Hadamard multiplication. Then $\mathcal{M}$ has a basis $\{A_i\}_{i=0}^d$ such that $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \le i, j \le d$. This basis is unique up to permutation of $A_0, A_1, \ldots, A_d$.*

Proof. For $A \in \mathcal{M}$ define the support set

$$\mathrm{Sup}(A) = \{(x, y) | x, y \in X, \;\; A_{x,y} \ne 0\}.$$

For nonzero $\alpha \in \mathbb{C}$ we have

$$\mathrm{Sup}(\alpha A) = \mathrm{Sup}(A).$$

For $A, B \in \mathcal{M}$ we have

$$\mathrm{Sup}(A \circ B) = \mathrm{Sup}(A) \cap \mathrm{Sup}(B).$$

In particular,

$$\mathrm{Sup}(A \circ A) = \mathrm{Sup}(A).$$

For $A \in \mathcal{M}$, we say that A is *minimal* whenever (i) $A \ne 0$; and (ii) there does not exist a nonzero $B \in \mathcal{M}$ such that $\mathrm{Sup}(B) \subsetneq \mathrm{Sup}(A)$. Assume that $A \in \mathcal{M}$ is minimal. Then for all $B \in \mathcal{M}$, either $\mathrm{Sup}(A) \subseteq \mathrm{Sup}(B)$ or $\mathrm{Sup}(A) \cap \mathrm{Sup}(B) = \emptyset$. For minimal elements $A, B \in \mathcal{M}$, either $\mathrm{Sup}(A) = \mathrm{Sup}(B)$ or $\mathrm{Sup}(A) \cap \mathrm{Sup}(B) = \emptyset$. For minimal elements $A, B \in \mathcal{M}$ such that $\mathrm{Sup}(A) = \mathrm{Sup}(B)$, there exists a nonzero $\alpha \in \mathbb{C}$ such that $B = \alpha A$; otherwise there exists a linear combination of A, B that is nonzero and has its support properly contained in the common support of A and B. For a minimal element $A \in \mathcal{M}$ the nonzero entries of A are all the same; otherwise the previous assertion is contradicted with $B = A \circ A$. A minimal element $A \in \mathcal{M}$ is called *normalized* whenever its nonzero entries are equal to 1. Every minimal element of $\mathcal{M}$ is a scalar multiple of a normalized minimal element. Let $\{A_i\}_{i=0}^d$ denote an ordering of the normalized minimal elements of $\mathcal{M}$. By construction $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \le i, j \le d$. Consequently $\{A_i\}_{i=0}^d$ are linearly independent. For $A \in \mathcal{M}$ we have

$$A \in \mathrm{Span}\{A_i | 0 \le i \le d, \ \ \mathrm{Sup}(A_i) \subseteq \mathrm{Sup}(A)\}.$$

By these comments $\{A\}_{i=0}^d$ is a basis for the vector space $\mathcal{M}$. The uniqueness assertion is clear. □

Lemma 2.2. *For $A \in M_X(\mathbb{C})$ the following are equivalent:*

(i) *the diagonal entries of A are all the same;*

(ii) *$I \circ A$ is a scalar multiple of I.*

Proof. Routine. □

Definition 2.3. A subspace $\mathcal{M}$ of $M_X(\mathbb{C})$ is *homogeneous* whenever each $A \in \mathcal{M}$ satisfies the equivalent conditions (i), (ii) in Lemma 2.2.

Proposition 2.4. *Let $\mathcal{M}$ denote a subspace of the vector space $M_X(\mathbb{C})$ that satisfies* (i)–(v) *below:*

(i) *$I, J \in \mathcal{M}$;*

(ii) *$\mathcal{M}$ is closed under matrix multiplication;*

(iii) *$\mathcal{M}$ is closed under Hadamard multiplication;*

(iv) *$\mathcal{M}$ is closed under the transpose map;*

(v) *$\mathcal{M}$ is homogeneous.*

Then there exists an association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ that has adjacency algebra $\mathcal{M}$. Also, $\mathcal{X}$ is commutative if and only if $AB = BA$ for all $A, B \in \mathcal{M}$. Moreover, $\mathcal{X}$ is symmetric if and only if $A^t = A$ for all $A \in \mathcal{M}$.

Proof. Since $\mathcal{M}$ is closed under Hadamard multiplication, by Lemma 2.1 there exists a basis $\{A_i\}_{i=0}^d$ for $\mathcal{M}$ such that $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \le i, j \le d$. Since $\mathcal{M}$ contains J, we have $J = \sum_{i=0}^d A_i$. Since $\mathcal{M}$ is homogeneous and contains I, we see that one of the matrices $\{A_i\}_{i=0}^d$ must equal I; without loss we many assume that $A_0 = I$. Since $\mathcal{M}$ is closed under the transpose map, $\mathcal{M}$ contains the matrices $\{A_i^t\}_{i=0}^d$. Observe that the matrices $\{A_i^t\}_{i=0}^d$ form a basis for $\mathcal{M}$, and satisfy $A_i^t \circ A_j^t = \delta_{i,j} A_i^t$ for $0 \le i, j \le d$. By the uniqueness assertion in Lemma 2.1, the sequence $\{A_i^t\}_{i=0}^d$ is a permutation of the sequence $\{A_i\}_{i=0}^d$. In other words, for $0 \le i \le d$ there exists $i' \in \{0, 1, \dots, d\}$ such that $A_i^t = A_{i'}$. Since $\mathcal{M}$ is closed under matrix multiplication, for $0 \le i, j \le d$ there exist scalars $p_{i,j}^k \in \mathbb{C}$ $(0 \le k \le d)$ such that

$$A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k.$$

For $0 \leq k \leq d$ we have $p^k_{i,j} \in \mathbb{N}$ because the nonzero entries of A_i, A_j, A_k are equal to 1. For $0 \leq i \leq d$ define

$$R_i = \{(x, y) | A_i(x, y) = 1\}.$$

By the above comments, the sequence $(X, \{R_i\}_{i=0}^d)$ is an association scheme, with associate matrices $\{A_i\}_{i=0}^d$ and adjacency algebra $\mathcal{M}$. The assertions about commutativity and symmetry are clear. □

We mention some concepts for later use. Let $\mathbb{R}$ denote the field of real numbers.

Let X denote a nonempty finite set. Let $V = \mathbb{C}^X$ denote the $\mathbb{C}$-vector space consisting of the column vectors that have coordinates indexed by X and entries in $\mathbb{C}$. Note that $M_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with a bilinear form $\langle\ ,\ \rangle$ such that $\langle u, v\rangle = u^t\overline{v}$ for all $u, v \in V$. Abbreviate $\|u\|^2 = \langle u, u\rangle$. For $u, v, w \in V$ and $\alpha \in \mathbb{C}$, we have

$$\begin{aligned} &\langle v, u\rangle = \overline{\langle u, v\rangle}, && \langle \alpha u, v\rangle = \alpha\langle u, v\rangle, \\ &\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle, && \|u\|^2 \in \mathbb{R}, \\ &\|u\|^2 \geq 0, && \|u\|^2 = 0 \ \text{ iff } \ u = 0. \end{aligned}$$

For $u, v \in V$ and $A \in M_X(\mathbb{C})$ we have

$$\langle Au, v\rangle = \langle u, \overline{A}^t v\rangle. \tag{6}$$

For a subspace $U \subseteq V$ define

$$U^\perp = \{v \in V | \langle u, v\rangle = 0 \ \forall\, u \in U\}.$$

Note that

$$V = U + U^\perp \qquad \text{(orthogonal direct sum)}.$$

We call $U^\perp$ the *orthgonal complement of* U.

3 Commutative association schemes

Throughout this section, we assume that $\mathcal{X} = (X\{R_i\}_{i=0}^d)$ is a commutative association scheme. By assumption,

$$p^k_{i,j} = p^k_{j,i} \qquad\qquad (0 \leq i, j, k \leq d).$$

Recall that for $x, y \in X$ and $0 \leq i \leq d$,

$$(x, y) \in R_i \ \text{ iff } \ (y, x) \in R_{i'}.$$

For $x \in X$ and $0 \leq i \leq d$ define

$$\Gamma_i(x) = \{y \in X | (x, y) \in R_i\}.$$

For $0 \le i, j, k \le d$ and $(x, y) \in R_k$,

$$p^k_{i,j} = |\Gamma_i(x) \cap \Gamma_{j'}(y)|.$$

Define

$$k_i = p^0_{i,i'} \qquad (0 \le i \le d). \tag{7}$$

For $x \in X$,

$$k_i = |\Gamma_i(x)| \qquad (0 \le i \le d).$$

Lemma 3.1. *We have*

(i) $k_0 = 1$;

(ii) $k_i = k_{i'} \qquad (0 \le i \le d)$;

(iii) $|X| = \sum_{i=0}^d k_i$;

(iv) $k_i \neq 0 \qquad (0 \le i \le d)$.

Proof. Routine. □

Lecture 4

Proposition 3.2. *We have*

(i) $p^k_{i,0} = \delta_{i,k} \qquad (0 \le i, k \le d)$;

(ii) $p^k_{0,j} = \delta_{j,k} \qquad (0 \le j, k \le d)$;

(iii) $p^0_{i,j} = \delta_{i,j'} k_i \qquad (0 \le i, j \le d)$;

(iv) $p^k_{i,j} = p^{k'}_{i',j'} \qquad (0 \le i, j, k \le d)$;

(v) $k_i = \sum_{j=0}^d p^k_{i,j} \qquad (0 \le i, k \le d)$;

(vi) $k_\ell p^\ell_{i,j} = k_i p^i_{\ell,j'} = k_j p^j_{i',\ell} \qquad (0 \le i, j, \ell \le d)$;

(vii) $\sum_{\alpha=0}^d p^\alpha_{i,j} p^\ell_{k,\alpha} = \sum_{\alpha=0}^d p^\alpha_{k,i} p^\ell_{\alpha,j} \qquad (0 \le i, j, k, \ell \le d)$.

Proof. (i)–(iv) Routine.
(v) Fix $(x, y) \in R_k$. Partition $\Gamma_i(x)$ according to how its elements are related to y. This gives

$$\Gamma_i(x) = \cup_{j=0}^d (\Gamma_i(x) \cap \Gamma_{j'}(y)) \qquad \text{(disjoint union)}.$$

In this equation, take the cardinality of each side.
(vi) The three common values are equal to $|X|^{-1}$ times the number of 3-tuples (x, y, z) such that $(x, y) \in R_\ell$ and $(x, z) \in R_i$ and $(z, y) \in R_j$.
(vii) In the equation $A_k(A_iA_j) = (A_kA_i)A_j$, write each side as a linear combination of $\{A_\ell\}_{\ell=0}^d$, and compare coefficients. □

As we study the Bose-Mesner algebra of $\mathcal{X}$, the following results will be useful.

Lemma 3.3. *Let $\mathcal{M}$ denote a nonzero subspace of the vector space $M_X(\mathbb{C})$. Assume that:*

(i) *$\mathcal{M}$ is closed under matrix multiplication;*

(ii) *$AB = BA$ for all $A, B \in \mathcal{M}$;*

(iii) *$\mathcal{M}$ is closed under the conjugate-transpose map.*

Then $\mathcal{M}$ has a basis $\{E_i\}_{i=0}^d$ such that $E_iE_j = \delta_{i,j}E_i$ for $0 \le i, j \le d$. This basis is unique up to permutation of $E_0, E_1, \ldots, E_d$.

Proof. Consider the action of $\mathcal{M}$ on the standard module V. We claim that V has a basis consisting of common eigenvectors for $\mathcal{M}$. Let U denote the sum of the common eigenspaces for $\mathcal{M}$. It suffices to show that $U = V$. Suppose $U \subsetneq V$. We have an orthogonal direct sum $V = U + U^\perp$. By construction U is $\mathcal{M}$-invariant. By this and (6) the subspace $U^\perp$ is $\mathcal{M}$-invariant. Since the elements of $\mathcal{M}$ mutually commute, there exists $0 \ne v \in U^\perp$ that is a common eigenvector for $\mathcal{M}$. The vector v is contained in a common eigenspace for $\mathcal{M}$, so $v \in U$. Now $v \in U \cap U^\perp = 0$, for a contradiction. By these comments $U = V$, and the claim is proved. By the claim, there exists an invertible $S \in M_X(\mathbb{C})$ such that SAS^{-1} is diagonal for all $A \in \mathcal{M}$. By construction, $S\mathcal{M}S^{-1}$ is a nonzero subspace of $M_X(\mathbb{C})$ that is closed under matrix multiplication and has all elements diagonal. For diagonal matrices $A, B \in M_X(\mathbb{C})$ we have $AB = A \circ B$. Therefore $S\mathcal{M}S^{-1}$ is closed under Hadamard multiplication. By Lemma 2.1 the subspace $S\mathcal{M}S^{-1}$ has a basis $\{A_i\}_{i=0}^d$ such that $A_i \circ A_j = \delta_{i,j}A_i$ for $0 \le i, j \le d$. Note that $A_iA_j = \delta_{i,j}A_i$ for $0 \le i, j \le d$. Define $E_i = S^{-1}A_iS$ for $0 \le i \le d$. Then $\{E_i\}_{i=0}^d$ is a basis for $\mathcal{M}$ such that $E_iE_j = \delta_{i,j}E_i$ for $0 \le i, j \le d$. The uniqueness assertion is routinely checked. □

Definition 3.4. Referring to Lemma 3.3, we call $\{E_i\}_{i=0}^d$ the *primitive idempotents* of $\mathcal{M}$.

Lemma 3.5. *For the subspace $\mathcal{M}$ in Lemma 3.3, its primitive idempotents satisfy $\overline{E}_i^t = E_i$ for $0 \le i \le d$.*

Proof. The subspace $\mathcal{M}$ contains $\overline{E}_i^t$ for $0 \le i \le d$. The matrices $\{\overline{E}_i^t\}_{i=0}^d$ form a basis for $\mathcal{M}$ such that $\overline{E}_i^t\overline{E}_j^t = \delta_{i,j}\overline{E}_i^t$ for $0 \le i, j \le d$. By the uniqueness statement in Lemma 3.3, the sequence $\{\overline{E}_i^t\}_{i=0}^d$ is a permutation of the sequence $\{E_i\}_{i=0}^d$. For $0 \le i \le d$ we have $\overline{E}_i^tE_i \ne 0$, so $\overline{E}_i^t = E_i$. □

Lemma 3.6. *We refer to the subspace $\mathcal{M}$ in Lemma 3.3.*

(i) *Assume that $I \in \mathcal{M}$. Then $I = \sum_{i=0}^d E_i$.*

(ii) *Assume that $J \in \mathcal{M}$. Then $|X|^{-1}J$ is a primitive idempotent of $\mathcal{M}$ (denoted E_0).*

(iii) *Assume that $\mathcal{M}$ is closed under both the transpose map and complex conjugation. Then for $0 \le i \le d$ there exists $\hat{i} \in \{0, 1, \ldots, d\}$ such that $E_i^t = E_{\hat{i}} = \overline{E}_i$.*

Proof. (i) There exists scalars $\{\alpha_i\}_{i=0}^d$ in $\mathbb{C}$ such that $I = \sum_{i=0}^d \alpha_i E_i$. For $0 \le i \le d$ we have

$$E_i = E_i I = E_i \sum_{j=0}^d \alpha_j E_j = \alpha_i E_i,$$

so $\alpha_i = 1$.
(ii) There exists scalars $\{\beta_i\}_{i=0}^d$ in $\mathbb{C}$ such that $J = \sum_{i=0}^d \beta_i E_i$. At least one of $\{\beta_i\}_{i=0}^d$ is nonzero. Without loss, we may assume $\beta_0 \neq 0$. We have $JE_0 = \beta_0 E_0$. Note that $J^2 = |X|J$, so

$$JE_0 = |X|^{-1}J^2E_0 = |X|^{-1}JE_0J = |X|^{-1}sJ,$$

where s is the sum of all the entries of E_0. By these comments E_0 is a scalar multiple of J. Using $E_0^2 = E_0$ we obtain $E_0 = |X|^{-1}J$.
(iii) The subspace $\mathcal{M}$ contains E_i^t for $0 \le i \le d$. The matrices $\{E_i^t\}_{i=0}^d$ form a basis for $\mathcal{M}$ such that $E_i^t E_j^t = \delta_{i,j} E_i^t$ for $0 \le i, j \le d$. By the uniqueness statement in Lemma 3.3, the sequence $\{E_i^t\}_{i=0}^d$ is a permutation of the sequence $\{E_i\}_{i=0}^d$. In other words, for $0 \le i \le d$ there exists $\hat{i} \in \{0, 1, \dots, d\}$ such that $E_i^t = E_{\hat{i}}$. By Lemma 3.5 we have $\overline{E}_i = E_i^t = E_{\hat{i}}$. □

We return our attention to the commutative association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$.

Proposition 3.7. *The Bose-Mesner algebra $\mathcal{M}$ of $\mathcal{X}$ has a basis $\{E_i\}_{i=0}^d$ that satisfies*

(i) $E_0 = |X|^{-1}J$;

(ii) $E_iE_j = \delta_{i,j}E_i$ *for* $0 \le i, j \le d$;

(iii) $I = \sum_{i=0}^d E_i$;

(iv) *for* $0 \le i \le d$ *there exists* $\hat{i} \in \{0, 1, \dots, d\}$ *such that* $E_i^t = E_{\hat{i}} = \overline{E}_i$.

Proof. Note that $\mathcal{M}$ satisfies the conditions of Lemma 3.3 and Lemma 3.6. □

The matrices $\{E_i\}_{i=0}^d$ form a basis for $\mathcal{M}$. Since $\mathcal{M}$ is closed under Hadamard multiplication, for $0 \le i, j \le d$ there exist $q_{i,j}^k \in \mathbb{C}$ $(0 \le k \le d)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^d q_{i,j}^k E_k. \tag{8}$$

By construction,

$$q_{i,j}^k = q_{j,i}^k \qquad (0 \le i, j, k \le d). \tag{9}$$

The scalars $q_{i,j}^k$ are called the *Krein parameters* of $\mathcal{X}$. Shortly we will show that $q_{i,j}^k$ is real and nonnegative for $0 \le i, j, k \le d$.

4 Character tables for commutative association schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. Recall the standard module $V = \mathbb{C}^X$. Define the vector $\mathbf{1} \in V$ that has all entries 1.

Lemma 4.1. *We have*

$$V = \sum_{i=0}^{d} E_i V \qquad \text{(orthogonal direct sum).}$$

For $0 \le i \le d$ the subspace E_iV is a common eigenspace for $\mathcal{M}$, and E_i is the projection onto this eigenspace. Moreover $\mathbf{1}$ is a basis for E_0V.

Proof. Routine consequence of Proposition 3.7. □

The Bose-Mesner algebra $\mathcal{M}$ has a basis $\{A_i\}_{i=0}^d$ and a basis $\{E_i\}_{i=0}^d$. Let us consider how these bases are related. We have

$$A_i = \sum_{j=0}^{d} P_i(j) E_j \qquad (0 \le i \le d), \tag{10}$$

$$E_i = |X|^{-1} \sum_{j=0}^{d} Q_i(j) A_j \qquad (0 \le i \le d), \tag{11}$$

$$P_i(j) \in \mathbb{C}, \qquad Q_i(j) \in \mathbb{C} \qquad (0 \le i, j \le d). \tag{12}$$

For $0 \le i, j \le d$ the scalar $P_i(j)$ is the eigenvalue of A_i for the common eigenspace E_jV.

Let $M_{d+1}(\mathbb{C})$ denote the algebra over $\mathbb{C}$ consisting of the $d+1$ by $d+1$ matrices that have all entries in $\mathbb{C}$. We index the rows and columns by $0, 1, \ldots, d$. Define $P \in M_{d+1}(\mathbb{C})$ that has (i,j)-entry $P_j(i)$ for $0 \le i, j \le d$. Define $Q \in M_{d+1}(\mathbb{C})$ that has (i,j)-entry $Q_j(i)$ for $0 \le i, j \le d$. By construction,

$$PQ = |X| I = QP. \tag{13}$$

We call P the *first eigenmatrix* (or *first character table*) of $\mathcal{X}$. We call Q the *second eigenmatrix* (or *second character table*) of $\mathcal{X}$.

Lecture 5

Lemma 4.2. *The following hold for $0 \le i \le d$:*

(i) $\operatorname{tr}(A_i) = \delta_{i,0}|X|$;

(ii) $A_i J = J A_i = k_i J$.

Proof. (i) We have $A_0 = I$. For $1 \leq i \leq d$ the diagonal entries of A_i are zero.
(ii) The matrix A_i has constant row sum k_i by the definition of k_i. The matrix A_i has constant column sum k_i because $A_i^t = A_{i'}$ and $k_{i'} = k_i$. $\square$

Define

$$m_i = \dim E_i V \qquad (0 \leq i \leq d). \tag{14}$$

Lemma 4.3. *We have*

(i) $m_0 = 1$;

(ii) $m_i = m_{\hat{i}} \qquad (0 \leq i \leq d)$;

(iii) $|X| = \sum_{i=0}^{d} m_i$;

(iv) $m_i \neq 0 \qquad (0 \leq i \leq d)$.

Proof. (i) E_0 has basis $\mathbf{1}$.
(ii) Note that m_i is equal to the rank of E_i. The matrices E_i, E_i^t have the same rank, and $E_i^t = E_{\hat{i}}$.
(iii) By Lemma 4.1 and (14).
(iv) By construction. $\square$

Lemma 4.4. *The following hold for* $0 \leq i \leq d$:

(i) $\mathrm{tr}(E_i) = m_i$;

(ii) $E_i J = J E_i = \delta_{i,0} J$.

Proof. (i) The matrix E_i is similar to a diagonal matrix that has m_i diagonal entries 1 and all other diagonal entries 0.
(ii) Use $E_i E_0 = E_0 E_i = \delta_{i,0} E_0$ and $E_0 = |X|^{-1} J$. $\square$

Proposition 4.5. *The following hold for* $0 \leq i \leq d$:

(i) $P_0(i) = 1$;

(ii) $P_i(0) = k_i$;

(iii) $Q_0(i) = 1$;

(iv) $Q_i(0) = m_i$.

Proof. (i) Since $A_0 = I = \sum_{i=0}^{d} E_i$.
(ii) In the equation $A_i = \sum_{j=0}^{d} P_i(j) E_j$, multiply each side on the right by E_0 and simplify using $A_i E_0 = k_i E_0$.
(iii) Since $|X| E_0 = J = \sum_{i=0}^{d} A_i$.
(iv) In the equation $E_i = |X|^{-1} \sum_{j=0}^{d} Q_i(j) A_j$, take the trace of each side and evaluate the result using Lemmas 4.2, 4.4. $\square$

Theorem 4.6. *The following hold for* $0 \le i, j \le d$*:*

(i) $P_{i'}(j) = \overline{P_i(j)}$;

(ii) $Q_{\hat{i}}(j) = \overline{Q_i(j)}$;

(iii) $\frac{Q_j(i)}{m_j} = \frac{\overline{P_i(j)}}{k_i}$;

(iv) $\sum_{\ell=0}^{d} P_\ell(i)\overline{P_\ell(j)}k_\ell^{-1} = \delta_{i,j}|X|m_i^{-1}$ (first orthogonality relation);

(v) $\sum_{\ell=0}^{d} P_i(\ell)\overline{P_j(\ell)}m_\ell = \delta_{i,j}|X|k_i$ (second orthogonality relation).

Proof. (i) In the equation $A_i = \sum_{j=0}^{d} P_i(j)E_j$, take the conjugate-transpose of each side and simplify the result using $\overline{E}_j^t = E_j$ along with $A_i^t = A_{i'} = \sum_{j=0}^{d} P_{i'}(j)E_j$.
(ii) In the equation $E_i = |X|^{-1}\sum_{j=0}^{d} Q_i(j)A_j$, take the complex conjugate of each side and simplify the result using $\overline{E}_i = E_{\hat{i}} = |X|^{-1}\sum_{j=0}^{d} Q_{\hat{i}}(j)A_j$.
(iii) We compute the trace of $A_{i'}E_j$ in two ways. On one hand,

$$\mathrm{tr}(A_{i'}E_j) = P_{i'}(j)\mathrm{tr}(E_j) = \overline{P_i(j)}\mathrm{tr}(E_j) = \overline{P_i(j)}m_j.$$

On the other hand,

$$\begin{aligned}\mathrm{tr}(A_{i'}E_j) &= |X|^{-1}\mathrm{tr}\left(A_{i'}\sum_{\ell=0}^{d} Q_j(\ell)A_\ell\right) = |X|^{-1}\mathrm{tr}\left(\sum_{\ell=0}^{d} Q_j(\ell)\sum_{h=0}^{d} p_{i',\ell}^{h}A_h\right)\\ &= \sum_{\ell=0}^{d} Q_j(\ell)p_{i',\ell}^{0} = \sum_{\ell=0}^{d} Q_j(\ell)\delta_{i,\ell}k_i = Q_j(i)k_i.\end{aligned}$$

(iv) In the equation $PQ = |X|I$, compute the (i,j)-entry of each side, and evaluate the result using (iii) above.
(v) In the equation $QP = |X|I$, compute the (j,i)-entry of each side, and evaluate the result using (iii) above. ◻

Theorem 4.7. *The following hold for* $0 \le i, j, \ell \le d$*:*

(i) $P_i(\ell)P_j(\ell) = \sum_{k=0}^{d} p_{i,j}^{k}P_k(\ell)$;

(ii) $Q_i(\ell)Q_j(\ell) = \sum_{k=0}^{d} q_{i,j}^{k}Q_k(\ell)$;

(iii) $P_i(j)Q_j(\ell) = \sum_{k=0}^{d} p_{i,k}^{\ell}Q_j(k)$.

Proof. (i) In the equation $A_iA_j = \sum_{k=0}^{d} p_{i,j}^{k}A_k$, write each side as a linear combination of the primitive idempotents using (10).
(ii) In the equation $E_i \circ E_j = |X|^{-1}\sum_{k=0}^{d} q_{i,j}^{k}E_k$, write each side as a linear combination of the associate matrices using (11).
(iii) By (i) above,

$$P_{i'}(j)P_\ell(j) = \sum_{k=0}^{d} p_{i',\ell}^{k}P_k(j).$$

In this equation, eliminate $P_\ell(j)$ and $P_k(j)$ using Theorem 4.6(iii), and simplify the result using Proposition 3.2(vi). □

Our next goal is to compute the intersection numbers and Krein parameters from the character tables.

Lemma 4.8. *For $0 \le i, j, \ell \le d$ we have*

$$p^\ell_{i,j} = |X|^{-1} k_\ell^{-1} \mathrm{tr}\big(A_i A_j A_{\ell'}\big); \tag{15}$$

$$q^\ell_{i,j} = |X| m_\ell^{-1} \mathrm{tr}\big((E_i \circ E_j) E_\ell\big). \tag{16}$$

Proof. Concerning (15), we have

$$\begin{aligned}\mathrm{tr}\big(A_i A_j A_{\ell'}\big) &= \mathrm{tr}\left(\sum_{h=0}^d p^h_{i,j} A_h A_{\ell'}\right) = \mathrm{tr}\left(\sum_{h=0}^d \sum_{\nu=0}^d p^h_{i,j} p^\nu_{h,\ell'} A_\nu\right) \\ &= |X| \sum_{h=0}^d p^h_{i,j} p^0_{h,\ell'} = |X| \sum_{h=0}^d p^h_{i,j} \delta_{h,\ell} k_\ell = |X| k_\ell p^\ell_{i,j}.\end{aligned}$$

Concerning (16), we have

$$\mathrm{tr}\big((E_i \circ E_j) E_\ell\big) = \mathrm{tr}\left(|X|^{-1} \sum_{h=0}^d q^h_{i,j} E_h E_\ell\right) = |X|^{-1} q^\ell_{i,j} \mathrm{tr}(E_\ell) = |X|^{-1} m_\ell q^\ell_{i,j}.$$

□

Lemma 4.9. *We have $q^\ell_{i,j} \in \mathbb{R}$ for $0 \le i, j, \ell \le d$.*

Proof. By (16) we have

$$\begin{aligned}|X|^{-1} m_\ell \overline{q^\ell_{i,j}} &= \mathrm{tr}\big((\overline{E_i} \circ \overline{E_j})\overline{E_\ell}\big) = \mathrm{tr}\big((E_i^t \circ E_j^t) E_\ell^t\big) = \mathrm{tr}\big((E_i \circ E_j)^t E_\ell^t\big) \\ &= \mathrm{tr}\big(E_\ell (E_i \circ E_j)\big)^t = \mathrm{tr}\big(E_\ell (E_i \circ E_j)\big) = \mathrm{tr}\big((E_i \circ E_j) E_\ell\big) = |X|^{-1} m_\ell q^\ell_{i,j}.\end{aligned}$$

□

Theorem 4.10. *For $0 \le i, j, \ell \le d$ we have*

$$p^\ell_{i,j} = |X|^{-1} k_\ell^{-1} \sum_{\nu=0}^d P_i(\nu) P_j(\nu) \overline{P_\ell(\nu)} m_\nu \tag{17}$$

$$= |X|^{-1} k_i k_j \sum_{\nu=0}^d Q_\nu(i) Q_\nu(j) \overline{Q_\nu(\ell)} m_\nu^{-2}; \tag{18}$$

$$q^\ell_{i,j} = |X|^{-1} m_\ell^{-1} \sum_{\nu=0}^d Q_i(\nu) Q_j(\nu) \overline{Q_\ell(\nu)} k_\nu \tag{19}$$

$$= |X|^{-1} m_i m_j \sum_{\nu=0}^d P_\nu(i) P_\nu(j) \overline{P_\nu(\ell)} k_\nu^{-2}. \tag{20}$$

Proof. To get (17), eliminate the associate matrices in (15) using (10), and simplify the result. To get (18) , evaluate (17) using Theorem 4.6(iii), and use the fact that $p^\ell_{i,j}$ is real. To get (19), eliminate the primitive idempotents E_i, E_j in (16) using (11), and simplify the result. To get (20) , evaluate (19) using Theorem 4.6(iii), and use the fact that $q^\ell_{i,j}$ is real. □

Lecture 6

We will use the following fact from linear algebra.

Lemma 4.11. *For $A, B, C \in M_X(\mathbb{C})$,*

$$\sum_{x\in X}\sum_{y\in X} A_{x,y}B_{x,y}C_{x,y} = \operatorname{tr}\big((A\circ B)C^t\big) = \operatorname{tr}\big((B\circ C)A^t\big) = \operatorname{tr}\big((C\circ A)B^t\big)$$
$$= \operatorname{tr}\big((A^t\circ B^t)C\big) = \operatorname{tr}\big((B^t\circ C^t)A\big) = \operatorname{tr}\big((C^t\circ A^t)B\big).$$

Proof. Use matrix multiplication. □

Proposition 4.12. *We have*

(i) $q^k_{i,0} = \delta_{i,k} \qquad (0 \le i, k \le d)$;

(ii) $q^k_{0,j} = \delta_{j,k} \qquad (0 \le j, k \le d)$;

(iii) $q^0_{i,j} = \delta_{i,\hat{j}} m_i \qquad (0 \le i, j \le d)$;

(iv) $q^k_{i,j} = q^{\hat{k}}_{\hat{i},\hat{j}} \qquad (0 \le i, j, k \le d)$;

(v) $m_i = \sum_{j=0}^d q^k_{i,j} \qquad (0 \le i, k \le d)$;

(vi) $m_\ell q^\ell_{i,j} = m_i q^i_{\ell,\hat{j}} = m_j q^j_{\hat{i},\ell} \qquad (0 \le i, j, \ell \le d)$;

(vii) $\sum_{\alpha=0}^d q^\alpha_{i,j} q^\ell_{k,\alpha} = \sum_{\alpha=0}^d q^\alpha_{k,i} q^\ell_{\alpha,j} \qquad (0 \le i, j, k, \ell \le d)$.

Proof. (i)–(iii) Routine application of (16) and Lemma 4.11.
(iv) Using (16),

$$q^{\hat{k}}_{\hat{i},\hat{j}} = |X| m_k^{-1}\operatorname{tr}\big((E_{\hat{i}}\circ E_{\hat{j}})E_{\hat{k}}\big) = |X| m_k^{-1}\operatorname{tr}\big((\overline{E_i}\circ \overline{E_j})\overline{E_k}\big)$$
$$= |X| m_k^{-1}\operatorname{tr}\big(\overline{(E_i\circ E_j)E_k}\big) = \overline{q^k_{i,j}} = q^k_{i,j}.$$

(v) Using (16),

$$\sum_{j=0}^d q^k_{i,j} = |X| m_k^{-1}\sum_{j=0}^d \operatorname{tr}\big((E_i\circ E_j)E_k\big) = |X| m_k^{-1}\operatorname{tr}\big((E_i\circ I)E_k\big) = m_k^{-1}\operatorname{tr}\big((m_i I)E_k\big) = m_i.$$

(vi) Use (16) and Lemma 4.11.
(vii) In the equation

$$(E_k\circ E_i)\circ E_j = E_k\circ(E_i\circ E_j),$$

write each side as a linear combination of $\{E_\ell\}_{\ell=0}^d$, and compare coefficients. □

Our next goal is to show that the Krein parameters are nonnegative. As a warmup, let us review some facts about Hermitean matrices.

A matrix $A \in M_X(\mathbb{C})$ is *Hermitean* whenever $\overline{A}^t = A$. For example, the primitive idempotents $\{E_i\}_{i=0}^d$ are Hermitean. Assume that $A, B \in M_X(\mathbb{C})$ are Hermitean. Then $A \circ B$ is Hermitean. Assume that $A \in M_X(\mathbb{C})$ is Hermitean. Then A diagonalizable, and its eigenspaces are mutually orthogonal. Moreoever, the eigenvalues of A are real. We say that A is *positive semidefinite* (or *PSD*) whenever the eigenvalues of A are nonnegative. One checks that A is PSD if and only if $\overline{v}^t A v \geq 0$ for all $v \in V$, where V is the standard module. Let $\mathbb{A}$ denote a principle submatrix of A (rows/cols of $\mathbb{A}$ indexed by the same subset of X). Note that $\mathbb{A}$ is Hermitean. If A is PSD then $\mathbb{A}$ is PSD.

Lemma 4.13. *Given PSD Hermitean matrices $A, B \in M_X(\mathbb{C})$. Then $A \circ B$ is PSD.*

Proof. Define a matrix $A \otimes B \in M_{X\times X}(\mathbb{C})$ as follows. For vertices $r = (x, y) \in X \times X$ and $s = (u, v) \in X \times X$, the (r, s)-entry of $A \otimes B$ is $A_{x,u}B_{y,v}$. The matrix $A \otimes B$ is Hermitean. The characteristic polynomial of $A\otimes B$ has roots $\{\lambda_x \mu_y | (x, y) \in X \times X\}$, where $\{\lambda_x | x \in X\}$ (resp. $\{\mu_y | y \in X\}$) are the roots of the characteristic polynomial of A (resp. B). Therefore $A \otimes B$ is PSD. The matrix $A \circ B$ is the principle submatrix of $A \otimes B$ with rows/columns indexed by $\{(z, z)|z \in X\}$. By these comments $A \circ B$ is PSD. □

Theorem 4.14. (the Krein condition) *We have $q^k_{i,j} \geq 0$ for $0 \leq i, j, k \leq d$.*

Proof. Let i, j be given. We show that $q^k_{i,j} \geq 0$ for $0 \leq k \leq d$. The matrices E_i and E_j are Hermitean. They are both PSD, because their eigenvalues are zero or one. Therefore $E_i \circ E_j$ is PSD by Lemma 4.13. Recall that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q^k_{i,j} E_k.$$

So for $0 \leq k \leq d$, the scalar $|X|^{-1}q^k_{i,j}$ is the eigenvalue of $E_i \circ E_j$ for the common eigenspace E_kV of $\mathcal{M}$. This eigenvalue is nonnegative, so $q^k_{i,j} \geq 0$. □

We mention some more inequalities.

Theorem 4.15. *For $0 \leq i, j \leq d$ we have*

$$|P_i(j)| \leq k_i, \qquad\qquad |Q_j(i)| \leq m_j. \tag{21}$$

Proof. We first prove the inequality on the left. Recall that $A_iE_j = P_i(j)E_j$. Pick a nonzero $v \in E_jV$, where V is the standard module. Then $A_iv = P_i(j)v$. In this equation, we compare the entries on either side. For $x \in X$, let v_x denote the x-entry of v. From the x-entry in $A_iv = P_i(j)v$, we obtain

$$\sum_{y\in\Gamma_i(x)} v_y = P_i(j)v_x. \tag{22}$$

Pick $x \in X$ such that $|v_x| \geq |v_y|$ for all $y \in X$. Observe that

$$|P_i(j)||v_x| = |P_i(j)v_x| = \left| \sum_{y \in \Gamma_i(x)} v_y \right| \leq \sum_{y \in \Gamma_i(x)} |v_y| \leq \sum_{y \in \Gamma_i(x)} |v_x| = k_i |v_x|.$$

We have $|v_x| > 0$ since $v \neq 0$, so $|P_i(j)| \leq k_i$. We also have

$$|Q_j(i)| = \left| \frac{\overline{P_i(j)} m_j}{k_i} \right| = \frac{|P_i(j)| m_j}{k_i} \leq m_j.$$

□

5 The intersection matrices

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

Earlier we computed the intersection numbers and Krein parameters in terms of the character tables. Our goal in this section is to compute the character tables in terms of the intersection numbers and also the Krein parameters.

Definition 5.1. For $0 \leq i \leq d$, define a matrix $B_i \in M_{d+1}(\mathbb{C})$ with (j,k)-entry $p^k_{i,j}$ for $0 \leq j, k \leq d$. We call B_i the i^{th} *intersection matrix* of $\mathcal{X}$.

Lemma 5.2. *The following* (i)–(vi) *hold.*

(i) $B_0 = I$.

(ii) *For* $0 \leq i \leq d$, *the top row of* B_i *is* $(0, \ldots, 0, 1, 0, \ldots, 0)$, *with the 1 in column* i.

(iii) $\{B_i\}_{i=0}^d$ *are linearly independent.*

(iv) *For* $0 \leq i, j \leq d$,

$$B_i B_j = \sum_{k=0}^d p^k_{i,j} B_k.$$

(v) $B_i B_j = B_j B_i$ *for* $0 \leq i, j \leq d$.

(vi) *For* $0 \leq i \leq d$ *we have* $B_i^t = K^{-1} B_{i'} K$ *where* $K = \operatorname{diag}(k_0, k_1, \ldots, k_d)$.

Proof. (i) For $0 \leq j, k \leq d$ the (j,k)-entry of B_0 is $p^k_{0,j} = \delta_{j,k}$.
(ii) For $0 \leq k \leq d$ the $(0,k)$-entry of B_i is $p^k_{i,0} = \delta_{i,k}$.
(iii) By (ii) above.
(iv) Compare the entries on either side using Proposition 3.2(vii).
(v) By (iv) and since $p^k_{i,j} = p^k_{j,i}$ for $0 \leq k \leq d$.
(vi) To get $K B_i^t = B_{i'} K$, compare the entries of each side using Proposition 3.2(vi). □

Definition 5.3. By Lemma 5.2 the matrices $\{B_i\}_{i=0}^d$ form a basis for a commutative subalgebra $\mathcal{B}$ of $M_{d+1}(\mathbb{C})$. We call $\mathcal{B}$ the *intersection algebra* of $\mathcal{X}$.

Theorem 5.4. *There exists an algebra isomorphism $\mathcal{M} \to \mathcal{B}$ that sends $A_i \mapsto B_i$ for $0 \leq i \leq d$.*

Proof. Clear from Lemma 5.2(iv). $\square$

Lecture 7

Let us recall some linear algebra. For the moment, let W denote any finite-dimensional vector space over $\mathbb{C}$, and let $\{w_i\}_{i=1}^n$ denote a basis for W. Let $A : W \to W$ denote a $\mathbb{C}$-linear map. There exists a unique $n \times n$ matrix B such that

$$Aw_j = \sum_{i=1}^{n} B_{i,j} w_i \qquad (1 \leq j \leq n).$$

We say that B *represents* A *with respect to* $\{w_i\}_{i=1}^n$. Let $\{w'_i\}_{i=1}^n$ denote a second basis for W. There exists a unique $n \times n$ matrix S such that

$$w'_j = \sum_{i=1}^{n} S_{i,j} w_i \qquad (1 \leq j \leq n).$$

The matrix S is invertible. We call S the *transition matrix from* $\{w_i\}_{i=1}^n$ *to* $\{w'_i\}_{i=1}^n$. By linear algebra, the matrix $S^{-1}BS$ represents A with respect to $\{w'_i\}_{i=1}^n$.

We return our attention to $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. The Bose-Mesner algebra $\mathcal{M}$ has bases $\{A_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$. Recall the first and second eigenmatrices P, Q. Then P is the transition matix from $\{E_i\}_{i=0}^d$ to $\{A_i\}_{i=0}^d$. Moreover, $|X|^{-1}Q$ is the transition matrix from $\{A_i\}_{i=0}^d$ to $\{E_i\}_{i=0}^d$.

For $A \in \mathcal{M}$, there exists a $\mathbb{C}$-linear map $L_A : \mathcal{M} \to \mathcal{M}$ that sends $B \mapsto AB$ for all $B \in \mathcal{M}$.

Theorem 5.5. *With the above notation, the following* (i)–(iv) *hold for $0 \leq i \leq d$:*

(i) *B_i^t represents L_{A_i} with respect to the basis $\{A_\ell\}_{\ell=0}^d$;*

(ii) *the matrix* $\mathrm{diag}(P_i(0), P_i(1), \ldots, P_i(d))$ *represents L_{A_i} with respect to $\{E_\ell\}_{\ell=0}^d$;*

(iii) *$PB_i^tP^{-1} = \mathrm{diag}(P_i(0), P_i(1), \ldots, P_i(d))$;*

(iv) *the scalars $P_i(0), P_i(1), \ldots, P_i(d)$ are the roots of the characteristic polynomial of B_i.*

Proof. (i) By Definition 5.1 and

$$A_iA_j = \sum_{k=0}^{d} p_{i,j}^k A_k \qquad (0 \leq j \leq d).$$

(ii) Since $A_iE_j = P_i(j)E_j$ for $0 \leq j \leq d$.
(iii) By (i), (ii) and the comments above the theorem statement.
(iv) B_i and B_i^t have the same characteristic polynomial. The result follows in view of (iii). $\square$

Definition 5.6. For $0 \le i \le d$, define a matrix $B_i^* \in M_{d+1}(\mathbb{C})$ with (j,k)-entry $q_{i,j}^k$ for $0 \le j,k \le d$. We call B_i^* the i^{th} *dual intersection matrix* of $\mathcal{X}$.

Lemma 5.7. *The following* (i)–(vi) *hold.*

(i) $B_0^* = I$.

(ii) *For* $0 \le i \le d$, *the top row of* B_i^* *is* $(0, \ldots, 0, 1, 0, \ldots, 0)$, *with the 1 in column* i.

(iii) $\{B_i^*\}_{i=0}^d$ *are linearly independent.*

(iv) *For* $0 \le i, j \le d$,

$$B_i^* B_j^* = \sum_{k=0}^{d} q_{i,j}^k B_k^*.$$

(v) $B_i^* B_j^* = B_j^* B_i^*$ *for* $0 \le i, j \le d$.

(vi) *For* $0 \le i \le d$ *we have* $(B_i^*)^t = M^{-1} B_i^* M$ *where* $M = \operatorname{diag}(m_0, m_1, \ldots, m_d)$.

Proof. Similar to the proof of Lemma 5.2. □

Definition 5.8. By Lemma 5.7 the matrices $\{B_i^*\}_{i=0}^d$ form a basis for a commutative subalgebra $\mathcal{B}^*$ of $M_{d+1}(\mathbb{C})$. We call $\mathcal{B}^*$ the *dual intersection algebra* of $\mathcal{X}$.

Definition 5.9. *Let* $\mathcal{M}^\circ$ *denote the algebra over* $\mathbb{C}$ *consisting of the vector space* $\mathcal{M}$ *together with the Hadamard multiplication* $\circ$. *The algebra* $\mathcal{M}^\circ$ *is commutative. Note that* J *is the multiplicative identity in* $\mathcal{M}^\circ$.

Theorem 5.10. *There exists an algebra isomorphism* $\mathcal{M}^\circ \to \mathcal{B}^*$ *that sends* $E_i \mapsto |X|^{-1} B_i^*$ *for* $0 \le i \le d$.

Proof. For $0 \le i, j \le d$ we have

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{i,j}^k E_k.$$

Compare this with Lemma 5.7(iv). □

For $A \in \mathcal{M}^\circ$, there exists a $\mathbb{C}$-linear map $L_A^\circ : \mathcal{M}^\circ \to \mathcal{M}^\circ$ that sends $B \mapsto A \circ B$ for all $B \in \mathcal{M}^\circ$.

Theorem 5.11. *With the above notation, the following* (i)–(iv) *hold for* $0 \le i \le d$:

(i) $|X|^{-1} (B_i^*)^t$ *represents* $L_{E_i}^\circ$ *with respect to the basis* $\{E_\ell\}_{\ell=0}^d$;

(ii) *the matrix* $|X|^{-1} \operatorname{diag}(Q_i(0), Q_i(1), \ldots, Q_i(d))$ *represents* $L_{E_i}^\circ$ *with respect to* $\{A_\ell\}_{\ell=0}^d$;

(iii) $Q (B_i^*)^t Q^{-1} = \operatorname{diag}(Q_i(0), Q_i(1), \ldots, Q_i(d))$;

(iv) *the scalars* $Q_i(0), Q_i(1), \ldots, Q_i(d)$ *are the roots of the characteristic polynomial of* B_i^*.

Proof. (i) By Definition 5.6 and

$$E_i \circ E_j = \sum_{k=0}^{d} q_{i,j}^k E_k \qquad (0 \leq j \leq d).$$

(ii) We have

$$E_i = |X|^{-1} \sum_{j=0}^{d} Q_i(j) A_j.$$

Therefore

$$E_i \circ A_j = |X|^{-1} Q_i(j) A_j \qquad (0 \leq j \leq d).$$

The result follows.
(iii) By (i), (ii) and since $|X|^{-1}Q$ is the transition matrix from the basis $\{A_\ell\}_{\ell=0}^d$ to the basis $\{E_\ell\}_{\ell=0}^d$.
(iv) B_i^* and $(B_i^*)^t$ have the same characteristic polynomial. The result follows in view of (iii). □

We have a comment.

Proposition 5.12. *We have* $KQ = \overline{P}^t M$, *where*

$$K = \mathrm{diag}(k_0, k_1, \ldots, k_d), \qquad M = \mathrm{diag}(m_0, m_1, \ldots, m_d).$$

Proof. We saw earlier that $k_i Q_j(i) = \overline{P_i(j)} m_j$ for $0 \leq i, j \leq d$. □

6 The dual Bose-Mesner algebra and the subconstituent algebra

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. Recall the standard module $V = \mathbb{C}^X$. For $y \in X$ define $\hat{y} \in V$ that has y-entry 1 and all other entries 0. The vectors $\{\hat{y}\}_{y \in X}$ is an orthonormal basis for V. We have

$$\mathbf{1} = \sum_{y \in X} \hat{y}.$$

For $z \in X$,

$$A_i \hat{z} = \sum_{(y,z) \in R_i} \hat{y} \qquad (0 \leq i \leq d).$$

Definition 6.1. Throughout this section, we fix a vertex $x \in X$. We call x the *base vertex.*

Definition 6.2. *For $0 \le i \le d$ we define a diagonal matrix $E_i^* = E_i^*(x)$ in $M_X(\mathbb{C})$ that has (y, y)-entry*

$$E_i^*(y,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in R_i; \\ 0 & \text{if} \quad (x,y) \notin R_i \end{cases} \qquad\qquad y \in X.$$

Lemma 6.3. *With reference to Definition 6.2,*

(i) $E_i^* E_j^* = \delta_{i,j} E_i^* \qquad (0 \le i, j \le d)$*;*

(ii) $I = \sum_{i=0}^{d} E_i^*$*;*

(iii) *the matrices $\{E_i^*\}_{i=0}^{d}$ are linearly independent.*

Proof. By Definition 6.2. □

Definition 6.4. By Lemma 6.3, the matrices $\{E_i^*\}_{i=0}^{d}$ form a basis for a commutative subalgebra $\mathcal{M}^* = \mathcal{M}^*(x)$ of $M_X(\mathbb{C})$. We call $\mathcal{M}^*$ the *dual Bose-Mesner algebra of $\mathcal{X}$ with respect to x.* We call E_i^* the i^{th} *dual primitive idempotent of $\mathcal{X}$ with respect to x.*

Lemma 6.5. *We have*

$$V = \sum_{i=0}^{d} E_i^* V \qquad\qquad \text{(orthogonal direct sum).}$$

For $0 \le i \le d$ the subspace $E_i^* V$ is a common eigenspace for $\mathcal{M}^*$, and E_i^* is the projection onto this eigenspace. The subspace $E_i^* V$ has basis $\{\hat{y} | y \in \Gamma_i(x)\}$. Moreover $k_i = \dim E_i^* V$. The vector $\hat{x}$ is a basis for $E_0^* V$.

Proof. Routine consequence of Definition 6.2. □

Referring to Lemma 6.5, we call $E_i^* V$ the i^{th} *subconstituent of $\mathcal{X}$ with respect to x.*

Next we describe how the algebras $\mathcal{M}^\circ$ and $\mathcal{M}^*$ are related.

Lemma 6.6. *There exists an algebra isomorphism $\natural : \mathcal{M}^\circ \to \mathcal{M}^*$ that sends $A_i \mapsto E_i^*$ for $0 \le i \le d$.*

Proof. For $0 \le i, j \le d$ we have $A_i \circ A_j = \delta_{i,j} A_i$ and $E_i^* E_j^* = \delta_{i,j} E_i^*$. □

We emphasize the nature of $\natural$. For $A, B \in \mathcal{M}$ we have

$$(A \circ B)^\natural = A^\natural B^\natural. \tag{23}$$

Lemma 6.7. *For $A \in \mathcal{M}$,*

$$(A^\natural)_{y,y} = A_{x,y} \qquad\qquad (y \in X). \tag{24}$$

Proof. Without loss, we may assume that A is an associate matrix A_i. In this case $A^\natural = E_i^*$. Now (24) holds by the definitions of A_i and E_i^*. $\square$

Definition 6.8. For $0 \le i \le d$ let $A_i^* \in \mathcal{M}^*$ be the image of $|X|E_i$ under the map $\natural$ from Lemma 6.6. We call A_i^* the i^{th} *dual associate matrix of* $\mathcal{X}$ *with respect to* x.

Lemma 6.9. *For* $0 \le i \le d$,

$$(A_i^*)_{y,y} = |X|(E_i)_{x,y} \qquad\qquad (y \in X).$$

Proof. By Lemma 6.7 with $A = |X|E_i$. $\square$

Lemma 6.10. *The matrices* $\{A_i^*\}_{i=0}^d$ *form a basis for* $\mathcal{M}^*$. *Moreover*

$$A_i^* = \sum_{j=0}^{d} Q_i(j) E_j^* \qquad\qquad (0 \le i \le d), \tag{25}$$

$$E_i^* = |X|^{-1} \sum_{j=0}^{d} P_i(j) A_j^* \qquad\qquad (0 \le i \le d). \tag{26}$$

Proof. The first assertion holds because $\natural : \mathcal{M}^\circ \to \mathcal{M}^*$ is a bijection and $\{E_i\}_{i=0}^d$ is a basis for $\mathcal{M}^\circ$. To get (25), (26) we apply $\natural$ to each side of (10), (11). $\square$

Lemma 6.11. *For* $0 \le i,j \le d$ *the scalar* $Q_i(j)$ *is the eigenvalue of* A_i^* *associated to the common eigenspace* E_j^*V *of* $\mathcal{M}^*$.

Proof. By (25). $\square$

Proposition 6.12. *The following* (i)–(iv) *hold:*

(i) $A_0^* = I$;

(ii) $|X|E_0^* = \sum_{i=0}^d A_i^*$;

(iii) $\overline{A_i^*} = A_i^* \qquad (0 \le i \le d)$;

(iv) *for* $0 \le i,j \le d$,

$$A_i^* A_j^* = \sum_{k=0}^{d} q_{i,j}^k A_k^*.$$

Proof. (i) We have

$$A_0^* = |X|(E_0)^\natural = J^\natural = I.$$

(ii) We have

$$\sum_{i=0}^{d} A_i^* = |X|(E_0 + E_1 + \cdots + E_d)^\natural = |X| I^\natural = |X| E_0^*.$$

(iii) We have

$$\overline{A_i^*} = |X|\overline{(E_i)^\natural} = |X|(\overline{E_i})^\natural = |X|(E_{\hat{i}})^\natural = A_{\hat{i}}^*.$$

(iv) Apply $\natural$ to each side of

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{i,j}^k E_k.$$

$\square$

Next, we consider how $\mathcal{M}$ and $\mathcal{M}^*$ are related.

Definition 6.13. Let $T = T(x)$ denote the subalgebra of $M_X(\mathbb{C})$ generated by $\mathcal{M}$ and $\mathcal{M}^*$. We call T the *subconstituent algebra of $\mathcal{X}$ with respect to x.*

We have some comments. By construction, the algebra T is finite-dimensional. Moreover T is noncommutative in general. The algebra T is closed under both the transpose map and complex-conjugation, because $\mathcal{M}$ and $\mathcal{M}^*$ are closed under both the transpose map and complex-conjugation.

Lecture 8

We are going to show that for $0 \le \alpha, \beta, \gamma \le d$,

$$E_\alpha^* A_\beta E_\gamma^* = 0 \text{ iff } p_{\alpha,\beta}^\gamma = 0;$$
$$E_\alpha A_\beta^* E_\gamma = 0 \text{ iff } q_{\alpha,\beta}^\gamma = 0.$$

The above equations are called the *triple product relations.*

To obtain the triple product relations, we endow the vector space $M_X(\mathbb{C})$ with a bilinear form $(\, , \,)$ such that $(A, B) = \operatorname{tr}(A^t \overline{B})$ for all $A, B \in M_X(\mathbb{C})$. Abbreviate $\|A\|^2 = (A, A)$. For $A, B, C \in M_X(\mathbb{C})$ and $\alpha \in \mathbb{C}$, we have

$$(B, A) = \overline{(A, B)}, \qquad (\alpha A, B) = \alpha(A, B),$$
$$(A + B, C) = (A, C) + (B, C), \qquad \|A\|^2 \in \mathbb{R},$$
$$\|A\|^2 \ge 0, \qquad \|A\|^2 = 0 \text{ iff } A = 0,$$
$$(AB, C) = (B, \overline{A}^t C) = (A, C\overline{B}^t).$$

Lemma 6.14. *For $0 \le \alpha, \beta, \gamma, i, j, k \le d$ we have*

(i) $(E_\alpha^* A_\beta E_\gamma^*, E_i^* A_j E_k^*) = \delta_{\alpha,i}\delta_{\beta,j}\delta_{\gamma,k} k_\gamma p_{\alpha,\beta}^\gamma$;

(ii) $(E_\alpha A_\beta^* E_\gamma, E_i A_j^* E_k) = \delta_{\alpha,i}\delta_{\beta,j}\delta_{\gamma,k} m_\gamma q_{\alpha,\beta}^\gamma$.

Proof. (i) Using $\mathrm{tr}(BC) = \mathrm{tr}(CB)$,

$$\begin{aligned}(E^*_\alpha A_\beta E^*_\gamma, E^*_i A_j E^*_k) &= \mathrm{tr}\big((E^*_\alpha A_\beta E^*_\gamma)^t \overline{E^*_i A_j E^*_k}\big)\\ &= \mathrm{tr}\big(E^*_\gamma A_{\beta'} E^*_\alpha E^*_i A_j E^*_k\big)\\ &= \delta_{\alpha,i}\delta_{\gamma,k}\mathrm{tr}\big(E^*_\gamma A_{\beta'} E^*_\alpha A_j\big)\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\big(E^*_\gamma A_{\beta'} E^*_\alpha A_j\big) &= \sum_{y\in X}\sum_{z\in X}(E^*_\gamma)_{y,y}(A_{\beta'})_{y,z}(E^*_\alpha)_{z,z}(A_j)_{z,y}\\ &= \sum_{y\in X}\sum_{z\in X}(E^*_\gamma)_{y,y}(A_{\beta'}\circ A_{j'})_{y,z}(E^*_\alpha)_{z,z}\\ &= \delta_{\beta,j}\sum_{y\in X}\sum_{z\in X}(E^*_\gamma)_{y,y}(A_{\beta'})_{y,z}(E^*_\alpha)_{z,z}\\ &= \delta_{\beta,j}\sum_{\substack{y\in\Gamma_\gamma(x),\\ z\in\Gamma_\alpha(x)\cap\Gamma_{\beta'}(y)}} 1\\ &= \delta_{\beta,j}k_\gamma p^\gamma_{\alpha,\beta}.\end{aligned}$$

(ii) We have

$$\begin{aligned}(E_\alpha A^*_\beta E_\gamma, E_i A^*_j E_k) &= \mathrm{tr}\big((E_\alpha A^*_\beta E_\gamma)^t \overline{E_i A^*_j E_k}\big)\\ &= \mathrm{tr}\big(E_{\hat\gamma} A^*_\beta E_{\hat\alpha} E_{\hat\imath} A^*_{\hat\jmath} E_{\hat k}\big)\\ &= \delta_{\alpha,i}\delta_{\gamma,k}\mathrm{tr}\big(E_{\hat\gamma} A^*_\beta E_{\hat\alpha} A^*_{\hat\jmath}\big)\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\big(E_{\hat\gamma} A^*_\beta E_{\hat\alpha} A^*_{\hat\jmath}\big) &= \sum_{y\in X}\sum_{z\in X}(E_{\hat\gamma})_{y,z}(A^*_\beta)_{z,z}(E_{\hat\alpha})_{z,y}(A^*_{\hat\jmath})_{y,y}\\ &= |X|^2\sum_{y\in X}\sum_{z\in X}(E_{\hat\gamma})_{y,z}(E_\beta)_{x,z}(E_{\hat\alpha})_{z,y}(E_{\hat\jmath})_{x,y}\\ &= |X|^2\sum_{y\in X}\sum_{z\in X}(E_{\hat\jmath})_{x,y}(E_{\hat\gamma}\circ E_\alpha)_{y,z}(E_{\hat\beta})_{z,x}\\ &= |X|^2\Big((x,x)\text{-entry of } E_{\hat\jmath}(E_{\hat\gamma}\circ E_\alpha)E_{\hat\beta}\Big)\\ &= |X|\mathrm{tr}\big(E_{\hat\jmath}(E_{\hat\gamma}\circ E_\alpha)E_{\hat\beta}\big)\\ &= |X|\mathrm{tr}\big((E_{\hat\gamma}\circ E_\alpha)E_{\hat\beta}E_{\hat\jmath}\big)\\ &= \delta_{\beta,j}|X|\mathrm{tr}\big((E_{\hat\gamma}\circ E_\alpha)E_{\hat\beta}\big)\\ &= \delta_{\beta,j}m_{\hat\beta}q^{\hat\beta}_{\hat\gamma,\alpha}\\ &= \delta_{\beta,j}m_\gamma q^\gamma_{\alpha,\beta}.\end{aligned}$$

□

Corollary 6.15. *For* $0 \le \alpha, \beta, \gamma \le d$ *we have*

(i) $\|E^*_\alpha A_\beta E^*_\gamma\|^2 = k_\gamma p^\gamma_{\alpha,\beta}$;

(ii) $\|E_\alpha A^*_\beta E_\gamma\|^2 = m_\gamma q^\gamma_{\alpha,\beta}$.

Proof. Set $i = \alpha$, $j = \beta$, $k = \gamma$ in Lemma 6.14. □

Corollary 6.15(ii) gives a second proof of the Krein condition.

Theorem 6.16. (Triple product relations). *For* $0 \le \alpha, \beta, \gamma \le d$ *we have*

(i) $E^*_\alpha A_\beta E^*_\gamma = 0$ iff $p^\gamma_{\alpha,\beta} = 0$;

(ii) $E_\alpha A^*_\beta E_\gamma = 0$ iff $q^\gamma_{\alpha,\beta} = 0$.

Proof. By Corollary 6.15. □

We bring in some notation. For subspaces R, S of $M_X(\mathbb{C})$, define

$$RS = \text{Span}\{rs | r \in R, \ s \in S\}.$$

Theorem 6.17. *With the above notation,*

(i) *the vector space* $\mathcal{M}^*\mathcal{M}\mathcal{M}^*$ *has an orthogonal basis*

$$\{E^*_\alpha A_\beta E^*_\gamma | 0 \le \alpha, \beta, \gamma \le d, \ p^\gamma_{\alpha,\beta} \ne 0\};$$

(ii) *the vector space* $\mathcal{M}\mathcal{M}^*\mathcal{M}$ *has an orthogonal basis*

$$\{E_\alpha A^*_\beta E_\gamma | 0 \le \alpha, \beta, \gamma \le d, \ q^\gamma_{\alpha,\beta} \ne 0\}.$$

Proof. By Lemma 6.14 and Theorem 6.16. □

We mention a consequence of Theorem 6.16. Recall the standard module V.

Proposition 6.18. *For* $0 \le j, k \le d$ *we have*

$$A_j E^*_k V \subseteq \sum_{\substack{0 \le i \le d, \\ p^k_{i,j} \ne 0}} E^*_i V, \qquad A^*_j E_k V \subseteq \sum_{\substack{0 \le i \le d, \\ q^k_{i,j} \ne 0}} E_i V. \tag{27}$$

Proof. Concerning the containment on the left in (27),

$$A_j E^*_k V = I A_j E^*_k V = \sum_{i=0}^{d} E^*_i A_j E^*_k V = \sum_{\substack{0 \le i \le d, \\ p^k_{i,j} \ne 0}} E^*_i A_j E^*_k V \subseteq \sum_{\substack{0 \le i \le d, \\ p^k_{i,j} \ne 0}} E^*_i V.$$

The containment on the right in (27) is similarly obtained. □

Next, we consider how the algebra T acts on the standard module V. By a *T-module* we mean a subspace $W \subseteq V$ such that $TW \subseteq W$. A T-module W is *irreducible* whenever W is nonzero, and W does not contain a T-module besides 0 and W.

Lemma 6.19. *Let W denote a T-module. Then the orthogonal complement $W^\perp$ is a T-module.*

Proof. For $A \in T$ we have $\overline{A}^t \in T$. Also

$$\langle Au, v \rangle = \langle u, \overline{A}^t v \rangle \qquad\qquad u, v \in V.$$

By these comments we obtain the result. □

Corollary 6.20. *The standard module V is an orthogonal direct sum of irreducible T-modules.*

Proof. Use Lemma 6.19. □

Next, we describe a particular irreducible T-module called the primary T-module. Recall the vector $\mathbf{1} = \sum_{y \in X} \hat{y}$. For $0 \le i \le d$ define the vector

$$\mathbf{1}_i = \sum_{y \in \Gamma_i(x)} \hat{y}.$$

Observe that

$$E_i^* \mathbf{1} = \mathbf{1}_i = A_{i'} \hat{x} \qquad\qquad (0 \le i \le d).$$

Consequently

$$\mathcal{M}^* E_0 V = \mathcal{M} E_0^* V. \tag{28}$$

Lemma 6.21. *The vector space $\mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$ is an irreducible T-module.*

Proof. Define $\mathcal{V} = \mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$. We have $\mathcal{M}\mathcal{V} \subseteq \mathcal{V}$ since $\mathcal{V} = \mathcal{M} E_0^* V$. We have $\mathcal{M}^* \mathcal{V} \subseteq \mathcal{V}$ since $\mathcal{V} = \mathcal{M}^* E_0 V$. Therefore $T\mathcal{V} \subseteq \mathcal{V}$, so $\mathcal{V}$ is a T-module. We show that the T-module $\mathcal{V}$ is irreducible. The standard T-module V is a direct sum of irreducible T-modules. There exists an irreducible T-module that is not orthogonal to $\hat{x}$. This T-module is closed under E_0^*, so it contains $\hat{x}$ and also $\mathcal{M}\hat{x} = \mathcal{V}$. This T-module must equal $\mathcal{V}$ by irreducibility. □

Lecture 9

Definition 6.22. Define $\mathcal{V} = \mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$. The T-module $\mathcal{V}$ is called *primary*.

Lemma 6.23. *For $0 \le i \le d$ we have*

$$|X| E_i \hat{x} = A_i^* \mathbf{1}. \tag{29}$$

Proof. Both vectors in (29) have y-coordinate $|X|(E_i)_{y,x}$ for $y \in X$. □

Definition 6.24. For $0 \le i \le d$ let $\mathbf{1}_i^*$ denote the common vector in (29).

We clarify the definitions. Note that $\mathbf{1}_0 = \hat{x}$ and $\mathbf{1}_0^* = \mathbf{1}$. Moreover

$$\mathbf{1}_0^* = \sum_{i=0}^{d} \mathbf{1}_i, \qquad \mathbf{1}_0 = |X|^{-1} \sum_{i=0}^{d} \mathbf{1}_i^*.$$

The following result is routinely verified.

Lemma 6.25. *For the primary T-module $\mathcal{V}$,*

(i) $\mathbf{1}_i$ *is a basis for* $E_i^*\mathcal{V}$ $(0 \le i \le d)$*;*

(ii) $\{\mathbf{1}_i\}_{i=0}^d$ *is a basis for* $\mathcal{V}$*;*

(iii) $\mathbf{1}_i^*$ *is a basis for* $E_i\mathcal{V}$ $(0 \le i \le d)$*;*

(iv) $\{\mathbf{1}_i^*\}_{i=0}^d$ *is a basis for* $\mathcal{V}$*.*

Next, we describe how the bases $\{\mathbf{1}_i\}_{i=0}^d$ and $\{\mathbf{1}_i^*\}_{i=0}^d$ are related.

Lemma 6.26. *For $0 \le j \le d$ we have*

(i) $\mathbf{1}_j = |X|^{-1} \sum_{i=0}^d \overline{P_j(i)} \mathbf{1}_i^*$*;*

(ii) $\mathbf{1}_j^* = \sum_{i=0}^d \overline{Q_j(i)} \mathbf{1}_i$*.*

Proof. (i) Observe

$$\mathbf{1}_j = E_j^*\mathbf{1} = |X|^{-1} \sum_{i=0}^{d} P_j(i) A_i^* \mathbf{1} = |X|^{-1} \sum_{i=0}^{d} P_j(i) \mathbf{1}_{\hat{i}}^*$$
$$= |X|^{-1} \sum_{i=0}^{d} P_j(\hat{i}) \mathbf{1}_i^* = |X|^{-1} \sum_{i=0}^{d} \overline{P_j(i)} \mathbf{1}_i^*.$$

(ii) Observe

$$\mathbf{1}_j^* = |X| E_j \hat{x} = \sum_{i=0}^{d} Q_j(i) A_i \hat{x} = \sum_{i=0}^{d} Q_j(i) \mathbf{1}_{i'} = \sum_{i=0}^{d} Q_j(i') \mathbf{1}_i = \sum_{i=0}^{d} \overline{Q_j(i)} \mathbf{1}_i.$$

□

Next we describe how the algebra T acts on the bases $\{\mathbf{1}_i\}_{i=0}^d$ and $\{\mathbf{1}_i^*\}_{i=0}^d$.

Lemma 6.27. *For $0 \le i, j \le d$ we have*

(i) $E_i^* \mathbf{1}_j = \delta_{i,j} \mathbf{1}_j$*;*

(ii) $A_i^* \mathbf{1}_j = Q_i(j) \mathbf{1}_j$*;*

(iii) $E_i \mathbf{1}_j = |X|^{-1}\overline{P_j(i)} \sum_{h=0}^{d} \overline{Q_i(h)} \mathbf{1}_h$;

(iv) $A_i \mathbf{1}_j = \sum_{k=0}^{d} p^k_{i',j} \mathbf{1}_k$.

Proof. (i) Clear.
(ii) Observe

$$A_i^* \mathbf{1}_j = A_i^* E_j^* \mathbf{1} = Q_i(j) E_j^* \mathbf{1} = Q_i(j) \mathbf{1}_j.$$

(iii) Observe

$$\begin{aligned} E_i \mathbf{1}_j &= E_i A_{j'} \hat{x} = A_{j'} E_i \hat{x} = P_{j'}(i) E_i \hat{x} = \overline{P_j(i)} E_i \hat{x} \\ &= |X|^{-1} \overline{P_j(i)} \mathbf{1}_i^* = |X|^{-1} \overline{P_j(i)} \sum_{h=0}^{d} \overline{Q_i(h)} \mathbf{1}_h. \end{aligned}$$

(iv) Observe

$$A_i \mathbf{1}_j = A_i A_{j'} \hat{x} = \sum_{k=0}^{d} p^k_{i,j'} A_k \hat{x} = \sum_{k=0}^{d} p^k_{i,j'} \mathbf{1}_{k'} = \sum_{k=0}^{d} p^{k'}_{i,j'} \mathbf{1}_k = \sum_{k=0}^{d} p^k_{i',j} \mathbf{1}_k.$$

□

Lemma 6.28. *For $0 \le i, j \le d$ we have*

(i) $E_i \mathbf{1}_j^* = \delta_{i,j} \mathbf{1}_j^*$;

(ii) $A_i \mathbf{1}_j^* = P_i(j) \mathbf{1}_j^*$;

(iii) $E_i^* \mathbf{1}_j^* = |X|^{-1}\overline{Q_j(i)} \sum_{h=0}^{d} \overline{P_i(h)} \mathbf{1}_h^*$;

(iv) $A_i^* \mathbf{1}_j^* = \sum_{k=0}^{D} q^k_{\hat{\imath},j} \mathbf{1}_k^*$.

Proof. Similar to the proof of Lemma 6.27. (i) Clear.
(ii) Observe

$$A_i \mathbf{1}_j^* = |X| A_i E_j \hat{x} = |X| P_i(j) E_j \hat{x} = P_i(j) \mathbf{1}_j^*.$$

(iii) Observe

$$E_i^* \mathbf{1}_j^* = E_i^* A_{\hat{\jmath}}^* \mathbf{1} = A_{\hat{\jmath}}^* E_i^* \mathbf{1} = A_{\hat{\jmath}}^* \mathbf{1}_i = Q_{\hat{\jmath}}(i) \mathbf{1}_i = \overline{Q_j(i)} \mathbf{1}_i = |X|^{-1} \overline{Q_j(i)} \sum_{h=0}^{d} \overline{P_i(h)} \mathbf{1}_h^*.$$

(iv) Observe

$$A_i^* \mathbf{1}_j^* = A_i^* A_{\hat{\jmath}}^* \mathbf{1} = \sum_{k=0}^{d} q^k_{i,\hat{\jmath}} A_k^* \mathbf{1} = \sum_{k=0}^{d} q^k_{i,\hat{\jmath}} \mathbf{1}_{\hat{k}}^* = \sum_{k=0}^{d} q^{\hat{k}}_{i,\hat{\jmath}} \mathbf{1}_k^* = \sum_{k=0}^{d} q^k_{\hat{\imath},j} \mathbf{1}_k^*.$$

□

Next we bring in the bilinear form.

Lemma 6.29. *For $0 \leq i, j \leq d$ we have*

(i) $\langle \mathbf{1}_i, \mathbf{1}_j \rangle = \delta_{i,j} k_i$;

(ii) $\langle \mathbf{1}^*_i, \mathbf{1}^*_j \rangle = \delta_{i,j} |X| m_i$;

(iii) $\langle \mathbf{1}_i, \mathbf{1}^*_j \rangle = \overline{P_i(j)} m_j = Q_j(i) k_i$.

Proof. (i) Routine.
(ii) Observe

$$\langle \mathbf{1}^*_i, \mathbf{1}^*_j \rangle = |X|^2 \langle E_i \hat{x}, E_j \hat{x} \rangle = |X|^2 \langle \hat{x}, E_i E_j \hat{x} \rangle = \delta_{i,j} |X|^2 \langle \hat{x}, E_i \hat{x} \rangle = \delta_{i,j} |X| m_i.$$

(iii) Observe

$$\begin{aligned} \langle \mathbf{1}_i, \mathbf{1}^*_j \rangle &= |X| \langle A_{i'} \hat{x}, E_j \hat{x} \rangle = |X| \langle \hat{x}, \overline{(A_{i'})^t} E_j \hat{x} \rangle = |X| \langle \hat{x}, A_i E_j \hat{x} \rangle \\ &= |X| \overline{P_i(j)} \langle \hat{x}, E_j \hat{x} \rangle = \overline{P_i(j)} m_j = Q_j(i) k_i. \end{aligned}$$

□

7 Duality for commutative association schemes

In this section we discuss the concept of duality for commutative association schemes. To motivate things, we start with a small example.

Consider the group $G = \mathbb{Z}/3\mathbb{Z}$ with three elements. Of course G is abelian, so each conjugacy class contains one element. Consider the conjugacy-class association scheme $\mathcal{X}$ for G. The associate matrices of $\mathcal{X}$ are

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We have $A_2 = A_1^2$ and $A_1^3 = I$. Let $\omega \in \mathbb{C}$ denote a primitive third root of unity. Note that

$$\overline{\omega} = \omega^2 = \omega^{-1}, \qquad\qquad 1 + \omega + \omega^2 = 0.$$

The primitive idempotents of $\mathcal{X}$ are

$$E_0 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad E_1 = \frac{1}{3} \begin{pmatrix} 1 & w^2 & w \\ w & 1 & w^2 \\ w^2 & w & 1 \end{pmatrix}, \qquad E_2 = \frac{1}{3} \begin{pmatrix} 1 & w & w^2 \\ w^2 & 1 & w \\ w & w^2 & 1 \end{pmatrix}.$$

The first and second eigenmatrices of $\mathcal{X}$ are

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \qquad\qquad Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

Note that

$$P = \overline{Q}. \tag{30}$$

We will interpret (30) using duality.

For the rest of this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

Definition 7.1. A *duality* of $\mathcal{X}$ is a $\mathbb{C}$-linear bijection $\Psi : \mathcal{M} \to \mathcal{M}$ that satisfies (i), (ii) below:

(i) $\Psi(AB) = \Psi(A) \circ \Psi(B)$ for all $A, B \in \mathcal{M}$;

(ii) $\Psi(\Psi(A)) = |X| A^t$ for all $A \in \mathcal{M}$.

We say that $\mathcal{X}$ is *self-dual* whenever $\mathcal{X}$ has a duality.

Lemma 7.2. *Assume that $\mathcal{X}$ has a duality Ψ. Then*

(i) $\Psi(A \circ B) = |X|^{-1}\Psi(A)\Psi(B)$ *for all* $A, B \in \mathcal{M}$*;*

(ii) $\Psi(A^t) = \big(\Psi(A)\big)^t$ *for all* $A \in \mathcal{M}$*.*

Proof. (i) Each side is equal to $|X|\Psi^{-1}(A^t \circ B^t)$.
(ii) Each side is equal to $|X|^{-1}\Psi^3(A)$. □

Lemma 7.3. *Assume that $\mathcal{X}$ has a duality Ψ. Then*

(i) $\Psi(I) = J$*;*

(ii) $\Psi(J) = |X| I$*.*

Proof. (i) For $A \in \mathcal{M}$,

$$\Psi(A) = \Psi(AI) = \Psi(A) \circ \Psi(I).$$

The result follows.
(ii) For $A \in \mathcal{M}$,

$$\Psi(A) = \Psi(A \circ J) = |X|^{-1}\Psi(A)\Psi(J).$$

So $|X|^{-1}\Psi(J) = I$. The result follows. □

Lemma 7.4. *Assume that $\mathcal{X}$ has a duality Ψ. Then there exits an ordering $\{R_i\}_{i=0}^d$ of the relations such that*

$$\Psi(E_i) = A_i \qquad (0 \le i \le d).$$

Proof. For $0 \leq i, j \leq d$ we have $E_i E_j = \delta_{i,j} E_i$. In this equation we apply Ψ to each side; this yields

$$\delta_{i,j} \Psi(E_i) = \Psi(E_i E_j) = \Psi(E_i) \circ \Psi(E_j).$$

By these comments, the sequence $\{\Psi(E_i)\}_{i=0}^d$ is a permutation of the sequence $\{A_i\}_{i=0}^d$. The result follows. $\square$

Lecture 10

Lemma 7.5. *Assume that $\mathcal{X}$ has a duality Ψ such that $\Psi(E_i) = A_i$ for $0 \leq i \leq d$. Then* (i)–(iv) *hold below:*

(i) $\Psi(A_i) = |X| E_i^t \qquad (0 \leq i \leq d)$;

(ii) $i' = \hat{\imath} \qquad (0 \leq i \leq d)$;

(iii) $P = \overline{Q}$;

(iv) $p^k_{i,j} = q^k_{i,j} \qquad (0 \leq i, j, k \leq d)$.

Proof. (i) We have

$$|X| E_i^t = \Psi(\Psi(E_i)) = \Psi(A_i).$$

(ii) We have

$$A_{\hat{\imath}} = \Psi(E_{\hat{\imath}}) = \Psi(E_i^t) = \big(\Psi(E_i)\big)^t = A_i^t = A_{i'}.$$

(iii) For $0 \leq i \leq d$ we have $A_i = \sum_{j=0}^d P_i(j) E_j$. In this equation we apply Ψ to each side; this yields

$$|X| E_i^t = \sum_{j=0}^d P_i(j) A_j.$$

We may now argue

$$\sum_{j=0}^d P_i(j) A_j = |X| E_i^t = |X| \overline{E_i} = \sum_{j=0}^d \overline{Q_i(j)} A_j$$

Therefore $P_i(j) = \overline{Q_i(j)}$ for $0 \leq i, j \leq d$. Consequently $P = \overline{Q}$.
(iv) We have

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^d q^k_{i,j} E_k \qquad (0 \leq i, j \leq d).$$

In this equation, we apply Ψ to each side and evaluate the result; this yields

$$A_iA_j = \sum_{k=0}^{d} q_{i,j}^k A_k \qquad (0 \leq i, j \leq d).$$

Recall that

$$A_iA_j = \sum_{k=0}^{d} p_{i,j}^k A_k \qquad (0 \leq i, j \leq d).$$

Therefore $p_{i,j}^k = q_{i,j}^k$ for $0 \leq i, j, k \leq d$. □

Next, we give a characterization of the self-dual condition.

Proposition 7.6. *The following are equivalent:*

(i) $\mathcal{X}$ *is self-dual;*

(ii) *there exists an ordering* $\{R_i\}_{i=0}^d$ *of the relations such that* $P = \overline{Q}$.

Proof. (i) ⇒ (ii) By Lemmas 7.4, 7.5.
(ii) ⇒ (i) Without loss, we may assume that $P = \overline{Q}$. The vector space $\mathcal{M}$ has a basis $\{A_i\}_{i=0}^d$ and a basis $\{E_i\}_{i=0}^d$. There exists a $\mathbb{C}$-linear bijection $\Psi : \mathcal{M} \to \mathcal{M}$ such that $\Psi(E_i) = A_i$ for $0 \leq i \leq d$. For $0 \leq i \leq d$ we have

$$A_i = \sum_{j=0}^{d} P_i(j) E_j$$

and also

$$|X| E_i^t = |X| \overline{E_i} = \sum_{j=0}^{d} \overline{Q_i(j)} A_j = \sum_{j=0}^{d} P_i(j) A_j.$$

Comparing the above two equations, we find that $\Psi(A_i) = |X| E_i^t$ for $0 \leq i \leq d$. By these comments Ψ satisfies the two conditions in Definition 7.1. Therefore Ψ is a duality, and $\mathcal{X}$ is self-dual. □

Problem 7.7. Let G denote a finite abelian group. Consider the conjugacy-class association scheme for G. Show that this association scheme has a duality.

Problem 7.8. Recall the Hamming association scheme $H(d, q)$. Let us take $d = 1$. The scheme $K_q = H(1, q)$ is called complete. Show that for K_q,

$$P = \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix} = Q.$$

In particular, K_q is self-dual.

Shortly, we will see that $H(d, q)$ is self-dual for all $d, q \geq 1$.

8 Fusion for commutative association schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

Our next goal is to construct new commutative association schemes by "fusing together" some of the relations $\{R_i\}_{i=0}^d$.

Definition 8.1. A *fusion scheme* for $\mathcal{X}$ is an association scheme $\tilde{\mathcal{X}} = (X, \{\tilde{R}_i\}_{i=0}^{\tilde{d}})$ such that

$$\tilde{R}_i = \cup_{j \in \Lambda_i} R_j \qquad (0 \le i \le \tilde{d}),$$

where $\{\Lambda_i\}_{i=0}^{\tilde{d}}$ is a partition of $\{0, 1, \ldots, d\}$ into nonempty subsets with $\Lambda_0 = \{0\}$.

By construction, a fusion scheme for $\mathcal{X}$ is commutative.

Let $\{\Lambda_i\}_{i=0}^{\tilde{d}}$ denote any partition of $\{0, 1, \ldots, d\}$ into nonempty subsets, such that $\Lambda_0 = \{0\}$. In general, the corresponding fusion scheme does not exist. However, in the following case it does exist.

Example 8.2. Define $\tilde{d} = 1$. Define $\Lambda_0 = \{0\}$ and $\Lambda_1 = \{1, 2, \ldots, d\}$. Then $\tilde{\mathcal{X}} = (X, \{\tilde{R}_i\}_{i=0}^{\tilde{d}})$ is the complete association scheme.

Let $\{\Lambda_i\}_{i=0}^{\tilde{d}}$ denote any partition of $\{0, 1, \ldots, d\}$ into nonempty subsets, such that $\Lambda_0 = \{0\}$. In the next result, we give necessary and sufficient conditions for the corresponding fusion scheme to exist. Recall the first eigenmatrix P for $\mathcal{X}$.

Proposition 8.3. *For the above partition $\{\Lambda_i\}_{i=0}^{\tilde{d}}$, the corresponding fusion scheme $\tilde{\mathcal{X}} = (X, \{\tilde{R}_i\}_{i=0}^{\tilde{d}})$ exists iff* (i), (ii) *hold below:*

(i) *the sequence $\{\tilde{R}_i^t\}_{i=0}^{\tilde{d}}$ is a permutation of the sequence $\{\tilde{R}_i\}_{i=0}^{\tilde{d}}$;*

(ii) *there exists a partition $\{F_i\}_{i=0}^{\tilde{d}}$ of $\{0, 1, \ldots, d\}$ into nonempty subsets such that $P|_{F_i \times \Lambda_j}$ has constant row sum for $0 \le i, j \le \tilde{d}$. By definition $P|_{F_i \times \Lambda_j}$ is the submatrix of P with row set F_i and column set Λ_j.*

Assume that (i), (ii) *hold. Then after permuting $\{F_i\}_{i=0}^{\tilde{d}}$ as necessary, we have:*

(a) $F_0 = \{0\}$*;*

(b) *the associate matrices of $\tilde{\mathcal{X}}$ are*

$$\tilde{A}_i = \sum_{j \in \Lambda_i} A_j \qquad (0 \le i \le \tilde{d}); \tag{31}$$

(c) *the primitive idempotents of $\tilde{\mathcal{X}}$ are*

$$\tilde{E}_i = \sum_{j \in F_i} E_j \qquad (0 \le i \le \tilde{d}); \tag{32}$$

(d) *for the first eigenmatrix* $\tilde{P}$ *of* $\tilde{\mathcal{X}}$ *and for* $0 \leq i, j \leq \tilde{d}$*, the submatrix* $P|_{F_i \times \Lambda_j}$ *has constant row sum* $\tilde{P}_j(i)$*;*

(e) *the sequence* $\{\tilde{A}_i^t\}_{i=0}^{\tilde{d}}$ *is a permutation of the sequence* $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$*;*

(f) *the sequence* $\{\tilde{E}_i^t\}_{i=0}^{\tilde{d}}$ *is a permutation of the sequence* $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$*.*

Proof. First assume that the association scheme $\tilde{\mathcal{X}}$ exists. Then (b) holds. The Bose-Mesner algebra $\tilde{\mathcal{M}}$ of $\tilde{\mathcal{X}}$ has basis $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$. The vector space $\tilde{\mathcal{M}}$ is closed under the transpose map, so (e) holds. Let $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ denote the primitive idempotents of $\tilde{\mathcal{X}}$. The matrices $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ form a basis for $\tilde{\mathcal{M}}$. For $0 \leq i \leq \tilde{d}$ the matrix $\tilde{E}_i$ is a linear combination of $\{E_\ell\}_{\ell=0}^{d}$. In this linear combination, the coefficients are all in $\{0, 1\}$ because $(\tilde{E}_i)^2 = \tilde{E}_i$. Therefore, there exists a subset $F_i \subseteq \{0, 1, \ldots, d\}$ such that $\tilde{E}_i = \sum_{j \in F_i} E_j$. Recall that $I = \sum_{\ell=0}^{d} E_\ell$, and $E_r E_s = \delta_{r,s} E_r$ for $0 \leq r, s \leq d$. Consequently the subsets $\{F_i\}_{i=0}^{\tilde{d}}$ partition $\{0, 1, \ldots, d\}$. This gives (c). Let $\tilde{P}$ denote the first eigenmatrix of $\tilde{\mathcal{X}}$. By construction

$$\tilde{A}_j = \sum_{i=0}^{\tilde{d}} \tilde{P}_j(i) \tilde{E}_i \qquad (0 \leq j \leq \tilde{d}). \tag{33}$$

In this equation, we evaluate the left-hand side using (31) and the right-hand side using (32). The result shows that the submatrix $P|_{F_i \times \Lambda_j}$ has constant row sum $\tilde{P}_j(i)$ for $0 \leq i, j \leq \tilde{d}$. This gives (d). Note that $J \in \tilde{\mathcal{M}}$, since $J = \sum_{i=0}^{d} A_i = \sum_{i=0}^{\tilde{d}} \tilde{A}_i$. This gives (a). Item (f) holds because $\tilde{\mathcal{M}}$ is closed under the transpose map and $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ is a basis for $\tilde{\mathcal{M}}$. Items (i) and (ii) are implied by (e) and (d), respectively.

We are done in one logical direction. Next we reverse the logical direction. Assume that (i), (ii) hold. We show that the association scheme $\tilde{\mathcal{X}}$ exists. For $0 \leq i \leq \tilde{d}$ define $\tilde{A}_i$ as in (31), and note that $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$ are linearly independent. For $0 \leq i \leq \tilde{d}$ define $\tilde{E}_i$ as in (32), and note that $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ are linearly independent. For $0 \leq i, j \leq \tilde{d}$ let $\tilde{P}_j(i)$ denote the common row sum for $P|_{F_i \times \Lambda_j}$. Then (33) holds. Consequently $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$ and $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ are both bases for the same vector space $\tilde{\mathcal{M}}$. We show that $\tilde{\mathcal{M}}$ is a Bose-Mesner algebra. To do this, we show that $\tilde{\mathcal{M}}$ satisfies the conditions (i)–(v) in Proposition 2.4.

- We have $I = \tilde{A}_0 \in \tilde{\mathcal{M}}$. Also $J \in \tilde{\mathcal{M}}$ since $J = \sum_{i=0}^{d} A_i = \sum_{i=0}^{\tilde{d}} \tilde{A}_i$.
- $\tilde{\mathcal{M}}$ is closed under matrix multiplication, because the basis $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ satisfies $\tilde{E}_i \tilde{E}_j = \delta_{i,j} \tilde{E}_i$ for $0 \leq i, j \leq \tilde{d}$.
- $\tilde{\mathcal{M}}$ is closed under Hadamard multiplication, because the basis $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$ satisfies $\tilde{A}_i \circ \tilde{A}_j = \delta_{i,j} \tilde{A}_i$ for $0 \leq i, j \leq \tilde{d}$.
- $\tilde{\mathcal{M}}$ is closed under the transpose map by assumption (i).
- $\tilde{\mathcal{M}}$ is homogeneous, because $\tilde{\mathcal{M}} \subseteq \mathcal{M}$ and $\mathcal{M}$ is homogeneous.

We have shown that $\tilde{\mathcal{M}}$ is a Bose-Mesner algebra, and consequently the association scheme $\tilde{\mathcal{X}}$ exists. $\square$

Lecture 11

9 Primitive association schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

We will define a condition on $\mathcal{X}$ called primitivity. To motivate this condition, assume for the moment that $\mathcal{X}$ is the conjugacy-class association scheme for a finite group G. Then $\mathcal{X}$ is primitive if and only if G is simple.

Definition 9.1. For $1 \leq i \leq d$ we view the pair (X, R_i) as a directed graph with vertex set X; vertices x, y satisfy $x \to y$ whenever $(x, y) \in R_i$.

Definition 9.2. For an integer $\ell \geq 0$, a *path of length* ℓ in a directed graph is a sequence of vertices $\{x_i\}_{i=0}^{\ell}$ such that $x_{i-1} \to x_i$ for $1 \leq i \leq \ell$. This path goes *from* x_0 *to* x_ℓ. For example, there is a path of length zero from any vertex to itself. A directed graph is said to be *connected* whenever for all vertices x, y there is a path from x to y.

Lemma 9.3. Consider the graph (X, R_i) from Definition 9.1. For $x, y \in X$ and $\ell \in \mathbb{N}$ the following are equal:

(i) the number of paths of length ℓ from x to y;

(ii) the (x, y)-entry of A_i^ℓ.

Proof. Routine. □

Definition 9.4. The association scheme $\mathcal{X}$ is called *primitive* whenever the directed graph (X, R_i) is connected for $1 \leq i \leq d$. We say that $\mathcal{X}$ is *imprimitive* whenever $\mathcal{X}$ is not primitive.

As we investigate primitivity, the following notation will be useful. For a subset $\Omega \subseteq \{0, 1, \ldots, d\}$ define the relation

$$R_\Omega = \cup_{k \in \Omega} R_k. \tag{34}$$

We consider the case in which R_Ω is an equivalence relation. This happens if $\Omega = \{0\}$ or $\Omega = \{0, 1, \ldots, d\}$. We are going to show that $\mathcal{X}$ is imprimitive iff there exists $\{0\} \subsetneq \Omega \subsetneq \{0, 1, \ldots, d\}$ such that R_Ω is an equivalence relation.

Lemma 9.5. *We refer to the graph* (X, R_i) *from Definition 9.1. For* $x \in X$ *the set*

$$\Gamma^{(i)}(x) = \{y \in X | \text{there exists a path from } x \text{ to } y\}$$

is described as follows:

(i) *there exists a subset* $\Omega \subseteq \{0, 1, \ldots, d\}$ *such that*

$$\Gamma^{(i)}(x) = \cup_{k\in\Omega}\Gamma_k(x).$$

(ii) Ω *is the minimal subset of* $\{0, 1, \ldots, d\}$ *such that* (a) $0 \in \Omega$; (b) *for* $0 \leq j, k \leq d$, *if* $j \in \Omega$ *and* $p^k_{i,j} > 0$ *then* $k \in \Omega$.

(iii) Ω *is independent of* x.

(iv) $|\Gamma^{(i)}(x)|$ *is independent of* x.

Proof. (i) Observe that

$$\Gamma^{(i)}(x) = \{y \in X | \exists \ell \in \mathbb{N} \text{ such that } (A_i^\ell)_{x,y} > 0\}.$$

For $\ell \in \mathbb{N}$ the matrix A_i^ℓ is a linear combination of the associate matrices. The result follows.
(ii) By the definition of the intersection numbers.
(iii) By (ii) above.
(iv) By (i) and (iii) above. □

Corollary 9.6. *We refer to the graph* (X, R_i) *from Definition 9.1. For* $x, y \in X$ *the following are equivalent:*

(i) *there exists a path from* x *to* y;

(ii) *there exists a path from* y *to* x.

Proof. (i) ⇒ (ii) We have $y \in \Gamma^{(i)}(x)$. By construction $\Gamma^{(i)}(y) \subseteq \Gamma^{(i)}(x)$. In this inclusion, the two sets have the same size, so $\Gamma^{(i)}(x) = \Gamma^{(i)}(y)$. By this and since $x \in \Gamma^{(i)}(x)$ we see that $x \in \Gamma^{(i)}(y)$. Consequently there exists a path from y to x.
(ii) ⇒ (i) By symmetry. □

Corollary 9.7. *We refer to the graph* (X, R_i) *from Definition 9.1, and the corresponding set* Ω *from Lemma 9.5. Then* $k \in \Omega$ *implies* $k' \in \Omega$ *for* $0 \leq k \leq d$.

Proof. Assume that $k \in \Omega$. Pick $x, y \in X$ with $(x, y) \in R_k$. By assumption, there exists a path from x to y. So there exists a path from y to x. We have $(y, x) \in R_{k'}$. By these comments $k' \in \Omega$. □

Lemma 9.8. *We refer to the graph* (X, R_i) *from Definition 9.1, and the corresponding set* Ω *from Lemma 9.5.*

(i) *The relation* R_Ω *from* (34) *is an equivalence relation;*

(ii) *for* $x \in X$ *the set* $\Gamma^{(i)}(x)$ *is the equivalence class of* R_Ω *that contains* x.

Proof. By Lemma 9.5 and Corollary 9.7. □

We have been discussing the directed graph (X, R_i). We now introduce a directed graph with vertex set $\{0, 1, \ldots, d\}$.

Definition 9.9. For $1 \leq i \leq d$ we define a directed graph Δ_{A_i} with vertex set $\{0, 1, \ldots, d\}$; vertices j, k satisfy $j \to k$ whenever $p^k_{i,j} > 0$. Note that a vertex j of Δ_{A_i} has a loop $j \to j$ whenever $p^j_{i,j} > 0$. We call Δ_{A_i} the *A_i-distribution diagram for $\mathcal{X}$*.

The graph Δ_{A_i} is related to the graph (X, R_i) as follows.

Lemma 9.10. *With the above notation, the following are equivalent for $0 \leq a, b \leq d$:*

(i) *there exists a path in Δ_{A_i} from a to b;*

(ii) *for all $(x, y) \in R_a$ there exists $z \in \Gamma_b(x)$ such that $z \in \Gamma^{(i)}(y)$;*

(iii) *for all $(x, z) \in R_b$ there exists $y \in \Gamma_a(x)$ such that $z \in \Gamma^{(i)}(y)$;*

(iv) *there exists $(x, y) \in R_a$ and there exists $z \in \Gamma_b(x)$ such that $z \in \Gamma^{(i)}(y)$.*

Proof. (i) $\Rightarrow$ (ii) Call the path $\{a_j\}_{j=0}^{\ell}$. We have $a_0 = a$ and $a_\ell = b$. There exists a path $\{y_j\}_{j=0}^{\ell}$ in (X, R_i) such that $y_0 = y$ and $y_j \in \Gamma_{a_j}(x)$ for $0 \leq j \leq \ell$. Define $z = y_\ell$. By construction $z \in \Gamma_b(x)$ and $z \in \Gamma^{(i)}(y)$.
(ii) $\Rightarrow$ (iv) Clear.
(i) $\Rightarrow$ (iii) Similar to the proof of (i) $\Rightarrow$ (ii).
(iii) $\Rightarrow$ (iv) Clear.
(iv) $\Rightarrow$ (i) There exists a path in (X, R_i) from y to z. Call the path $\{y_j\}_{j=0}^{\ell}$. We have $y_0 = y$ and $y_\ell = z$. For $0 \leq j \leq \ell$ define $a_j \in \{0, 1, \ldots, d\}$ such that $y_j \in \Gamma_{a_j}(x)$. Note that $a_0 = a$ and $a_\ell = b$. By construction the sequence $\{a_j\}_{j=0}^{\ell}$ is a path in Δ_{A_i} from a to b. □

Corollary 9.11. *We refer to the distribution diagram Δ_{A_i} from Definition 9.9. For $0 \leq a, b \leq d$ the following are equivalent:*

(i) *there exists a path from a to b;*

(ii) *there exists a path from b to a.*

Proof. By Corollary 9.6 and Lemma 9.10(i),(iv). □

Lemma 9.12. *We refer to the distribution diagram Δ_{A_i} from Definition 9.9. The following sets are equal:*

(i) *the connected component of Δ_{A_i} that contains 0;*

(ii) *the set Ω from Lemma 9.5.*

Proof. By Lemma 9.10 with $a = 0$. □

Corollary 9.13. *For $1 \leq i \leq d$ the following are equivalent:*

(i) *the graph Δ_{A_i} is connected;*

(ii) *the graph (X, R_i) is connected.*

Proof. By Lemma 9.12 and the construction. □

Lecture 12

Proposition 9.14. *The following are equivalent:*

(i) *the association scheme $\mathcal{X}$ is imprimitive;*

(ii) *there exists a subset $\{0\} \subsetneq \Omega \subsetneq \{0, 1, \ldots, d\}$ such that R_Ω is an equivalence relation.*

Proof. (i) ⇒ (ii) There exists a relation R_i $(1 \leq i \leq d)$ such that the graph (X, R_i) is not connected. Consider the corresponding set Ω from Lemma 9.10. By Lemma 9.8 the relation R_Ω is an equivalence relation. By construction $\{0\} \subsetneq \Omega \subsetneq \{0, 1, \ldots, d\}$.
(ii) ⇒ (i) There exists $i \in \Omega$ with $i \neq 0$. Each equivalence class of R_Ω is a disjoint union of connected components for (X, R_i). The relation R_Ω has more than one equivalence class, so (X, R_i) is not connected. Therefore $\mathcal{X}$ is imprimitive. □

Earlier we used the intersection numbers to define the distribution diagrams of $\mathcal{X}$. Next we use the Krein parameters to define the representation diagrams of $\mathcal{X}$.

Definition 9.15. For $1 \leq i \leq d$ we define a directed graph Δ_{E_i} with vertex set $\{0, 1, \ldots, d\}$; vertices j, k satisfy $j \to k$ whenever $q^k_{i,j} > 0$. Note that a vertex j of Δ_{E_i} has a loop $j \to j$ whenever $q^j_{i,j} > 0$. We call Δ_{E_i} the E_i*-representation diagram for* $\mathcal{X}$.

Lemma 9.16. *We refer to the representation diagram Δ_{E_i} from Definition 9.15. For $0 \leq a, b \leq d$ the following are equivalent:*

(i) *there exists a path from a to b;*

(ii) *there exists a path from b to a.*

Proof. (i) ⇒ (ii) Fix $x \in X$, and consider the dual Bose-Mesner algebra $\mathcal{M}^* = \mathcal{M}^*(x)$. Since A^*_i is nonzero and diagonalizable, there exists $t \in \mathbb{N}$ such that $\operatorname{tr}\big((A^*_i)^{t+1}\big) \neq 0$. Recall that $\{A^*_j\}_{j=0}^d$ is a basis for $\mathcal{M}^*$. Also recall that $A^*_0 = I$. For $1 \leq j \leq d$ we have

$$\operatorname{tr}(A^*_j) = \operatorname{tr} \sum_{\ell=0}^{d} Q_j(\ell) E^*_\ell = \sum_{\ell=0}^{d} Q_j(\ell) k_\ell = m_j \sum_{\ell=0}^{d} \overline{P_\ell(j)} = m_j \sum_{\ell=0}^{d} P_\ell(0) \overline{P_\ell(j)} k_\ell^{-1} = 0.$$

Write $(A^*_i)^t = \sum_{j=0}^d \alpha_j A^*_j$ and $(A^*_i)^{t+1} = \sum_{j=0}^d \beta_j A^*_j$. So $A^*_i \sum_{j=0}^d \alpha_j A^*_j = \sum_{j=0}^d \beta_j A^*_j$. We have $\beta_0 \neq 0$ since the trace of $(A^*_i)^{t+1}$ is nonzero. Observe that

$$\beta_0 = \sum_{j=0}^{d} \alpha_j q^0_{i,j} = \sum_{j=0}^{d} \alpha_j \delta_{\hat{i},j} m_j = \alpha_{\hat{i}} m_i.$$

By these comments $\alpha_{\hat{i}} \neq 0$. Without loss, we may assume that $a \to b$ in Δ_{E_i}. So $q^b_{i,a} \neq 0$. Observe that $q^a_{b,\hat{i}} \neq 0$, so $E_b A^*_{\hat{i}} E_a \neq 0$. Therefore

$$E_b (A^*_i)^t E_a = \sum_{j=0}^{d} \alpha_j E_b A^*_j E_a = \alpha_{\hat{i}} E_b A^*_{\hat{i}} E_a + \text{orthogonal terms} \neq 0.$$

Observe that

$$E_b(A_i^*)^t E_a = E_b A_i^* \left(\sum_{b_1=0}^{d} E_{b_1}\right) A_i^* \left(\sum_{b_2=0}^{d} E_{b_2}\right) A_i^* \cdots A_i^* \left(\sum_{b_{t-1}=0}^{d} E_{b_{t-1}}\right) A_i^* E_a$$
$$= \sum E_b A_i^* E_{b_1} A_i^* E_{b_2} A_i^* \cdots A_i^* E_{b_{t-1}} A_i^* E_a,$$

where the sum is over all paths $b, b_1, b_2, \ldots, b_{t-1}, a$ in Δ_{E_i}. Such a path exists because $E_b(A_i^*)^t E_a \neq 0$. We have shown that there exists a path in Δ_{E_i} from b to a.
(ii) $\Rightarrow$ (i) By symmetry. □

To motivate the next result, we make some observations. Pick an integer i $(1 \leq i \leq d)$. Recall that

$$P_i(0) = k_i, \qquad Q_i(0) = m_i,$$
$$|P_i(j)| \leq k_i, \qquad |Q_i(j)| \leq m_i \qquad (0 \leq j \leq d).$$

The intersection matrix B_i has all entries real and nonnegative. It is diagonalizable, and its characteristic polynomial has roots $\{P_i(j)\}_{j=0}^{d}$. The dual intersection matrix B_i^* has all entries real and nonnegative. It is diagonalizable, and its characteristic polynomial has roots $\{Q_i(j)\}_{j=0}^{d}$. Recall the distribution diagram Δ_{A_i} and the representation diagram Δ_{E_i}. The following result is a special case of the Frobenius theory for nonnegative matrices.

Proposition 9.17. *With the above notation,*

(i) *the number of connected components of* Δ_{A_i} *is equal to*

$$|\{j | 0 \leq j \leq d,\ P_i(j) = k_i\}|; \qquad (35)$$

(ii) *the number of connected components of* Δ_{E_i} *is equal to*

$$|\{j | 0 \leq j \leq d,\ Q_i(j) = m_i\}|.$$

Proof. (i) Let W denote the k_i-eigenspace for B_i^t. The dimension of W is equal to (35). Let m denote the number of connected components for Δ_{A_i}. We will show that $\dim W = m$. For $0 \leq r, s \leq d$ define $r \sim s$ whenever r, s are in the same connected component of Δ_{A_i}. Let $v = (v_0, v_1, \ldots, v_d)^t \in \mathbb{C}^{d+1}$. We show that the following are equivalent:

(a) $v \in W$;

(b) $v_r = v_s$ if $r \sim s$ $(0 \leq r, s \leq d)$.

(a) $\Rightarrow$ (b) Let $C \subseteq \{0, 1, \ldots, d\}$ denote a connected component of Δ_{A_i}. We show that $v_r = v_s$ for $r, s \in C$. First assume that $v_r = 0$ for $r \in C$. Then certainly $v_r = v_s$ for $r, s \in C$. Next assume that $\{v_r\}_{r \in C}$ are not all 0. Multiplying v by a nonzero scalar if necessary, we may assume that $|v_r| \leq 1$ for $r \in C$, and also $v_r = 1$ for some $r \in C$. Define $C_1 = \{r \in C | v_r = 1\}$.

We show that $C_1 = C$. Suppose $C_1 \subsetneq C$. Since C is connected, there exists $r \in C_1$ and $s \in C \backslash C_1$ such that $s \to r$. We examine the r-coordinate in $B_i^t v = k_i v$; this gives

$$k_i = \sum_{j=0}^{d} (B_i^t)_{r,j} v_j = \sum_{j=0}^{d} (B_i)_{j,r} v_j = \sum_{j=0}^{d} p^r_{i,j} v_j. \tag{36}$$

For $0 \le j \le d$ we have $p^r_{i,j} = 0$ if $j \notin C$ and $|v_j| \le 1$ if $j \in C$; therefore $p^r_{i,j}|v_j| \le p^r_{i,j}$. Consequently

$$k_i = \left| \sum_{j=0}^{d} p^r_{i,j} v_j \right| \le \sum_{j=0}^{d} p^r_{i,j} |v_j| \le \sum_{j=0}^{d} p^r_{i,j} = k_i. \tag{37}$$

Combining (36), (37) we obtain $v_j = 1$ for all $j \in C$ such that $j \to r$. This fails for $j = s$, so we have a contradiction. Therefore $C_1 = C$, so $v_r = 1$ for $r \in C$. In particular $v_r = v_s$ for $r, s \in C$.
(b) $\Rightarrow$ (a) This holds because B_i^t has constant row sum k_i.
We have shown that (a), (b) are equivalent. Consequently $\dim W = m$, and the result follows.
(ii) Similar to the proof of (i) above. □

Lecture 13

Proposition 9.18. *The following are equivalent:*

(i) *there exists $i \in \{1, 2, \ldots, d\}$ such that Δ_{A_i} is disconnected;*

(ii) *there exists $j \in \{1, 2, \ldots, d\}$ such that Δ_{E_j} is disconnected.*

Proof. For $1 \le i, j \le d$ we have

$$\frac{P_i(j)}{k_i} = \frac{\overline{Q_j(i)}}{m_j}.$$

Therefore, $P_i(j) = k_i$ if and only if $Q_j(i) = m_j$. The result follows from this and Proposition 9.17. □

Proposition 9.19. *The following are equivalent:*

(i) *the association scheme $\mathcal{X}$ is primitive;*

(ii) *the distribution diagram Δ_{A_i} is connected for $1 \le i \le d$;*

(iii) *the representation diagram Δ_{E_j} is connected for $1 \le j \le d$.*

Proof. By Definition 9.4, Corollary 9.13, and Proposition 9.18. □

Problem 9.20. For a finite group G, show that the following are equivalent:

(i) the conjugacy–class association scheme for G is primitive;

(ii) G is simple.

10 Subschemes and quotient schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

Throughout this section, we assume that $\mathcal{X}$ is imprimitive. By Proposition 9.14 there exists a subset $\{0\} \subsetneq \Omega \subsetneq \{0, 1, \ldots, d\}$ such that R_Ω is an equivalence relation. Recall that

$$R_\Omega = \cup_{k\in\Omega} R_k.$$

Write $s+1 = |\Omega|$. Permuting the relations $\{R_i\}_{i=0}^d$ if necessary, we may assume without loss of generality that

$$\Omega = \{0, 1, \ldots, s\}.$$

Lemma 10.1. *The following hold:*

(i) *if* $0 \le i, j \le s$ *and* $p^k_{i,j} > 0$ *then* $0 \le k \le s \qquad (0 \le i, j, k \le d)$;

(ii) *for* $0 \le i \le s$ *we have* $0 \le i' \le s$.

Proof. (i) The relation R_Ω is transitive.
(ii) The relation R_Ω is symmetric. □

Lemma 10.2. *Let* $Y \subseteq X$ *denote an equivalence class of* R_Ω. *Then:*

(i) $\mathcal{Y} = (Y, \{R_i|_{Y\times Y}\}_{i=0}^s)$ *is a commutative association scheme;*

(ii) $p^k_{i,j}(\mathcal{Y}) = p^k_{i,j}(\mathcal{X}) \qquad (0 \le i, j, k \le s)$;

(iii) $k_i(\mathcal{Y}) = k_i(\mathcal{X}) \qquad (0 \le i \le s)$;

(iv) $|Y| = \sum_{i=0}^s k_i$ *where* $k_i = k_i(\mathcal{X}) = k_i(\mathcal{Y})$.

Proof. (i) We check that $\mathcal{Y}$ satisfies the four conditions in Definition 1.1.

- The trivial relation for $\mathcal{Y}$ is $R_0|_{Y\times Y} = \{(y, y)|y \in Y\}$.
- The relations $\{R_i|_{Y\times Y}\}_{i=0}^s$ partition $Y \times Y$, because for $(x, y) \in Y \times Y$ we have $(x, y) \in R_\Omega$, so there exists $i \in \Omega = \{0, 1, \ldots, s\}$ such that $(x, y) \in R_i$. This i is unique by construction.
- For $0 \le i \le s$ we have $0 \le i' \le s$ and
$$\left(R_i|_{Y\times Y}\right)^t = R_{i'}|_{Y\times Y}.$$
- For $0 \le i, j, k \le s$ and $x, y \in Y$ with $(x, y) \in R_k$,
$$\begin{aligned} p^k_{i,j}(\mathcal{X}) &= \left|\{z \in X|(x, z) \in R_i,\ (z, y) \in R_j\}\right| \\ &= \left|\{z \in Y|(x, z) \in R_i|_{Y\times Y},\ (z, y) \in R_j|_{Y\times Y}\}\right| \\ &= p^k_{i,j}(\mathcal{Y}). \end{aligned}$$

(ii) This was obtained in the proof of (i) above.
(iii) Since $k_i = p^0_{i,i'}$.
(iv) Clear. □

Definition 10.3. The association scheme $\mathcal{Y}$ in Lemma 10.2 is called the *subscheme of* $\mathcal{X}$ *induced on* Y.

Consider the subscheme $\mathcal{Y} = (Y, \{R_i|_{Y\times Y}\}_{i=0}^s)$ from Lemma 10.2. Our next general goal is to describe how the Bose-Mesner algebra of $\mathcal{Y}$ is related to $\mathcal{M}$.

For notational convenience, define

$$k_\Omega = \sum_{i=0}^{s} k_i, \qquad A_\Omega = \sum_{i=0}^{s} A_i.$$

Note that

$$A_i \circ A_\Omega = A_i \qquad (0 \le i \le s).$$

Moreover

$$A_\Omega \circ A_\Omega = A_\Omega. \tag{38}$$

Lemma 10.4. *The matrices* $\{A_i\}_{i=0}^s$ *form a basis for a subalgebra* $\mathcal{M}_\Omega$ *of* $\mathcal{M}$ *that is closed under Hadamard multiplication, complex conjugation, and the transpose map. With respect to Hadamard multiplication,* $\mathcal{M}_\Omega$ *is an algebra with multiplicative identity* A_Ω.

Proof. By Lemma 10.1 and the construction. □

By construction, the algebra $\mathcal{M}_\Omega$ in Lemma 10.4 is commutative.

Let $X_1, X_2, \dots, X_r$ denote an ordering of the equivalence classes of R_Ω. We have $|X_i| = k_\Omega$ for $1 \le i \le r$. The sets $\{X_i\}_{i=1}^r$ partition of X, so

$$r = k_\Omega^{-1}|X|.$$

Relative to the partition $\{X_i\}_{i=1}^r$, every matrix in $\mathcal{M}_\Omega$ is block-diagonal, with each diagonal block of dimension $k_\Omega \times k_\Omega$. For example A_Ω is block diagonal, with each diagonal block a copy of J with dimension $k_\Omega \times k_\Omega$. We have $J^2 = k_\Omega J$. Therefore

$$(A_\Omega)^2 = k_\Omega A_\Omega. \tag{39}$$

Lemma 10.5. *Let* $Y \subseteq X$ *denote an equivalence class of* R_Ω*, and consider the association scheme* $\mathcal{Y} = (Y, \{R_i|_{Y\times Y}\}_{i=0}^s)$ *from Lemma 10.2.*

(i) *The associate matrices of* $\mathcal{Y}$ *are* $\{A_i|_{Y\times Y}\}_{i=0}^s$.

(ii) *The Bose-Mesner algebra of* $\mathcal{Y}$ *is* $\mathcal{M}_\Omega|_{Y\times Y}$.

(iii) *The map*

$$\mathcal{M}_\Omega \to \mathcal{M}_\Omega|_{Y\times Y}$$
$$A \mapsto A|_{Y\times Y}$$

is an algebra isomorphism with respect to matrix multiplication and Hadamard multiplication.

(iv) *The above map sends* $k_\Omega^{-1}A_\Omega$ *to the trivial primitive idempotent for* $\mathcal{Y}$.

Proof. (i) By construction.
(ii) By (i) above.
(iii) The map is a bijection because it sends the basis $\{A_i\}_{i=0}^s$ of $\mathcal{M}_\Omega$ to the basis $\{A_i|_{Y\times Y}\}_{i=0}^s$ of $\mathcal{M}_\Omega|_{Y\times Y}$. The map is an algebra homomorphism by the block-diagonal nature of the matrices in $\mathcal{M}_\Omega$.
(iv) By the comments above (39). □

The algebra $\mathcal{M}_\Omega$ is closed under the conjugate-transpose map. By Lemma 3.3 the algebra $\mathcal{M}_\Omega$ has a basis of primitive idempotents. One of these primitive idempotents is $k_\Omega^{-1}A_\Omega$, in view of Lemma 10.5(iv).

Lemma 10.6. *For the subalgebra* $\mathcal{M}_\Omega$ *the primitive idempotents have the form* $\{E_{\Lambda_i}\}_{i=0}^s$ *such that:*

(i) $\{\Lambda_i\}_{i=0}^s$ *is a partition of* $\{0, 1, \ldots, d\}$ *into nonempty sets;*

(ii) $E_{\Lambda_i} = \sum_{j\in\Lambda_i} E_j \qquad (0 \le i \le s)$;

(iii) $0 \in \Lambda_0$;

(iv) $E_{\Lambda_0} = k_\Omega^{-1}A_\Omega$.

Proof. (i), (ii) Each primitive idempotent E of $\mathcal{M}_\Omega$ is a linear combination of $\{E_j\}_{j=0}^d$. In this linear combination, each coefficient is zero or one because $E^2 = E$. The result is a routine consequence of this.
(iii) There exists i $(0 \le i \le s)$ such that $0 \in \Lambda_i$. After relabelling, we may assume that $i = 0$.
(iv) There exists i $(0 \le i \le s)$ such that $E_{\Lambda_i} = k_\Omega^{-1}A_\Omega$. We have $i = 0$ by (iii) and $A_\Omega J \neq 0$. □

Lecture 14

Recall the eigenmatrices P, Q for $\mathcal{X}$. Let P_Ω, Q_Ω denote the eigenmatrices for $\mathcal{Y}$.

Proposition 10.7. *The following* (i)–(iv) *hold.*

(i) *For* $0 \le i \le s$ *the submatrix* $P|_{\Lambda_i\times\Omega}$ *has all rows identical.*

(ii) *For* $0 \le i, j \le s$ *the* (i, j)*-entry of* P_Ω *is equal to the* (α, j)*-entry of* P*, where* $\alpha \in \Lambda_i$.

(iii) *For* $0 \leq j \leq s$ *the submatrix* $Q|_{\{s+1,\ldots,d\}\times\Lambda_j}$ *has row sum 0.*

(iv) *For* $0 \leq i,j \leq s$ *the* (i,j)*-entry of* Q_Ω *is equal to* $|Y|/|X|$ *times the* i^{th} *row sum of* $Q|_{\Omega\times\Lambda_j}$*.*

Proof. (i), (ii) For $0 \leq j \leq s$ we have $A_j = \sum_{i=0}^{d} P_j(i)E_i$. In this equation the right-hand side is a linear combination of $\{E_{\Lambda_i}\}_{i=0}^{s}$. The result follows.
(iii), (iv) For $0 \leq j \leq s$ we have

$$E_{\Lambda_j} = \sum_{\alpha\in\Lambda_j} E_\alpha = |X|^{-1}\sum_{\alpha\in\Lambda_j}\sum_{i=0}^{d} Q_\alpha(i)A_i = |X|^{-1}\sum_{i=0}^{d} A_i \sum_{\alpha\in\Lambda_j} Q_\alpha(i).$$

The matrix E_{Λ_j} is a linear combination of $\{A_i\}_{i=0}^{s}$. The result follows. □

We have been discussing the subscheme $\mathcal{Y}$ of $\mathcal{X}$ induced by an equivalence class Y of R_Ω. Next we discuss the quotient association scheme induced by R_Ω. For the quotient association scheme the vertex set consists of the equivalence classes of R_Ω.

For notational convenience, we abbreviate $\Lambda = \Lambda_0$. We have $0 \in \Lambda$. Write $t+1 = |\Lambda|$. Permuting $\{E_i\}_{i=0}^{d}$ if necessary, we may assume without loss of generality that

$$\Lambda = \{0, 1, \ldots, t\}.$$

By construction,

$$k_\Omega^{-1} A_\Omega = \sum_{j=0}^{t} E_j = E_\Lambda. \tag{40}$$

Note that

$$E_j E_\Lambda = E_j \qquad (0 \leq j \leq t).$$

Lemma 10.8. *The following hold:*

(i) *if* $0 \leq i,j \leq t$ *and* $q^k_{i,j} > 0$ *then* $0 \leq k \leq t$ $\qquad (0 \leq i,j,k \leq d)$*;*

(ii) *for* $0 \leq i \leq t$ *we have* $0 \leq \hat{i} \leq t$*.*

Proof. (i) In the equation $A_\Omega \circ A_\Omega = A_\Omega$, use (40) to write each side as a linear combination of the primitive idempotents of $\mathcal{X}$.
(ii) By (40) and since A_Ω is symmetric. □

Lemma 10.9. *The matrices* $\{E_i\}_{i=0}^{t}$ *form a basis for a subalgebra* $\mathcal{M}^\circ_\Lambda$ *of* $\mathcal{M}^\circ$ *that is closed under matrix multiplication, complex conjugation, and the transpose map. With respect to matrix multiplication,* $\mathcal{M}^\circ_\Lambda$ *is a commutative algebra with multiplicative identity* E_Λ*.*

Proof. By Lemma 10.8 and the construction. □

Lemma 10.10. *The subalgebra* $\mathcal{M}^\circ_\Lambda$ *has a basis* $\{A_{\Omega_i}\}_{i=0}^{t}$ *such that:*

(i) $\{\Omega_i\}_{i=0}^t$ *is a partition of* $\{0, 1, \ldots, d\}$ *into nonempty sets;*

(ii) $A_{\Omega_i} = \sum_{j \in \Omega_i} A_j \qquad (0 \leq i \leq t)$*;*

(iii) $A_\Omega A_{\Omega_i} = A_{\Omega_i} A_\Omega = k_\Omega A_{\Omega_i} \qquad (0 \leq i \leq t)$*;*

(iv) $\Omega_0 = \Omega$.

Proof. (i), (ii) Because $\mathcal{M}_\Lambda^\circ$ is closed under Hadamard multiplication and contains $J = |X|E_0$.
(iii) Since $E_\Lambda = k_\Omega^{-1} A_\Omega$ is the multiplicative identity for $\mathcal{M}_\Lambda^\circ$.
(iv) Permuting $\{\Omega_i\}_{i=0}^t$ if necessary, we may assume that $0 \in \Omega_0$. Since $A_\Omega \in \mathcal{M}_\Lambda^\circ$, A_Ω is a linear combination of $\{A_{\Omega_i}\}_{i=0}^t$. In this linear combination each coefficient is 0 or 1, since A_Ω has all entries 0 or 1. So Ω is a union of some of $\{\Omega_i\}_{i=0}^t$. We have $0 \in \Omega$ and $0 \in \Omega_0$, so $\Omega_0 \subseteq \Omega$. We have

$$A_\Omega A_{\Omega_0} = k_\Omega A_{\Omega_0}.$$

Since $0 \in \Omega_0$, the product $A_\Omega A_0$ will contribute to $A_\Omega A_{\Omega_0}$. But $A_\Omega A_0 = A_\Omega I = A_\Omega$, so A_Ω will contribute to $A_\Omega A_{\Omega_0}$. Therefore $\Omega_0 = \Omega$. $\square$

Recall the equivalence classes $X_1, X_2, \ldots, X_r$ of R_Ω. The subsets $\{X_i\}_{i=1}^r$ partition X.

Lemma 10.11. *For* $0 \leq i \leq t$ *consider the matrix* A_{Ω_i}.

(i) *For* $1 \leq \alpha, \beta \leq r$ *the submatrix* $A_{\Omega_i}|_{X_\alpha \times X_\beta}$ *has all entries 0 or all entries 1.*

(ii) *Define an* $r \times r$ *matrix* D_i *with* (α, β)*-entry*

$$D_i(\alpha, \beta) = \begin{cases} 1 & \text{if } A_{\Omega_i}|_{X_\alpha \times X_\beta} \text{ has all entries 1;} \\ 0 & \text{if } A_{\Omega_i}|_{X_\alpha \times X_\beta} \text{ has all entries 0} \end{cases} \qquad (1 \leq \alpha, \beta \leq r).$$

Then $A_{\Omega_i} = D_i \otimes J$, *where* J *is* $k_\Omega \times k_\Omega$.

Proof. (i) The matrix A_Ω is block diagonal, with each diagonal block a copy of J. Every entry of A_{Ω_i} is 0 or 1. We have $A_\Omega A_{\Omega_i} = A_{\Omega_i} A_\Omega = k_\Omega A_{\Omega_i}$. The result follows by matrix multiplication.
(ii) This is a reformulation of (i) above. $\square$

We have a comment.

Lemma 10.12. *The following hold:*

(i) $D_0 = I$*;*

(ii) *the matrix* $\sum_{i=0}^t D_i$ *has all entries 1.*

Proof. (i) The matrix A_Ω is block diagonal, with each diagonal block a copy of J. Therefore $A_\Omega = I \otimes J$, so $D_0 = I$.
(ii) The matrix $\sum_{i=0}^d A_i$ has all entries 1, and

$$\sum_{i=0}^d A_i = \sum_{i=0}^t A_{\Omega_i} = \sum_{i=0}^t D_i \otimes J.$$

$\square$

Define the set

$$\tilde{X} = \{X_1, X_2, \ldots, X_r\}.$$

We view $D_i \in M_{\tilde{X}}(\mathbb{C})$ for $0 \leq i \leq t$. We are going to show that $\{D_i\}_{i=0}^t$ are the associate matrices for a commutative association scheme with vertex set $\tilde{X}$. This association scheme is the quotient scheme that was mentioned earlier.

Lemma 10.13. *For $A \in \mathcal{M}_\Lambda^\circ$ there exists a unique $\tilde{A} \in M_{\tilde{X}}(\mathbb{C})$ such that $A = \tilde{A} \otimes J$.*

Proof. By Lemma 10.11 and since $\{A_{\Omega_i}\}_{i=0}^t$ is a basis for $\mathcal{M}_\Lambda^\circ$. □

Lemma 10.14. *The following* (i)–(vii) *hold for $A, B \in \mathcal{M}_\Lambda^\circ$ and $\alpha \in \mathbb{C}$:*

(i) $\widetilde{A+B} = \tilde{A} + \tilde{B}$;

(ii) $\widetilde{\alpha A} = \alpha \tilde{A}$;

(iii) $\widetilde{A \circ B} = \tilde{A} \circ \tilde{B}$;

(iv) $\widetilde{AB} = k_\Omega \tilde{A}\tilde{B}$;

(v) $\tilde{A} = 0$ iff $A = 0$;

(vi) $\widetilde{A^t} = \tilde{A}^t$;

(vii) $\overline{\tilde{A}} = \tilde{\overline{A}}$.

Proof. These are readily checked using Lemma 10.13. □

Definition 10.15. *Define the vector space $\tilde{\mathcal{M}}_\Lambda^\circ = \{\tilde{A} | A \in \mathcal{M}_\Lambda^\circ\}$.*

Lemma 10.16. *The vector space $\tilde{\mathcal{M}}_\Lambda^\circ$ is closed under Hadamard multiplication, matrix multiplication, complex conjugation, and the transpose map. Moreover:*

(i) *The map*

$$\begin{aligned} \mathcal{M}_\Lambda^\circ &\to \tilde{\mathcal{M}}_\Lambda^\circ \\ A &\mapsto \tilde{A} \end{aligned}$$

is an isomorphism of algebras with respect to Hadamard multiplication.

(ii) *The map*

$$\begin{aligned} \mathcal{M}_\Lambda^\circ &\to \tilde{\mathcal{M}}_\Lambda^\circ \\ A &\mapsto k_\Omega \tilde{A} \end{aligned}$$

is an isomorphism of algebras with respect to matrix multiplication.

Proof. Immediate from Lemma 10.14. □

Lecture 15

Proposition 10.17. *$\tilde{\mathcal{M}}_\Lambda^\circ$ is the Bose-Mesner algebra of a commutative association scheme with vertex set $\tilde{X}$, associate matrices $\{D_i\}_{i=0}^t$, and primitive idempotents $\{k_\Omega \tilde{E}_i\}_{i=0}^t$.*

Proof. By Proposition 2.4 and Lemmas 10.12, 10.16. □

Definition 10.18. The association scheme in Proposiiton 10.17 is called the *quotient association scheme* of $\mathcal{X}$ induced by the relation R_Ω. We denote this association scheme by $\mathcal{Q}$.

Lemma 10.19. *The association schemes $\mathcal{Q}$ and $\mathcal{X}$ are related as follows:*

(i) $q_{i,j}^k(\mathcal{Q}) = q_{i,j}^k(\mathcal{X}) \qquad (0 \le i,j,k \le t)$;

(ii) $m_i(\mathcal{Q}) = m_i(\mathcal{X}) \qquad (0 \le i \le t)$;

(iii) $|\tilde{X}| = \sum_{i=0}^t m_i$, *where* $m_i = m_i(\mathcal{Q}) = m_i(\mathcal{X})$.

Proof. (i) For $0 \le i,j \le t$ we have

$$\begin{aligned} E_i \circ E_j &= |X|^{-1} \sum_{k=0}^{d} q_{i,j}^k(\mathcal{X}) E_k \\ &= |X|^{-1} \sum_{k=0}^{t} q_{i,j}^k(\mathcal{X}) E_k. \end{aligned}$$

In the above equation we apply the map $A \mapsto \tilde{A}$ to each side; this yields

$$\tilde{E}_i \circ \tilde{E}_j = |X|^{-1} \sum_{k=0}^{t} q_{i,j}^k(\mathcal{X}) \tilde{E}_k.$$

We also have

$$\left(k_\Omega \tilde{E}_i\right) \circ \left(k_\Omega \tilde{E}_j\right) = |\tilde{X}|^{-1} \sum_{k=0}^{t} q_{i,j}^k(\mathcal{Q}) k_\Omega \tilde{E}_i.$$

Comparing the above equations using $k_\Omega |\tilde{X}| = |X|$, we get the result.
(ii) Since $m_i = q_{i,\hat{i}}^0$.
(iii) Apply Lemma 4.3(iii) to the association scheme $\mathcal{Q}$. □

Corollary 10.20. *For the relation R_Ω,*

$$|X| = \left(\sum_{i=0}^{s} k_i\right)\left(\sum_{j=0}^{t} m_j\right).$$

Proof. By Lemma 10.19(iii) and since

$$|\tilde{X}| = k_\Omega^{-1}|X|, \qquad\qquad k_\Omega = \sum_{i=0}^{s} k_i.$$

□

Recall the eigenmatrices P, Q for $\mathcal{X}$. Let $\tilde{P}$ and $\tilde{Q}$ denote the eigenmatrices for $\mathcal{Q}$.

Proposition 10.21. *The following* (i)–(iv) *hold.*

(i) *For* $0 \leq i \leq t$ *the submatrix* $Q|_{\Omega_i \times \Lambda}$ *has all rows identical.*

(ii) *For* $0 \leq i, j \leq t$ *the* (i, j)*-entry of* $\tilde{Q}$ *is equal to the* (α, j)*-entry of* Q*, where* $\alpha \in \Omega_i$.

(iii) *For* $0 \leq j \leq t$ *the submatrix* $P|_{\{t+1,\ldots,d\} \times \Omega_j}$ *has row sum 0.*

(iv) *For* $0 \leq i, j \leq t$ *the* (i, j)*-entry of* $\tilde{P}$ *is equal to* k_Ω^{-1} *times the* i^{th} *row sum of* $P|_{\Lambda \times \Omega_j}$.

Proof. Similar to the proof of Proposition 10.7. □

11 Distance-regular graphs and P-polynomial association schemes

In this section, we restrict our attention to symmetric association schemes.

Throughout this section, $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denotes a symmetic association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. To avoid trivialities, we assume that $d \geq 1$. Since the matrices in $\mathcal{M}$ are symmetric, we have

$$i' = i, \qquad\qquad \hat{i} = i \qquad\qquad (0 \leq i \leq d).$$

Consequently

$$P_i(j) \in \mathbb{R}, \qquad Q_i(j) \in \mathbb{R} \qquad\qquad (0 \leq i, j \leq d).$$

Definition 11.1. The ordering $\{R_i\}_{i=0}^d$ is called *P-polynomial* whenever the following hold for $0 \leq i, j, k \leq d$:

(i) $p^k_{i,j} = 0$ if one of i, j, k is greater than the sum of the other two;

(ii) $p^k_{i,j} \neq 0$ if one of i, j, k is equal to the sum of the other two.

We say that $\mathcal{X}$ is *P-polynomial* whenever there exists a P-polynomial ordering of the relations.

Lemma 11.2. *Assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Then the first intersection matrix has the form*

$$B_1 = \begin{pmatrix} a_0 & c_1 & & & & \mathbf{0} \\ b_0 & a_1 & c_2 & & & \\ & b_1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & c_d \\ \mathbf{0} & & & & b_{d-1} & a_d \end{pmatrix},$$

where we abbreviate

$$c_i = p^i_{1,i-1}(1 \le i \le d), \qquad a_i = p^i_{1,i}(0 \le i \le d), \qquad b_i = p^i_{1,i+1}(1 \le i \le d-1).$$

Moreover $\{c_i\}_{i=1}^d$ and $\{b_i\}_{i=0}^{d-1}$ are nonzero.

Proof. By the definition of the intersection matrices. □

Until further notice, assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. We abbreviate $A = A_1$. Note that $a_0 = 0$ and $c_1 = 1$.

Lemma 11.3. *We have*

$$\begin{aligned} AA_i &= b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} && (1 \le i \le d-1), \\ AA_d &= b_{d-1}A_{d-1} + a_dA_d. \end{aligned}$$

Proof. This is $A_iA_j = \sum_{k=0}^d p^k_{i,j} A_k$ with $j = 1$. □

Let λ denote an indeterminate. Let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra of polynomials in λ that have all coefficients in $\mathbb{R}$.

Definition 11.4. We define some polynomials $\{v_i\}_{i=0}^{d+1}$ in $\mathbb{R}[\lambda]$ such that

$$\begin{aligned} v_0 = 1, &\qquad v_1 = \lambda, \\ \lambda v_i = b_{i-1}v_{i-1} + a_iv_i + c_{i+1}v_{i+1} & \qquad\qquad (1 \le i \le d), \end{aligned}$$

where $c_{d+1} = 1$.

Lemma 11.5. *The following* (i)–(iv) *hold:*

(i) $\deg v_i = i \quad (0 \le i \le d+1)$;

(ii) *the coefficient of λ^i in v_i is $(c_1c_2\cdots c_i)^{-1} \quad (0 \le i \le d+1)$;*

(iii) $v_i(A) = A_i \quad (0 \le i \le d)$;

(iv) $v_{d+1}(A) = 0$.

Proof. (i), (ii) By Definition 11.4.
(iii), (iv) Compare Lemma 11.3 and Definition 11.4. □

Corollary 11.6. *The following hold:*

(i) *the algebra $\mathcal{M}$ is generated by A;*

(ii) *the minimal polynomial of A is $c_1 c_2 \cdots c_d v_{d+1}$.*

Proof. By Lemma 11.5 and since $\{A_i\}_{i=0}^d$ is a basis for $\mathcal{M}$. □

Recall that

$$A_i = \sum_{j=0}^{d} P_i(j) E_j \qquad (0 \leq i \leq d).$$

Define

$$\theta_j = P_1(j) \qquad (0 \leq j \leq d).$$

Note that

$$A = \sum_{j=0}^{d} \theta_j E_j. \tag{41}$$

Lemma 11.7. *The following* (i)–(iii) *hold:*

(i) *the scalars $\{\theta_j\}_{j=0}^d$ are mutually distinct, and these are the roots of the polynomial v_{d+1};*

(ii) *the eigenspaces of A are $\{E_j V\}_{j=0}^d$;*

(iii) *for $0 \leq j \leq d$, θ_j is the eigenvalue of A for $E_j V$.*

Proof. (i) The roots of v_{d+1} are mutually distinct by Corollary 11.6(ii) and since A is diagonalizable. These roots are $\{\theta_j\}_{i=0}^d$ by (41) .
(ii), (iii) By (41). □

Lemma 11.8. *We have*

$$P_i(j) = v_i(\theta_j) \qquad (0 \leq i, j \leq d).$$

Proof. We have

$$\sum_{j=0}^{d} P_i(j) E_j = A_i = v_i(A) = \sum_{j=0}^{d} v_i(\theta_j) E_j.$$

□

We have been describing some features of the P-polynomial association scheme $\mathcal{X}$. Next, we use these features to characterize the P-polynomial property. Going forward, we no longer assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial.

We make some definitions. A matrix $B \in M_{d+1}(\mathbb{R})$ is called *tridiagonal* whenever

$$B_{i,j} = 0 \quad \text{if} \quad |i-j| > 1 \qquad (0 \leq i, j \leq d).$$

Assume that B is tridiagonal. Then B is called *irreducible* whenever

$$B_{i,i-1} \neq 0, \qquad B_{i-1,i} \neq 0 \qquad (1 \leq i \leq d).$$

Theorem 11.9. *For the symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ the following are equivalent:*

(i) *the ordering $\{R_i\}_{i=0}^d$ is P-polynomial;*

(ii) *the first intersection matrix B_1 is irreducible tridiagonal;*

(iii) *there exist polynomials $\{v_i\}_{i=0}^d$ in $\mathbb{R}[\lambda]$ such that v_i has degree i and $A_i = v_i(A)$ for $0 \le i \le d$, where $A = A_1$;*

(iv) *there exist polynomials $\{v_i\}_{i=0}^d$ in $\mathbb{R}[\lambda]$ such that v_i has degree i $(0 \le i \le d)$ and*

$$P_i(j) = v_i(\theta_j) \qquad (0 \le i, j \le d),$$

where $\theta_j = P_1(j)$ for $0 \le j \le d$.

Proof. (i) ⇒ (ii) By Lemma 11.2.
(ii) ⇒ (iii) In the proof of Lemma 11.5, we only used the fact that B_1 is irreducible tridiagonal.
(iii) ⇔ (iv) Use

$$A_i = \sum_{j=0}^{d} P_i(j) E_j \qquad (0 \le i \le d)$$

and $A = \sum_{j=0}^{d} \theta_j E_j$.
(iii) ⇒ (i) Recall that

$$k_\ell p_{i,j}^\ell = k_i p_{j,\ell}^i = k_j p_{\ell,i}^j \qquad (0 \le i, j, \ell \le d).$$

It suffices to show that for $0 \le i, j \le d$ with $i + j \le d$,

$$i + j = \max\{\ell | 0 \le \ell \le d,\ p_{i,j}^\ell > 0\}. \tag{42}$$

In the equation

$$A_i A_j = \sum_{\ell=0}^{d} p_{i,j}^\ell A_\ell,$$

we view each side as a polynomial in A. By comparing the degrees we routinely obtain (42). □

Lecture 16

Next, we explain how P-polynomial association schemes are related to distance-regular graphs.

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and adjacency relation $\mathcal{R}$. Vertices x, y are *adjacent* whenever $(x, y) \in \mathcal{R}$.

To avoid trivialities, we assume that $|X| \geq 2$. Let ∂ denote the path-length distance function for Γ, and define $d = \max\{\partial(x,y) | x, y \in X\}$. We call d the *diameter* of Γ. For $x \in X$ and an integer $i \geq 0$ define $\Gamma_i(x) = \{y \in X | \partial(x,y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$ we say that Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say that Γ is *distance-regular* whenever for $0 \leq i \leq d$ and $x, y \in X$ with $\partial(x,y) = i$, the constants

$$a_i = |\Gamma_i(x) \cap \Gamma(y)|, \qquad b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \qquad c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$$

depend only on i and not on the choice of x, y. Assume that Γ is distance-regular. By construction $a_0 = 0$, $b_d = 0$, $c_0 = 0$, $c_1 = 1$. Moreover

$$c_i > 0 \quad (1 \leq i \leq d), \qquad b_i > 0 \quad (0 \leq i \leq d-1).$$

The graph Γ is regular with valency $k = b_0$. Moreover,

$$c_i + a_i + b_i = k \qquad (0 \leq i \leq d).$$

Theorem 11.10. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter d. For $0 \leq i \leq d$ define*

$$R_i = \{(x,y) | x, y \in X,\ \partial(x,y) = i\}.$$

Then $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial.

Proof. For $0 \leq i \leq d$ define a matrix $A_i \in M_X(\mathbb{C})$ with (y,z)-entry

$$(A_i)_{y,z} = \begin{cases} 1, & \text{if } \partial(y,z) = i; \\ 0, & \text{if } \partial(y,z) \neq i \end{cases} \qquad (y, z \in X).$$

The matrix A_i is symmetric. Note that $A_0 = I$. Abbreviate $A = A_1$. The distance-regularity of Γ implies that

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1} \qquad (1 \leq i \leq d-1), \tag{43}$$

$$AA_d = b_{d-1}A_{d-1} + a_d A_d. \tag{44}$$

Let $\mathcal{M}$ denote the subalgebra of $M_X(\mathbb{C})$ generated by A. The algebra $\mathcal{M}$ is commutative. By (43), (44) the matrices $\{A_i\}_{i=0}^d$ form a basis for $\mathcal{M}$. Consequently $\mathcal{M}$ is closed under Hadamard multiplication. The algebra $\mathcal{M}$ contains $J = \sum_{i=0}^d A_i$. By these comments and Proposition 2.4, we see that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme. The ordering $\{R_i\}_{i=0}^d$ is P-polynomial because the distance function ∂ satisfies the triangle inequality. $\square$

Theorem 11.11. *Let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denote a symmetric association scheme such that $\{R_i\}_{i=0}^d$ is P-polynomial. Then the graph (X, R_1) is distance-regular with diameter d. Moreover*

$$R_i = \{(x,y) | x, y \in X,\ \partial(x,y) = i\} \qquad (0 \leq i \leq d).$$

Proof. Routine consequence of Lemma 11.3. □

Problem 11.12. Assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is P-polynomial with respect to the ordering $\{R_i\}_{i=0}^d$. Show that

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad (0 \le i \le d).$$

Problem 11.13. Show that the Hamming scheme $H(d, q)$ is P-polynomial, with valency $k = (q-1)d$ and intersection numbers

$$c_i = i, \qquad b_i = (q-1)(d-i), \qquad a_i = (q-2)i$$

for $0 \le i \le d$.

Problem 11.14. Show that the Johnson scheme $J(v, d)$ is P-polynomial, with valency $k = d(v-d)$ and intersection numbers

$$c_i = i^2, \qquad b_i = (d-i)(v-d-i), \qquad a_i = i(v-2i)$$

for $0 \le i \le d$.

12 Q-polynomial association schemes

In this section, we continue to discuss a symmetric association $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. We assume that $d \ge 1$.

Definition 12.1. The ordering $\{E_i\}_{i=0}^d$ is called *Q-polynomial* whenever the following hold for $0 \le i, j, k \le d$:

(i) $q_{i,j}^k = 0$ if one of i, j, k is greater than the sum of the other two;

(ii) $q_{i,j}^k \ne 0$ if one of i, j, k is equal to the sum of the other two.

We say that $\mathcal{X}$ is *Q-polynomial* whenever there exists a Q-polynomial ordering of the primitive idempotents.

Lemma 12.2. *Assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Then the first dual intersection matrix has the form*

$$B_1^* = \begin{pmatrix} a_0^* & c_1^* & & & & \mathbf{0} \\ b_0^* & a_1^* & c_2^* & & & \\ & b_1^* & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & c_d^* \\ \mathbf{0} & & & & b_{d-1}^* & a_d^* \end{pmatrix},$$

where we abbreviate

$$c_i^* = q_{1,i-1}^i (1 \le i \le d), \qquad a_i^* = q_{1,i}^i (0 \le i \le d), \qquad b_i^* = q_{1,i+1}^i (1 \le i \le d-1).$$

Moreover $\{c_i^*\}_{i=1}^d$ and $\{b_i^*\}_{i=0}^{d-1}$ are nonzero.

Proof. By the definition of the dual intersection matrices. □

Until further notice, assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. We fix $x \in X$, and consider the subconstituent algebra $T = T(x)$. We abbreviate $A^* = A_1^*$. Note that $a_0^* = 0$ and $c_1^* = 1$.

Lemma 12.3. *We have*

$$A^* A_i^* = b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^* \qquad (1 \le i \le d-1),$$
$$A^* A_d^* = b_{d-1}^* A_{d-1}^* + a_d^* A_d^*.$$

Proof. This is $A_i^* A_j^* = \sum_{k=0}^d q_{i,j}^k A_k^*$ with $j = 1$. □

Definition 12.4. We define some polynomials $\{v_i^*\}_{i=0}^{d+1}$ in $\mathbb{R}[\lambda]$ such that

$$v_0^* = 1, \qquad v_1^* = \lambda,$$
$$\lambda v_i^* = b_{i-1}^* v_{i-1}^* + a_i^* v_i^* + c_{i+1}^* v_{i+1}^* \qquad (1 \le i \le d),$$

where $c_{d+1}^* = 1$.

Lemma 12.5. *The following* (i)–(iv) *hold:*

(i) $\deg v_i^* = i \quad (0 \le i \le d+1)$*;*

(ii) *the coefficient of* λ^i *in* v_i^* *is* $(c_1^* c_2^* \cdots c_i^*)^{-1} \quad (0 \le i \le d+1)$*;*

(iii) $v_i^*(A^*) = A_i^* \quad (0 \le i \le d)$*;*

(iv) $v_{d+1}^*(A^*) = 0$.

Proof. Similar to the proof of Lemma 11.5. □

Corollary 12.6. *The following hold:*

(i) *the algebra* $\mathcal{M}^*$ *is generated by* A^**;*

(ii) *the minimal polynomial of* A^* *is* $c_1^* c_2^* \cdots c_d^* v_{d+1}^*$.

Proof. Similar to the proof of Lemma 11.6. □

Recall that

$$A_i^* = \sum_{j=0}^d Q_i(j) E_j^* \qquad (0 \le i \le d).$$

Define

$$\theta_j^* = Q_1(j) \qquad (0 \le j \le d).$$

Note that

$$A^* = \sum_{j=0}^d \theta_j^* E_j^*.$$

Lemma 12.7. *The following* (i)–(iii) *hold:*

(i) *the scalars* $\{\theta^*_j\}_{j=0}^d$ *are mutually distinct, and these are the roots of the polynomial* v^*_{d+1}*;*

(ii) *the eigenspaces of* A^* *are subconstituents* $\{E^*_jV\}_{j=0}^d$*;*

(iii) *for* $0 \leq j \leq d$*,* θ^*_j *is the eigenvalue of* A^* *for* E^*_jV*.*

Proof. Similar to the proof of Lemma 11.7. □

Lemma 12.8. *We have*

$$Q_i(j) = v^*_i(\theta^*_j) \qquad\qquad (0 \leq i, j \leq d).$$

Proof. We have

$$\sum_{j=0}^{d} Q_i(j)E^*_j = A^*_i = v^*_i(A^*) = \sum_{j=0}^{d} v_i(\theta^*_j)E^*_j.$$

□

We have been describing some features of the Q-polynomial association scheme $\mathcal{X}$. Next, we use these features to characterize the Q-polynomial property. Going forward, we no longer assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial.

Theorem 12.9. *For the symmetric association scheme* $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ *the following are equivalent:*

(i) *the ordering* $\{E_i\}_{i=0}^d$ *is* Q*-polynomial;*

(ii) *the first dual intersection matrix* B^*_1 *is irreducible tridiagonal;*

(iii) *there exist polynomials* $\{v^*_i\}_{i=0}^d$ *in* $\mathbb{R}[\lambda]$ *such that* v^*_i *has degree* i *and* $A^*_i = v^*_i(A^*)$ *for* $0 \leq i \leq d$*, where* $A^* = A^*_1$*;*

(iv) *there exist polynomials* $\{v^*_i\}_{i=0}^d$ *in* $\mathbb{R}[\lambda]$ *such that* v^*_i *has degree* i $(0 \leq i \leq d)$ *and*

$$Q_i(j) = v^*_i(\theta^*_j) \qquad\qquad (0 \leq i, j \leq d),$$

where $\theta^*_j = Q_1(j)$ *for* $0 \leq j \leq d$*.*

Proof. Similar to the proof of Theorem 11.9. □

As we will see, both the Hamming scheme $H(d, q)$ and the Johnson scheme $J(v, d)$ are Q-polynomial.

Problem 12.10. Assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$. Show that

$$m_i = \frac{b^*_0 b^*_1 \cdots b^*_{i-1}}{c^*_1 c^*_2 \cdots c^*_i} \qquad\qquad (0 \leq i \leq d).$$

Lecture 17

13 The conjugacy class association scheme for a finite abelian group

In this section we consider the conjugacy class association scheme for a finite abelian group G. Our goal is to show that this association scheme is self-dual.

Recall that any finite abelian group is a direct sum of cyclic groups. Write

$$G = \big(\mathbb{Z}/n_1\mathbb{Z}\big) \oplus \big(\mathbb{Z}/n_2\mathbb{Z}\big) \oplus \cdots \oplus \big(\mathbb{Z}/n_r\mathbb{Z}\big).$$

The group operation is expressed additively:

$$\begin{aligned} G \times G &\to G \\ (i,j) &\mapsto i+j \end{aligned}$$

For $1 \le i \le r$ let $\omega_i \in \mathbb{C}$ denote a primitive n_i^{th} root of unity. Thus $\omega_i^{n_i} = 1$, and $\omega_i^j \ne 1$ for $1 \le j \le n_i - 1$. Note that $\overline{\omega_i} = \omega_i^{-1}$ for $1 \le i \le r$.

We denote the group association scheme by $\mathcal{X} = (G, \{R_i\}_{i \in G})$. The associate matrices of $\mathcal{X}$ satisfy

$$\begin{aligned} A_i A_j &= A_{i+j} \qquad && (i,j \in G), \\ A_i^t &= A_{-i} \qquad && (i \in G). \end{aligned}$$

Recall the eigenmatrices P, Q of $\mathcal{X}$.

Lemma 13.1. *The matrices P, Q are described as follows after permuting the primitive idempotents as necessary. For $i = (i_1, i_2, \ldots, i_r) \in G$ and $j = (j_1, j_2, \ldots, j_r) \in G$ we have*

$$P_i(j) = \omega_1^{i_1 j_1} \omega_2^{i_2 j_2} \cdots \omega_r^{i_r j_r}, \qquad\qquad Q_i(j) = \omega_1^{-i_1 j_1} \omega_2^{-i_2 j_2} \cdots \omega_r^{-i_r j_r}.$$

Proof. Define

$$p_i(j) = \omega_1^{i_1 j_1} \omega_2^{i_2 j_2} \cdots \omega_r^{i_r j_r}, \qquad\qquad q_i(j) = \omega_1^{-i_1 j_1} \omega_2^{-i_2 j_2} \cdots \omega_r^{-i_r j_r}.$$

Note that $p_i(j) q_i(j) = 1$ and $\overline{p_i(j)} = q_i(j)$. For $a, b, i, j \in G$ we have

$$\begin{aligned}
&p_i(0) = 1, \qquad p_0(j) = 1, \qquad p_i(a)p_i(b) = p_i(a+b), \qquad p_a(j)p_b(j) = p_{a+b}(j), \\
&p_i(j) = p_j(i), \qquad p_{-i}(j) = q_i(j) = p_i(-j), \\
&q_i(0) = 1, \qquad q_0(j) = 1, \qquad q_i(a)q_i(b) = q_i(a+b), \qquad q_a(j)q_b(j) = q_{a+b}(j), \\
&q_i(j) = q_j(i), \qquad q_{-i}(j) = p_i(j) = q_i(-j).
\end{aligned}$$

It suffices to show that the matrices

$$E_i = |G|^{-1} \sum_{j \in G} q_i(j) A_j \qquad\qquad (i \in G)$$

are the primitive idempotents of $\mathcal{X}$, and that $A_r E_i = p_r(i) E_i$ for $i, r \in G$. For $i, r \in G$ we have

$$\begin{aligned} A_r E_i &= |G|^{-1} \sum_{j \in G} q_i(j) A_r A_j \\ &= |G|^{-1} \sum_{j \in G} q_i(j) A_{r+j} \\ &= |G|^{-1} \sum_{j \in G} q_i(j-r) A_j \\ &= |G|^{-1} \sum_{j \in G} q_i(j) q_i(-r) A_j \\ &= |G|^{-1} \sum_{j \in G} q_i(j) p_i(r) A_j \\ &= p_i(r) E_i \\ &= p_r(i) E_i. \end{aligned}$$

So far, we have shown that E_i is a scalar multiple of a primitive idempotent of $\mathcal{X}$. The scalar is equal to 1, because $\text{tr}(E_i) = 1$ and every primitive idempotent has trace 1. The result follows. □

Proposition 13.2. *We have $P = \overline{Q}$.*

Proof. By Lemma 13.1 we have $P_i(j) = \overline{Q_i(j)}$ for $i, j \in G$. □

Corollary 13.3. *The conjugacy class association scheme $\mathcal{X}$ is self-dual.*

Proof. By Proposition 7.6 and Proposition 13.2. □

14 The Hamming association scheme $H(d, q)$

In this section we consider the Hamming association scheme $H(d, q)$. We saw earlier that $H(d, q)$ is P-polynomial. Our next goal is to show that $H(d, q)$ is Q-polynomial. We will do this by showing that $H(d, q)$ is self-dual. Recall that $H(d, q)$ has valency $k = (q-1)d$ and intersection numbers

$$c_i = i, \qquad\qquad b_i = (q-1)(d-i), \qquad\qquad a_i = (q-2)i$$

for $0 \le i \le d$. We have

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} = (q-1)^i \binom{d}{i} \qquad\qquad (0 \le i \le d).$$

Recall the abbreviation

$$\theta_j = P_1(j) \qquad\qquad (0 \le j \le d).$$

For notational convenience, we order the primitive idempotents such that

$$\theta_0 > \theta_1 > \cdots > \theta_d.$$

Lemma 14.1. *For $0 \le j \le d$ we have*

$$\theta_j = (q-1)(d-j) - j, \qquad\qquad m_j = k_j.$$

Proof. The vertex set X of $H(d,q)$ is given by

$$X = F \times F \times \cdots \times F \qquad\qquad (d \text{ copies}),$$

where $|F| = q$. View F as the vertex set of the complete graph K_q. Let $\mathcal{A}$ denote the adjacency matrix of K_q. The matrix $\mathcal{A}$ has eigenvalues $q-1$ (with multiplicity 1) and -1 (with multiplicity $q-1$). Let $W = \mathbb{C}^F$ denote the standard module for K_q. Let W_0 (resp. W_1) denote the eigenspace of $\mathcal{A}$ with eigenvalue $q-1$ (resp. -1). The dimension of W_0 (resp. W_1) is 1 (resp. $q-1$). The sum $W = W_0 + W_1$ is direct. Recall the standard module V of $H(d,q)$. We view

$$V = W \otimes W \otimes \cdots \otimes W \qquad\qquad (d \text{ factors}).$$

From this point of view, the adjacency matrix $A = A_1$ of $H(d,q)$ satisfies

$$A = \sum_{i=1}^{d} I \otimes \cdots \otimes I \otimes \mathcal{A} \otimes I \otimes \cdots \otimes I,$$

where $\mathcal{A}$ is the i^{th} factor. We have

$$\begin{aligned} V &= (W_0 + W_1) \otimes (W_0 + W_1) \otimes \cdots \otimes (W_0 + W_1) \\ &= \sum U_1 \otimes U_2 \otimes \cdots \otimes U_d, \end{aligned}$$

where the sum is over all sequences $U_1, U_2, \ldots, U_d$ such that U_i is one of W_0, W_1 for $1 \le i \le d$. On each summand $U_1 \otimes U_2 \otimes \cdots \otimes U_d$ the matrix A acts as $(q-1)(d-j) - j$ times the identity, where

$$j = \big|\{i | 1 \le i \le d,\ U_i = W_1\}\big|.$$

Consequently

$$\theta_j = (q-1)(d-j) - j \qquad\qquad (0 \le j \le d).$$

Moreover, for $0 \le j \le d$ we have

$$E_j V = \sum U_1 \otimes U_2 \otimes \cdots \otimes U_d,$$

where the sum is over all sequences $U_1, U_2, \ldots, U_d$ involving j copies of W_1 and $d-j$ copies of W_0. This sum has $\binom{d}{j}$ summands, and each summand has dimension $(q-1)^j$. Therefore

$$m_j = \dim E_j V = (q-1)^j \binom{d}{j} = k_j.$$

□

Lemma 14.2. *For $0 \leq i, j \leq d$ we have*

$$\theta_j P_i(j) = b_{i-1} P_{i-1}(j) + a_i P_i(j) + c_{i+1} P_{i+1}(j),$$

where $P_{-1}(j) = 0$ and $P_{d+1}(j) = 0$.

Proof. We have

$$A A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1},$$

where $A_{-1} = 0$ and $A_{d+1} = 0$. In the above equation multiply each side by E_j, and evaluate the result. □

Recall the polynomial algebra $\mathbb{R}[\lambda]$. For $0 \leq i \leq d$ define the polynomial $K_i \in \mathbb{R}[\lambda]$ by

$$K_i = \sum_{\ell=0}^{i} (-1)^\ell (q-1)^{i-\ell} \binom{d-\lambda}{i-\ell} \binom{\lambda}{\ell}. \tag{45}$$

Note that K_i has degree i. We call K_i the i^{th} *Krawtchouk polynomial* with parameters d, q. For example

$$K_0 = 1, \qquad\qquad K_1 = (q-1)(d-\lambda) - \lambda.$$

We are going to show that $P_i(j) = K_i(j)$ for $0 \leq i, j \leq d$. It is convenient to use generating functions.

Lemma 14.3. Let z denote an indeterminate. Then

$$\sum_{i=0}^{d} K_i(j) z^i = \big(1-z\big)^j \big(1 + (q-1)z\big)^{d-j} \qquad\qquad (0 \leq j \leq d).$$

Proof. Consider the right-hand side of the above equation. For $0 \leq i \leq d$ compute the coefficient of z^i using the binomial theorem. Evaluate this coefficient using (45). □

Lemma 14.4. *For $0 \leq i, j \leq d$ we have*

$$\theta_j K_i(j) = b_{i-1} K_{i-1}(j) + a_i K_i(j) + c_{i+1} K_{i+1}(j),$$

where $K_{-1} = 0$ and $K_{d+1} = 0$.

Proof. We use the generating function in Lemma 14.3. For $0 \leq j \leq d$ define

$$G_j(z) = \big(1-z\big)^j \big(1+(q-1)z\big)^{d-j}.$$

Let $D = d/dz$ denote the derivative with respect to z. By elementary calculus,

$$\theta_j G_j(z) = (q-1)\big(dz - z^2 D\big)G_j(z) + (q-2)zDG_j(z) + DG_j(z).$$

One routinely checks that

$$\begin{aligned}
\theta_j G_j(z) &= \sum_{i=0}^{d} \theta_j K_i(j) z^i, \\
(q-1)\big(dz - z^2 D\big)G_j(z) &= \sum_{i=0}^{d} b_{i-1} K_{i-1}(j) z^i, \\
(q-2)zDG_j(z) &= \sum_{i=0}^{d} a_i K_i(j) z^i, \\
DG_j(z) &= \sum_{i=0}^{d} c_{i+1} K_{i+1}(j) z^i.
\end{aligned}$$

The result follows. □

Proposition 14.5. *We have*

$$P_i(j) = K_i(j) \qquad\qquad (0 \leq i, j \leq d).$$

Proof. View j as fixed, and consider the sequences $\{P_i(j)\}_{i=0}^d$, $\{K_i(j)\}_{i=0}^d$. These sequences satisfy the same 3-term recurrence. They also satisfy the same initial condition $P_0(j) = 1 = K_0(j)$. The result follows. □

Lecture 18

Our next goal is to show that $Q_i(j) = K_i(j)$ for $0 \leq i, j \leq d$.

Lemma 14.6. *For $0 \leq i, j \leq d$ we have*

$$\frac{K_i(j)}{k_i} = \frac{K_j(i)}{k_j}. \tag{46}$$

Proof. Using (45) we find that each side of (46) is equal to

$$\sum_{\ell} \frac{(-1)^\ell}{(q-1)^\ell} \frac{i!(d-i)!j!(d-j)!}{(i-\ell)!(j-\ell)!\ell!(d-i-j+\ell)!},$$

where the sum is over all nonnegative integers ℓ such that $i+j-d \leq \ell \leq \min(i,j)$. □

Proposition 14.7. *We have*

$$Q_i(j) = K_i(j) \qquad\qquad (0 \le i, j \le d).$$

Proof. We have

$$\frac{Q_i(j)}{m_i} = \frac{\overline{P_j(i)}}{k_j} = \frac{K_j(i)}{k_j} = \frac{K_i(j)}{k_i} = \frac{K_i(j)}{m_i}.$$

Therefore $Q_i(j) = K_i(j)$. □

Corollary 14.8. *The Hamming scheme $H(d, q)$ is self-dual.*

Proof. We have $P = Q = \overline{Q}$. □

We mention some alternative forms for the Krawtchouk polynomials.

Lemma 14.9. For $0 \le i \le d$ we have

$$K_i = \sum_{\ell=0}^{i} (-1)^\ell q^\ell (q-1)^{i-\ell} \binom{d-\ell}{i-\ell} \binom{\lambda}{\ell}, \tag{47}$$

$$K_i = \sum_{\ell=0}^{i} (-1)^\ell q^{i-\ell} \binom{d-i+\ell}{\ell} \binom{d-\lambda}{i-\ell}. \tag{48}$$

Proof. We will use generating functions. First we consider (47). For $0 \le j \le d$ we have

$$\begin{aligned}
&\sum_{i=0}^{d} z^i \sum_{\ell=0}^{i} (-1)^\ell q^\ell (q-1)^{i-\ell} \binom{d-\ell}{i-\ell} \binom{j}{\ell} \\
&= \sum_{\ell=0}^{j} \sum_{i=\ell}^{d} z^i (-1)^\ell q^\ell (q-1)^{i-\ell} \binom{d-\ell}{i-\ell} \binom{j}{\ell} \\
&= \sum_{\ell=0}^{j} (-1)^\ell q^\ell z^\ell \binom{j}{\ell} \sum_{i=\ell}^{d} z^{i-\ell} (q-1)^{i-\ell} \binom{d-\ell}{i-\ell} \\
&= \sum_{\ell=0}^{j} (-1)^\ell q^\ell z^\ell \binom{j}{\ell} \sum_{r=0}^{d-\ell} z^r (q-1)^r \binom{d-\ell}{r} \qquad\qquad r = i-\ell \\
&= \sum_{\ell=0}^{j} (-1)^\ell q^\ell z^\ell \binom{j}{\ell} \big(1 + (q-1)z\big)^{d-\ell} \\
&= \big(1 + (q-1)z\big)^d \sum_{\ell=0}^{j} (-1)^\ell q^\ell z^\ell \binom{j}{\ell} \big(1 + (q-1)z\big)^{-\ell} \\
&= \big(1 + (q-1)z\big)^d \left(1 - \frac{qz}{1 + (q-1)z}\right)^j \\
&= \big(1 + (q-1)z\big)^d \left(\frac{1-z}{1 + (q-1)z}\right)^j \\
&= \big(1 - z\big)^j \big(1 + (q-1)z\big)^{d-j}.
\end{aligned}$$

We now consider (48). For $0 \le j \le d$ we have

$$\begin{aligned}
&\sum_{i=0}^{d} z^i \sum_{\ell=0}^{i} (-1)^\ell q^{i-\ell} \binom{d-i+\ell}{\ell} \binom{d-j}{i-\ell} \\
&\qquad = \sum_{r=0}^{d-j} z^r q^r \binom{d-j}{r} \sum_{\ell=0}^{d-r} z^\ell (-1)^\ell \binom{d-r}{\ell} \qquad\qquad r = i-\ell \\
&\qquad = \sum_{r=0}^{d-j} z^r q^r \binom{d-j}{r} (1-z)^{d-r} \\
&\qquad = (1-z)^j \sum_{r=0}^{d-j} \binom{d-j}{r} q^r z^r (1-z)^{d-j-r} \\
&\qquad = \bigl(1-z\bigr)^j \bigl(1+(q-1)z\bigr)^{d-j}.
\end{aligned}$$

□

Remark 14.10. We mention an alternative proof of Proposition 14.5. We use the notation from the proof of Lemma 14.1. For $0 \le i \le d$ we have

$$A_i = \sum F_1 \otimes F_2 \otimes \cdots \otimes F_d, \tag{49}$$

where the sum is over all the sequences $F_1, F_2, \ldots, F_d$ involving i copies of $\mathcal{A}$ and $d-i$ copies of I. The sum (49) has $\binom{d}{i}$ summands. Recall that for $0 \le j \le d$ we have

$$E_j V = \sum U_1 \otimes U_2 \otimes \cdots \otimes U_d, \tag{50}$$

where the sum is over all the sequences $U_1, U_2, \ldots, U_d$ involving j copies of W_1 and $d-j$ copies of W_0. The sum (50) has $\binom{d}{j}$ summands. Fix one of these summands:

$$U_1 \otimes U_2 \otimes \cdots \otimes U_d. \tag{51}$$

We compute the action of A_i on (51). For the moment, pick a summand $F_1 \otimes F_2 \otimes \cdots \otimes F_d$ from (49). Let ℓ denote the number of integers r $(1 \le r \le d)$ such that $F_r = \mathcal{A}$ and $U_r = W_1$. Note that $0 \le \ell \le i$. The summand $F_1 \otimes F_2 \otimes \cdots \otimes F_d$ acts on (51) as $(-1)^\ell (q-1)^{i-\ell}$ times the identity. We call ℓ the *index* of $F_1 \otimes F_2 \otimes \cdots \otimes F_d$ on (51). For $0 \le \ell \le i$ there are exactly $\binom{d-j}{i-\ell}\binom{j}{\ell}$ summands in (49) that have index ℓ on (51). By these comments, A_i acts on (51) as the following scalar multiple of the identity:

$$\sum_{\ell=0}^{i} (-1)^\ell (q-1)^{i-\ell} \binom{d-j}{i-\ell} \binom{j}{\ell}.$$

Consequently

$$P_i(j) = \sum_{\ell=0}^{i} (-1)^\ell (q-1)^{i-\ell} \binom{d-j}{i-\ell} \binom{j}{\ell} \qquad\qquad (0 \le i, j \le d).$$

Remark 14.11. The formula (48) has the following combinatorial interpretation. We use the notation from the proof of Lemma 14.1. For $0 \le i \le d$ define

$$\Phi_i = \sum H_1 \otimes H_2 \otimes \cdots \otimes H_d, \tag{52}$$

where the sum is over all the sequences $H_1, H_2, \ldots, H_d$ involving i copies of $\mathcal{A} + I$ and $d - i$ copies of I. The sum (52) has $\binom{d}{i}$ summands. By combinatorial counting, we find

$$\Phi_i = \sum_{\ell=0}^{i} \binom{d-\ell}{i-\ell} A_\ell \qquad (0 \le i \le d).$$

Solving the above equations, we obtain

$$A_i = \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{d-\ell}{i-\ell} \Phi_\ell \qquad (0 \le i \le d). \tag{53}$$

For $0 \le j, \ell \le d$ we now compute the action of Φ_ℓ on the eigenspace $E_j V$. The matrix $\mathcal{A} + I$ acts on W_0 (resp. W_1) as q (resp. 0) times the identity. Therefore, Φ_ℓ acts on $E_j V$ as $q^\ell \binom{d-j}{\ell}$ times the identity. By this and (53) we find that for $0 \le i, j \le d$,

$$P_i(j) = \sum_{\ell=0}^{i} (-1)^{i-\ell} q^\ell \binom{d-\ell}{i-\ell} \binom{d-j}{\ell}.$$

In the above equation we make a change of variables $\ell \mapsto i - \ell$; this yields

$$P_i(j) = \sum_{\ell=0}^{i} (-1)^{\ell} q^{i-\ell} \binom{d-i+\ell}{\ell} \binom{d-j}{i-\ell}.$$

By this and $P_i(j) = K_i(j)$ we get (48).

For the rest of this section, we assume that $q = 2$. We view $F = \{0, 1\}$. The Hamming graph $H(d, 2)$ is often called the *binary Hamming graph*, or the *d-cube*, or a *hypercube*. We identify the vertices of $H(d, 2)$ with the subsets of $\{1, 2, \ldots d\}$. Vertices $y, z \in X$ are adjacent whenever one contains the other, and their cardinalities differ by one. The graph $H(d, 2)$ has valency $k = d$ and intersection numbers

$$c_i = i, \qquad b_i = d - i, \qquad a_i = 0$$

for $0 \le i \le d$. Moreover

$$k_i = \binom{d}{i} \qquad (0 \le i \le d).$$

The adjacency matrix A has eigenvalues

$$\theta_i = d - 2i \qquad (0 \le i \le d).$$

Since $H(d,2)$ is self-dual, we have

$$c_i^* = i, \qquad b_i^* = d - i, \qquad a_i^* = 0$$

for $0 \le i \le d$. Moreover

$$m_i = \binom{d}{i} \qquad (0 \le i \le d)$$

and

$$\theta_i^* = d - 2i \qquad (0 \le i \le d).$$

Fix the vertex $x = \emptyset$. For $y \in X$ we have $\partial(x,y) = |y|$.

Our next goal is to describe the subconstituent algebra $T = T(x)$.

Definition 14.12. For $y, z \in X$ we say that z *covers* y whenever $y \subseteq z$ and $|y| + 1 = |z|$.

define

$$R = \sum_{i=0}^{d-1} E_{i+1}^* A E_i^*, \qquad L = \sum_{i=1}^{d} E_{i-1}^* A E_i^*.$$

For $y \in X$ we have

$$R\hat{y} = \sum_{z \text{ covers } y} \hat{z}, \qquad L\hat{y} = \sum_{y \text{ covers } z} \hat{z}.$$

Note that

$$A = R + L, \qquad R^t = L.$$

Recall the dual adjacency matrix $A^* = A_1^*$ with respect to x. We have

$$A^* = \sum_{i=0}^{d} (d - 2i) E_i^*.$$

Lemma 14.13. *The matrices R, L, A^* satisfy*

$$A^*L - LA^* = 2L, \qquad A^*R - RA^* = -2R, \qquad LR - RL = A^*.$$

Proof. To verify these equations, for $y, z \in X$ compare the (y, z)-entry on either side. □

Lecture 19

Remark 14.14. The above equations are the defining relations for the Lie algebra $\mathfrak{sl}_2$. We briefly explain the details. The Lie algebra $\mathfrak{sl}_2$ consists of the 2×2 matrices over $\mathbb{C}$ that have trace 0, together with the Lie bracket $[r, s] = rs - sr$. The vector space $\mathfrak{sl}_2$ has a basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie bracket satisfies

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H. \tag{54}$$

By these comments, the standard module V of $H(d, 2)$ becomes an $\mathfrak{sl}_2$-module on which E, F, H act as L, R, A^* respectively.

Recall that the standard module V is an orthogonal direct sum of irreducible T-modules.

Definition 14.15. Let W denote an irreducible T-module. Define

$$r = \min\{i | 0 \le i \le d,\ E_i^*W \ne 0\}, \qquad \delta = \big|\{i | 0 \le i \le d,\ E_i^*W \ne 0\}\big| - 1.$$

We call r (resp. δ) the *endpoint* (resp. *diameter*) of W.

Proposition 14.16. *Let W denote an irreducible T-module. The endpoint r of W satisfies*

$$0 \le r \le d/2.$$

The diameter δ of W satisfies

$$\delta = d - 2r.$$

There exists a basis $\{w_i\}_{i=0}^{\delta}$ of W such that

$$\begin{aligned} w_i &\in E_{r+i}^*W && (0 \le i \le \delta), \\ Rw_i &= (i+1)w_{i+1} && (0 \le i \le \delta), \qquad Rw_\delta = 0, \\ Lw_i &= (\delta - i + 1)w_{i-1} && (1 \le i \le \delta), \qquad Lw_0 = 0. \end{aligned}$$

Proof. Pick $0 \neq w_0 \in E^*_r W$. For $0 \leq i \leq d-r$ define

$$w_i = \frac{R^i w_0}{i!}.$$

We have $w_i \in E^*_{r+i}W$, so

$$A^* w_i = (d - 2r - 2i) w_i.$$

By construction,

$$R w_i = (i+1) w_{i+1} \qquad (0 \leq i \leq d-r),$$

where $w_{d-r+1} = 0$. By the definition of w_0, we have $Lw_0 = 0$. By this and $LR - RL = A^*$ we find by induction on i that

$$L w_{i+1} = (d - 2r - i) w_i \qquad (0 \leq i \leq d-r).$$

There exists a unique integer s $(0 \leq s \leq d-r)$ such that $w_0, w_1, \ldots, w_s$ are nonzero and $w_{s+1} = 0$. We claim that $s = d - 2r$. To see this, note that

$$L w_{s+1} = (d - 2r - s) w_s.$$

In the above equation, we have $Lw_{s+1} = L0 = 0$ and $w_s \neq 0$, so $s = d - 2r$. The claim is proved. One readily checks that the vectors $\{w_i\}_{i=0}^{d-2r}$ form a basis for W, and the result follows. □

Definition 14.17. Decompose the standard module V into an orthogonal direct sum of irreducible T-modules. For an integer r $(0 \leq r \leq d/2)$ let $\text{mult}(r)$ denote the number of irreducible T-modules in this decomposition that have endpoint r.

Proposition 14.18. *With the above notation,*

$$\begin{aligned} &\text{mult}(0) = 1, \\ &\text{mult}(r) = \binom{d}{r} - \binom{d}{r-1} \qquad (1 \leq r \leq d/2). \end{aligned}$$

Proof. Using Proposition 14.16 we find that for $0 \leq i \leq d/2$,

$$\sum_{j=0}^{i} \text{mult}(j) = \dim E^*_i V = k_i = \binom{d}{i}.$$

The result follows. □

Recall that

$$A^*L - LA^* = 2L, \qquad A^*R - RA^* = -2R, \qquad LR - RL = A^*. \tag{55}$$

Next we express these relations in terms of A, A^*.

Proposition 14.19. *For the graph $H(d, 2)$ the adjacency matrix A and dual adjacency matrix A^* satisfy*

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^*,$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4A.$$

Proof. Recall $A = R + L$. Adding the first two equations in (55), we obtain

$$AA^* - A^*A = 2(R - L).$$

Combining this with $A = R + L$, we obtain

$$R = \frac{AA^* - A^*A + 2A}{4}, \qquad L = \frac{A^*A - AA^* + 2A}{4}.$$

Use these equations to eliminate R, L in the first two relations from (55). The result follows. □

For more information about $H(d, 2)$ see

Junie Go. The Terwilliger algebra of the hypercube. Europ. J. Combin. (2002) 399–429.

15 The Johnson scheme $J(v, d)$

In this section we consider the Johnson association scheme $J(v, d)$. We mentioned earlier that $J(v, d)$ is P-polynomial; we view $J(v, d)$ as a distance-regular graph. This graph has valency $k = d(v - d)$ and intersection numbers

$$c_i = i^2, \qquad b_i = (d - i)(v - d - i), \qquad a_i = i(v - 2i)$$

for $0 \leq i \leq d$. We will show that $J(v, d)$ is Q-polynomial. As we will see, $J(v, d)$ is not self-dual. Note that

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} = \binom{d}{i}\binom{v-d}{i} \qquad (0 \leq i \leq d).$$

The vertex set X of $J(v, d)$ consists of the d-subsets of $\{1, 2, \ldots, v\}$. Consequently, we may identify X with the d^{th} subconstituent of $H(v, 2)$ with respect to the vertex $\emptyset$. We will use bold face notation for $H(v, 2)$. Thus

$$V = \mathbf{E}_d^* \mathbf{V}.$$

Lemma 15.1. *On V,*

$$\mathbf{RL} - d\mathbf{I} = \mathbf{LR} - (v - d)\mathbf{I}. \qquad (56)$$

Proof. We have $\mathbf{LR} - \mathbf{RL} = \mathbf{A}^*$. Also $\mathbf{A}^*$ acts on V as $(v - 2d)\mathbf{I}$. The result follows. □

Lemma 15.2. *The following are the same:*

(i) *the adjacency matrix* A *of* $J(v,d)$*;*

(ii) *the restriction of either side of* (56) *to* $X \times X$*.*

Proof. By the definition of adjacency for the graph $J(v,d)$. □

Recall the abbreviation

$$\theta_j = P_1(j) \qquad\qquad (0 \le j \le d).$$

For notational convenience, we order the primitive idempotents such that

$$\theta_0 > \theta_1 > \cdots > \theta_d.$$

Lemma 15.3. *For* $0 \le j \le d$ *we have*

$$E_j V = \mathbf{E}^*_d \mathbf{W}_j,$$

where $\mathbf{W}_j$ *is the sum of the irreducible* $\mathbf{T}$*-modules that have endpoint* j*. We have*

$$\theta_j = (d-j)(v-d-j) - j \qquad\qquad (0 \le j \le d).$$

Moreover, $m_0 = 1$ *and*

$$m_j = \binom{v}{j} - \binom{v}{j-1} \qquad\qquad (1 \le j \le d).$$

Proof. Let $\mathbf{W}$ denote an irreducible $\mathbf{T}$-module with endpoint j. Note that $\mathbf{RL} - d\mathbf{I}$ acts on $\mathbf{E}^*_d \mathbf{W}$ as $\alpha_j \mathbf{I}$, where

$$\alpha_j = (d-j)(v-d-j+1) - d = (d-j)(v-d-j) - j.$$

The result follows. □

Adjusting the above formula for m_j, we find

$$m_j = \binom{v}{j} \frac{v-2j+1}{v-j+1} \qquad\qquad (0 \le j \le d).$$

Our next goal is to compute $P_i(j)$ for $0 \le i,j \le d$. To do this, we describe A_i in terms of $\mathbf{R}$, $\mathbf{L}$.

Lemma 15.4. *For* $0 \le i \le d$ *the following holds on* V*:*

$$\frac{\mathbf{R}^i}{i!} \frac{\mathbf{L}^i}{i!} = \sum_{\ell=0}^{i} A_\ell \binom{d-\ell}{d-i}.$$

Proof. This result asserts that for subsets y, z of $\{1, 2, \ldots, v\}$ such that

$$|y| = d, \qquad |z| = d, \qquad |y \cap z| = d - \ell,$$

the number of subsets $w \subseteq \{1, 2, \ldots, v\}$ such that

$$w \subseteq y, \qquad w \subseteq z, \qquad |w| = d - i$$

is equal to

$$\binom{d-\ell}{d-i}.$$

This assertion is routinely checked. □

Lecture 20

Lemma 15.5. *For $0 \leq i \leq d$ the following holds on V:*

$$A_i = \sum_{\ell=0}^{i} \frac{\mathbf{R}^\ell}{\ell!} \frac{\mathbf{L}^\ell}{\ell!} (-1)^{i-\ell} \binom{d-\ell}{d-i}.$$

Proof. Use linear algebra to solve the system of linear equations in Lemma 15.4. □

Proposition 15.6. *For $J(v,d)$ the entries of P are given as follows. For $0 \leq i, j \leq d$,*

$$P_i(j) = \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{d-j}{\ell} \binom{v-d-j+\ell}{\ell} \binom{d-\ell}{d-i}.$$

Proof. Let $\mathbf{W}$ denote an irreducible $\mathbf{T}$-module with endpoint j. For $0 \leq \ell \leq d$ the matrix

$$\frac{\mathbf{R}^\ell}{\ell!} \frac{\mathbf{L}^\ell}{\ell!}$$

acts on $\mathbf{E}^*_d \mathbf{W}$ as $\gamma_\ell \mathbf{I}$, where

$$\gamma_\ell = \binom{d-j}{\ell} \binom{v-d-j+\ell}{\ell}.$$

The result follows. □

For $J(v,d)$ the matrix Q satisfies

$$Q_i(j) = \frac{m_i}{k_j} P_j(i) \qquad (0 \leq i, j \leq d).$$

In order to clarify our formulas, we bring in hypergeometric series. For $a \in \mathbb{C}$ define

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1) \qquad (n \in \mathbb{N}).$$

We interpret $(a)_0 = 1$. For $n, m \in \mathbb{N}$ we have

$$(-m)_n = \begin{cases} \neq 0 & \text{if } n \leq m; \\ 0 & \text{if } n \geq m+1. \end{cases}$$

For $r, s \in \mathbb{N}$ and complex scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_r; \qquad \beta_1, \beta_2, \ldots, \beta_s$$

the corresponding *hypergeometric series* is

$${}_rF_s\left(\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_s \end{matrix} \middle| \, z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_r)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_s)_n} \frac{z^n}{n!}.$$

If at least one of $\alpha_1, \alpha_2, \ldots, \alpha_r$ is an integer at most 0, then the above sum has finitely many nonzero summands.

Proposition 15.7. *For $0 \leq i, j \leq d$ we have*

$$\frac{P_i(j)}{k_i} = \frac{Q_j(i)}{m_j} = {}_3F_2\left(\begin{matrix} -i, -j, j-v-1 \\ d-v, -d \end{matrix} \middle| \, 1\right). \tag{57}$$

Proof. Use Proposition 15.6. □

Recall the abbreviation

$$\theta^*_i = Q_1(i) \qquad (0 \leq i \leq d).$$

Corollary 15.8. *For $J(v,d)$ we have*

$$\theta^*_i = v - 1 - \frac{v(v-1)i}{d(v-d)} \qquad (0 \leq i \leq d).$$

Proof. Set $j = 1$ in Proposition 15.7. □

Note that $\{\theta^*_i\}_{i=0}^d$ are mutually distinct.

The following definition is for notational convenience.

Definition 15.9. Define

$$s^* = \frac{v(1-v)}{d(v-d)}.$$

Further define

$$\varphi_i = s^* i(i-d-1)(i+d-v-1) \qquad (1 \leq i \leq d).$$

Note that $\varphi_i \neq 0$ for $1 \leq i \leq d$.

Proposition 15.10. *For $0 \le i, j \le d$ the common value in* (57) *is equal to*

$$\sum_{n=0}^{\min(i,j)} \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{n-1}^*)(\theta_j - \theta_0)(\theta_j - \theta_1)\cdots(\theta_j - \theta_{n-1})}{\varphi_1\varphi_2\cdots\varphi_n}.$$

Proof. For $0 \le n \le d$ we have

$$\begin{aligned}(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{n-1}^*) &= (-1)^n s^{*n}(-i)_n,\\ (\theta_j - \theta_0)(\theta_j - \theta_1)\cdots(\theta_j - \theta_{n-1}) &= (-1)^n(-j)_n(j-v-1)_n,\\ \varphi_1\varphi_2\cdots\varphi_n &= s^{*n}(d-v)_n(-d)_n n!\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{n-1}^*)(\theta_j - \theta_0)(\theta_j - \theta_1)\cdots(\theta_j - \theta_{n-1})}{\varphi_1\varphi_2\cdots\varphi_n}\\ &\qquad = \frac{(-i)_n(-j)_n(j-v-1)_n}{(d-v)_n(-d)_n n!}.\end{aligned}$$

The result follows from this and Proposition 15.7. □

Definition 15.11. For $0 \le i \le d$ define the polynomials $\tau_i, \tau_i^* \in \mathbb{R}[\lambda]$ by

$$\begin{aligned}\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{i-1}),\\ \tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*)\cdots(\lambda - \theta_{i-1}^*).\end{aligned}$$

Each of τ_i, τ_i^* is monic with degree i.

Lemma 15.12. *For $0 \le i, j \le d$ we have*

$$\tau_i(\theta_j) = \begin{cases} 0 & \text{if } j \le i-1; \\ \neq 0 & \text{if } j \ge i. \end{cases} \qquad \tau_i^*(\theta_j^*) = \begin{cases} 0 & \text{if } j \le i-1; \\ \neq 0 & \text{if } j \ge i. \end{cases}$$

Proof. Since $\{\theta_j\}_{j=0}^d$ are mutually distinct and $\{\theta_j^*\}_{j=0}^d$ are mutually distinct. □

Definition 15.13. For $0 \le i \le d$ define the polynomials $v_i, v_i^* \in \mathbb{R}[\lambda]$ by

$$v_i = k_i \sum_{n=0}^{i} \frac{\tau_n^*(\theta_i^*)\tau_n}{\varphi_1\varphi_2\cdots\varphi_n}, \qquad v_i^* = m_i \sum_{n=0}^{i} \frac{\tau_n(\theta_i)\tau_n^*}{\varphi_1\varphi_2\cdots\varphi_n}.$$

Each of v_i, v_i^* has degree i. We call v_i (resp. v_i^*) the i^{th} *Eberlein* (resp. *Hahn*) polynomial.

Proposition 15.14. *For $0 \le i, j \le d$ we have*

$$P_i(j) = v_i(\theta_j), \qquad Q_i(j) = v_i^*(\theta_j^*).$$

Proof. By Proposition 15.10, Lemma 15.12, and Definition 15.13 we obtain

$$P_i(j) = k_i \sum_{n=0}^{\min(i,j)} \frac{\tau_n^*(\theta_i^*)\tau_n(\theta_j)}{\varphi_1\varphi_2\cdots\varphi_n} = k_i \sum_{n=0}^{i} \frac{\tau_n^*(\theta_i^*)\tau_n(\theta_j)}{\varphi_1\varphi_2\cdots\varphi_n} = v_i(\theta_j).$$

The proof of $Q_i(j) = v_i^*(\theta_j^*)$ is similar. □

Corollary 15.15. *The Johnson graph $J(v, d)$ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$.*

Proof. For $0 \le i \le d$ we displayed a polynomial v_i^* of degree i such that $Q_i(j) = v_i^*(\theta_j^*)$ for $0 \le j \le d$. The result follows by Theorem 12.9. $\square$

Remark 15.16. The polynomials $\{v_i\}_{i=0}^d$ from Definition 15.13 are the same as the polynomials $\{v_i\}_{i=0}^d$ from Definition 11.4. The polynomials $\{v_i^*\}_{i=0}^d$ from Definition 15.13 are the same as the polynomials $\{v_i^*\}_{i=0}^d$ from Definition 12.4.

Next we compute the Krein parameters for $J(v, d)$.

Lemma 15.17. *For $J(v, d)$ we have*

$$
\begin{aligned}
b_i^* &= \frac{v(v-1)}{d(v-d)} \frac{(d-i)(v-i+1)(v-d-i)}{(v-2i)(v-2i+1)} && (0 \le i \le d-1), \\
c_i^* &= \frac{v(v-1)}{d(v-d)} \frac{i(d-i+1)(v-d-i+1)}{(v-2i+1)(v-2i+2)} && (1 \le i \le d), \\
a_i^* &= \frac{(v-1)(v-2d)^2 i(v-i+1)}{d(v-d)(v-2i)(v-2i+2)} && (0 \le i \le d-1), \\
a_d^* &= \frac{(v-1)(v-2d)(v-d+1)}{(v-d)(v-2d+2)}.
\end{aligned}
$$

Proof. Evaluate the 3-term recurrence given in Definition 12.4 using the formulas in Definition 15.13. $\square$

Problem 15.18. Show that for $H(d, q)$ the following holds for $0 \le i, j \le d$:

$$
\frac{P_i(j)}{k_i} = \frac{Q_j(i)}{m_j} = {}_2F_1\left(\left. \begin{matrix} -i, -j \\ -d \end{matrix} \right| \frac{q}{q-1} \right).
$$

Lecture 21

16 Embeddings into spheres

Throughout this section we consider a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Recall that

$$i' = i, \qquad \hat{i} = i \qquad (0 \leq i \leq d),$$

and that

$$P_i(j) \in \mathbb{R}, \qquad Q_i(j) \in \mathbb{R} \qquad (0 \leq i, j \leq d).$$

To avoid trivialities, we assume that $d \geq 1$.

It will be convenient to work over the field $\mathbb{R}$ instead of $\mathbb{C}$. We take the standard module to be $V = \mathbb{R}^X$. We endow V with a bilinear form $\langle\, ,\, \rangle$ such that $\langle u, v \rangle = u^t v$ for all $u, v \in V$. Abbreviate $\|u\|^2 = \langle u, u \rangle$.

Throughout this section we fix a nontrivial primitive idempotent E. Without loss of generality, we may assume that $E = E_1$. We abbreviate

$$\theta_i^* = Q_1(i) \qquad (0 \leq i \leq d).$$

Note that

$$E = |X|^{-1} \sum_{i=0}^{d} \theta_i^* A_i.$$

Recall that EV is a common eigenspace for the Bose-Mesner algebra $\mathcal{M}$.

Definition 16.1. We define the map

$$\rho : \quad \begin{array}{c} X \to EV \\ y \mapsto |X|^{1/2} E\hat{y} \end{array}$$

We call ρ the *spherical representation of* $\mathcal{X}$ associated with E.

By construction,

$$EV = \mathrm{Span}\{\rho(y) | y \in X\}.$$

Lemma 16.2. *The following hold for* $0 \leq i \leq d$.

(i) *For* $y, z \in X$ *such that* $(y, z) \in R_i$,

$$\langle \rho(y), \rho(z) \rangle = \theta_i^*.$$

(ii) *For $y \in X$,*

$$\sum_{z \in \Gamma_i(y)} \rho(z) = P_i(1)\rho(y),$$

where we recall

$$\Gamma_i(y) = \{z \in X | (y,z) \in R_i\}.$$

Proof. (i) We have

$$\begin{aligned}\langle \rho(y), \rho(z) \rangle &= |X| \langle E\hat{y}, E\hat{z} \rangle \\ &= |X| \langle \hat{y}, E^t E \hat{z} \rangle \\ &= |X| \langle \hat{y}, E^2 \hat{z} \rangle \\ &= |X| \langle \hat{y}, E \hat{z} \rangle \\ &= |X| \Big((y,z)\text{-entry of } E \Big) \\ &= \theta_i^*.\end{aligned}$$

(ii) We have

$$\begin{aligned}\sum_{z \in \Gamma_i(y)} \rho(z) &= |X|^{1/2} \sum_{z \in \Gamma_i(y)} E\hat{z} \\ &= |X|^{1/2} E \sum_{z \in \Gamma_i(y)} \hat{z} \\ &= |X|^{1/2} E A_i \hat{y} \\ &= |X|^{1/2} A_i E \hat{y} \\ &= |X|^{1/2} P_i(1) E \hat{y} \\ &= P_i(1)\rho(y).\end{aligned}$$

□

Note that $\theta_0^* = Q_1(0) = m_1$. By Lemma 16.2(i),

$$\|\rho(y)\|^2 = \theta_0^* \qquad (y \in X).$$

Lemma 16.3. *For $y, z \in X$ the angle between $\rho(y), \rho(z)$ has cosine θ_i^*/θ_0^*, where $(y,z) \in R_i$.*

Proof. By Lemma 16.2(i) and the comment above the lemma statement. □

Definition 16.4. The spherical representation ρ is said to be *nondegenerate* whenever $\{\theta_i^*\}_{i=0}^d$ are mutually distinct.

Recall the Q-polynomial property from Definition 12.1.

Definition 16.5. We say that $\mathcal{X}$ is *Q-polynomial with respect to E* whenever the there exists a Q-polynomial ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents such that $E = E_1$.

Lemma 16.6. *Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then ρ is nondegenerate.*

Proof. We saw earlier that $\{\theta^*_i\}_{i=0}^d$ are mutually distinct. □

Definition 16.7. The spherical representation ρ is said to be *weakly nondegenerate* whenever $\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$.

Let us clarify the meaning of weakly nondegenerate.

Lemma 16.8. *The spherical representation ρ is weakly nondegenerate if and only if the vectors $\{\rho(y)|y \in X\}$ are mutually distinct.*

Proof. By Lemma 16.3. □

Recall the representation diagram Δ_E from Definition 9.15.

Lemma 16.9. *The spherical representation ρ is weakly nondegenerate if and only if Δ_E is connected.*

Proof. By Proposition 9.17. □

Definition 16.10. The spherical representation ρ is said to be *balanced* whenever:

(i) ρ is weakly nondegenerate;

(ii) for distinct $y, z \in X$ and $0 \leq i, j \leq d$ we have

$$\sum_{w \in \Gamma_i(y)\cap\Gamma_j(z)} \rho(w) - \sum_{w \in \Gamma_j(y)\cap\Gamma_i(z)} \rho(w) \in \operatorname{Span}\big(\rho(y) - \rho(z)\big). \tag{58}$$

The equation (58) is called the *balanced set condition*.

Lemma 16.11. *Assume that ρ is balanced, and pick distinct $y, z \in X$. For $0 \leq i, j \leq d$ we have*

$$\sum_{w \in \Gamma_i(y)\cap\Gamma_j(z)} \rho(w) - \sum_{w \in \Gamma_j(y)\cap\Gamma_i(z)} \rho(w) = r^k_{i,j}\big(\rho(y) - \rho(z)\big),$$

where $(y, z) \in R_k$ and

$$r^k_{i,j} = p^k_{i,j}\frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_k}.$$

Proof. The left-hand side of (58) is a scalar multiple of $\rho(y) - \rho(z)$; denote the scalar by α. To compute α, take the inner product of $\rho(y)$ with each side of (58). We have

$$\Big\langle \rho(y), \sum_{w \in \Gamma_i(y)\cap\Gamma_j(z)} \rho(w)\Big\rangle = \sum_{w \in \Gamma_i(y)\cap\Gamma_j(z)} \langle\rho(y), \rho(w)\rangle = \sum_{w \in \Gamma_i(y)\cap\Gamma_j(z)} \theta^*_i = p^k_{i,j}\theta^*_i.$$

Similarly

$$\left\langle \rho(y), \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} \rho(w) \right\rangle = p^k_{i,j}\theta^*_j.$$

We also have

$$\langle \rho(y), \rho(y) \rangle = \theta^*_0, \qquad \langle \rho(y), \rho(z) \rangle = \theta^*_k.$$

By these comments,

$$p^k_{i,j}(\theta^*_i - \theta^*_j) = \alpha(\theta^*_0 - \theta^*_k).$$

The result follows. □

We have some comments about the representation diagram Δ_E. This diagram has vertex set $0, 1, 2, \ldots, d$. Since $\mathcal{X}$ is symmetric, the edges in Δ_E are undirected. Some of the vertices might have a loop. Let Δ^R_E denote the diagram obtained from Δ_E by removing the loops. We call Δ^R_E the *reduced representation diagram* for E.

We now state the next main result.

Theorem 16.12. *The following are equivalent:*

(i) ρ *is balanced;*

(ii) Δ^R_E *is a tree.*

We will prove Theorem 16.12 shortly. First we mention a corollary. Note that Δ^R_E is a path if and only if $\mathcal{X}$ is Q-polynomial with respect to E.

Corollary 16.13. *Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then ρ is balanced.*

Proof. The diagram Δ^R_E is a path and hence a tree. □

Lecture 22

To prove Theorem 16.12, we will use the subconstituent algebra. For the rest of this section, fix a vertex $x \in X$. Recall that $T = T(x)$ is generated by $\mathcal{M}$ and $\mathcal{M}^* = \mathcal{M}^*(x)$. Abbreviate $A^* = A^*_1 \in \mathcal{M}^*$. By construction

$$A^* = \sum_{i=0}^{d} \theta^*_i E^*_i.$$

We define a subspace $\mathcal{L}$ of the vector space T:

$$\mathcal{L} = \text{Span}\{MA^*N - NA^*M | M, N \in \mathcal{M}\}.$$

Each of $\{A_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$ is a basis for $\mathcal{M}$. Therefore

$$\mathcal{L} = \text{Span}\{A_iA^*A_j - A_jA^*A_i | 0 \leq i, j \leq d\} \tag{59}$$

and

$$\mathcal{L} = \text{Span}\{E_iA^*E_j - E_jA^*E_i | 0 \leq i, j \leq d\}. \tag{60}$$

We now give a refined version of (60).

Lemma 16.14. *The set*

$$\{E_iA^*E_j - E_jA^*E_i | 0 \le i < j \le d,\ q^1_{i,j} \neq 0\} \tag{61}$$

is a basis for $\mathcal{L}$*.*

Proof. We saw earlier that for $0 \le i, j \le d$ the matrix $E_iA^*E_j = 0$ if and only if $q^1_{i,j} = 0$. Also, the nonzero matrices among $\{E_iA^*E_j | 0 \le i, j \le d\}$ are mutually orthogonal, and therefore linearly independent. □

Corollary 16.15. *The dimension of* $\mathcal{L}$ *is equal to the number of edges in* Δ^R_E*.*

Proof. For $0 \le i < j \le d$ the vertices i, j of Δ^R_E are adjacent if and only if $q^1_{i,j} \neq 0$. The result follows from this and Lemma 16.14. □

Corollary 16.16. *Assume that* ρ *is weakly nondegenerate. Then* $d \le \dim \mathcal{L}$*, with equality if and only if* Δ^R_E *is a tree.*

Proof. The graph Δ^R_E is connected. An undirected connected graph with $d + 1$ vertices has at least d edges, with equality if and only if the graph is a tree. The result follows. □

Lemma 16.17. *Assume that* ρ *is weakly nondegenerate. Then the set* $\{A^*A_k - A_kA^* | 1 \le k \le d\}$ *is a linearly independent subset of* $\mathcal{L}$*.*

Proof. By construction the given set is contained in $\mathcal{L}$. For $1 \le k \le d$ and $y \in X$ we compute the (x, y)-entry of $A^*A_k - A_kA^*$. This entry is equal to

$$A^*_{x,x}(A_k)_{x,y} - (A_k)_{x,y}A^*_{y,y} = \begin{cases} \theta^*_0 - \theta^*_k & \text{if } (x, y) \in R_k; \\ 0 & \text{if } (x, y) \notin R_k. \end{cases}$$

The linear independence is a routine consequence of this. □

Corollary 16.18. *Assume that* ρ *is weakly nondegenerate. Then the following are equivalent:*

(i) $\dim \mathcal{L} = d$*;*

(ii) Δ^R_E *is a tree;*

(iii) *the matrices* $\{A^*A_k - A_kA^* | 1 \le k \le d\}$ *span* $\mathcal{L}$*;*

(iv) *the matrices* $\{A^*A_k - A_kA^* | 1 \le k \le d\}$ *form a basis for* $\mathcal{L}$*.*

Proof. By Corollary 16.15 and Lemma 16.17. □

Proof of Theorem 16.12. (i) ⇒ (ii) We assume ρ is balanced, so ρ is weakly nondegenerate. We show that for $0 \le i, j \le d$,

$$A_iA^*A_j - A_jA^*A_i = \sum_{k=1}^{d} r^k_{i,j}(A^*A_k - A_kA^*), \tag{62}$$

where

$$r^k_{i,j} = p^k_{i,j}\frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_k} \qquad\qquad (1 \le k \le d).$$

To establish (62), we will show that the following matrix is equal to 0:

$$A_iA^*A_j - A_jA^*A_i - \sum_{k=1}^{d} r^k_{i,j}(A^*A_k - A_kA^*). \tag{63}$$

For $y, z \in X$ we compute the (y, z)-entry of (63). The (y, z)-entry of $A_iA^*A_j$ is equal to

$$\begin{aligned}\sum_{w\in X}(A_i)_{y,w}A^*_{w,w}(A_j)_{w,z} &= \sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} A^*_{w,w}\\ &= \sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \langle\rho(x), \rho(w)\rangle\\ &= \Big\langle \rho(x), \sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \rho(w)\Big\rangle.\end{aligned}$$

Similarly, the (y, z)-entry of $A_jA^*A_i$ is equal to

$$\Big\langle \rho(x), \sum_{w\in\Gamma_j(y)\cap\Gamma_i(z)} \rho(w)\Big\rangle.$$

For $1 \le k \le d$ the (y, z)-entry of $A^*A_k - A_kA^*$ is equal to

$$A^*_{y,y}(A_k)_{y,z} - (A_k)_{y,z}A^*_{z,z} = \begin{cases}\langle\rho(x), \rho(y) - \rho(z)\rangle & \text{if } (y, z) \in R_k;\\ 0 & \text{if } (y, z) \notin R_k.\end{cases}$$

By these comments, the (y, z)-entry of (63) is equal to

$$\Big\langle \rho(x), \sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \rho(w) - \sum_{w\in\Gamma_j(y)\cap\Gamma_i(z)} \rho(w) - r^k_{i,j}\big(\rho(y) - \rho(z)\big)\Big\rangle, \tag{64}$$

provided that $y \ne z$ and $(y, z) \in R_k$. In this case the scalar (64) is equal to zero, because in the inner product the vector on the right is zero. Note that for $y = z$ the (y, z)-entry of (63) is equal to zero. We have shown that the matrix (63) is equal to zero, so (62) holds. By (59) and (62) we get Corollary 16.18(iii), which implies Corollary 16.18(ii). We have shown that Δ^R_E is a tree.

(ii) ⇒ (i) The graph Δ^R_E is connected since it is a tree. Therefore ρ is weakly nondegenerate. We show that ρ satisfies the balanced set condition. By Corollary 16.18 the matrices $\{A^*A_k - A_kA^* | 1 \le k \le d\}$ form a basis for $\mathcal{L}$. Consequently, for $0 \le i, j \le d$ there exist $r^k_{i,j} \in \mathbb{R}$ $(1 \le k \le d)$ such that

$$A_iA^*A_j - A_jA^*A_i = \sum_{k=1}^{d} r^k_{i,j}(A^*A_k - A_kA^*). \tag{65}$$

For distinct $y, z \in X$ we examine the (y, z)-entry in (65). The result shows that $\rho(x)$ is orthogonal to

$$\sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \rho(w) - \sum_{w\in\Gamma_j(y)\cap\Gamma_i(z)} \rho(w) - r^k_{i,j}\big(\rho(y) - \rho(z)\big), \tag{66}$$

where $(y, z) \in R_k$. Since the choice of x is arbitrary, the vector (66) must be orthogonal to EV. The vector (66) is contained in EV, so the vector (66) is equal to zero. Therefore

$$\sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \rho(w) - \sum_{w\in\Gamma_j(y)\cap\Gamma_i(z)} \rho(w) = r^k_{i,j}\big(\rho(y) - \rho(z)\big).$$

Consequently ρ satisfies the balanced set condition. We conclude that ρ is balanced. □

As we saw earlier, if $\mathcal{X}$ is Q-polynomial with respect to E, then ρ is balanced. We are going to show that the converse is true, provided that $\mathcal{X}$ is P-polynomial. This converse is implied by the following theorem. To avoid trivialities, we will assume $d \geq 3$.

Theorem 16.19. *Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\rho = \rho_E$ satisfies:*

(i) *ρ is weakly nondegenerate;*

(ii) *for all $y, z \in X$,*

$$\sum_{w\in\Gamma(y)\cap\Gamma_2(z)} \rho(w) - \sum_{w\in\Gamma_2(y)\cap\Gamma(z)} \rho(w) \in \operatorname{Span}\big(\rho(y) - \rho(z)\big).$$

Then $\mathcal{X}$ is Q-polynomial with respect to E.

Proof. We abbreviate $A = A_1$. Fix $x \in X$ and write $T = T(x)$. We assume that ρ is weakly nondegenerate, so $\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$.
Claim 1. Pick an integer k ($1 \leq k \leq d$) and $y, z \in X$ such that $(y, z) \in R_k$. Then

$$\sum_{w\in\Gamma(y)\cap\Gamma_2(z)} \rho(w) - \sum_{w\in\Gamma_2(y)\cap\Gamma(z)} \rho(w) = r^k_{1,2}\big(\rho(y) - \rho(z)\big),$$

where

$$r^k_{1,2} = p^k_{1,2}\frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_k}. \tag{67}$$

Proof of Claim 1. By assumption there exists $\alpha \in \mathbb{R}$ such that

$$\sum_{w\in\Gamma(y)\cap\Gamma_2(z)} \rho(w) - \sum_{w\in\Gamma_2(y)\cap\Gamma(z)} \rho(w) = \alpha\big(\rho(y) - \rho(z)\big).$$

For each term in the above equation, take the inner product with $\rho(y)$. A brief calculation yields

$$p^k_{1,2}(\theta^*_1 - \theta^*_2) = \alpha(\theta^*_0 - \theta^*_k).$$

Therefore

$$\alpha = p^k_{1,2} \frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_k},$$

and Claim 1 is proved.

Lecture 23

Claim 2. We have

$$AA^*A_2 - A_2A^*A = \sum_{k=1}^{d} r^k_{1,2}(A^*A_k - A_kA^*). \tag{68}$$

Proof of Claim 2. For $y, z \in X$ we compute the (y, z)-entry of the left-hand side of (68) minus the right-hand side of (68). For $y = z$ the (y, z)-entry is zero. For $y \neq z$ the (y, z)-entry is equal to

$$\left\langle \rho(x), \sum_{w \in \Gamma(y)\cap\Gamma_2(z)} \rho(w) - \sum_{w \in \Gamma_2(y)\cap\Gamma(z)} \rho(w) - r^k_{1,2}\big(\rho(y) - \rho(z)\big) \right\rangle,$$

where $(y, z) \in R_k$. The above scalar is zero by Claim 1. Claim 2 is proved.
Conceivably $\theta^*_1 = \theta^*_2$. In this case $r^k_{1,2} = 0$ for $1 \leq k \leq d$. So by Claim 2, $AA^*A_2 = A_2A^*A$. In this equation we eliminate A_2 using $A_2 = (A^2 - a_1A - \kappa I)/c_2$ $(\kappa = b_0)$ and get

$$A^2A^*A - AA^*A^2 = \kappa(A^*A - AA^*). \tag{69}$$

We will return to this equation shortly.
Claim 3. Assume that $\theta^*_1 \neq \theta^*_2$. Then there exist scalars $\beta, \gamma, \varrho \in \mathbb{R}$ such that

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*], \tag{70}$$

where $[r, s] = rs - sr$.
Proof of Claim 3. Referring to (67), the scalar $p^k_{1,2}$ is zero if $k > 3$ and nonzero if $k = 3$. Therefore $r^k_{1,2}$ is zero if $k > 3$ and nonzero if $k = 3$. The matrices A_2 and A_3 appear in (68). Recall that A_2 and A_3 are polynomials in A that have degrees 2 and 3, respectively. Evaluating (68) using this fact, we obtain

$$A^3A^* - A^*A^3 \in \operatorname{Span}\Big(A^2A^*A - AA^*A^2, A^2A^* - A^*A^2, AA^* - A^*A\Big).$$

Therefore there exist $\beta, \gamma, \varrho \in \mathbb{R}$ such that

$$A^3A^* - A^*A^3 = (\beta + 1)(A^2A^*A - AA^*A^2) + \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A).$$

In this equation we rearrange the terms to obtain (70). Claim 3 is proved.
For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A for E_i. Recall the reduced representation diagram Δ^R_E. The graph Δ^R_E is connected since ρ is weakly nondegenerate. Recall that in Δ^R_E, vertex 0 is adjacent to vertex 1 and no other vertex. We will show that Δ^R_E is a path. To do this, it suffices to show that each vertex i in Δ^R_E is adjacent to at most 2 vertices in Δ^R_E.
Claim 4. For distinct vertices i, j in Δ^R_E that are adjacent,

(i) if $\theta^*_1 = \theta^*_2$ then $\theta_i\theta_j = -\kappa$;

(ii) if $\theta^*_1 \neq \theta^*_2$ then $\mathcal{P}(\theta_i, \theta_j) = 0$, where

$$\mathcal{P}(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \varrho.$$

Proof of Claim 4. First assume that $\theta^*_1 = \theta^*_2$. Then (69) holds. In (69), multiply each term on the left by E_i and on the right by E_j. Simplify the result to get

$$0 = E_iA^*E_j(\theta_i - \theta_j)(\theta_i\theta_j + \kappa).$$

We have $E_iA^*E_j \neq 0$ since i, j are adjacent in Δ^R_E. The scalar $\theta_i - \theta_j$ is nonzero since $i \neq j$. Therefore $\theta_i\theta_j + \kappa = 0$ so $\theta_i\theta_j = -\kappa$. Next assume that $\theta^*_1 \neq \theta^*_2$. Then (70) holds. In (70), multiply each term on the left by E_i and the right by E_j. Simplify the result to get

$$0 = E_iA^*E_j(\theta_i - \theta_j)\mathcal{P}(\theta_i, \theta_j).$$

We have $E_iA^*E_j \neq 0$ since i, j are adjacent in Δ^R_E. The scalar $\theta_i - \theta_j$ is nonzero since $i \neq j$. Therefore $\mathcal{P}(\theta_i, \theta_j) = 0$.
Claim 5. We have $\theta^*_1 \neq \theta^*_2$.
Proof of Claim 5. Suppose that $\theta^*_1 = \theta^*_2$. By Claim 4 and since vertex 0 is adjacent to vertex 1, we have $\theta_0\theta_1 = -\kappa$. We have $\theta_0 = \kappa$ so $\theta_1 = -1$. The graph Δ^R_E is connected, so vertex 1 is adjacent to some nonzero vertex j. By Claim 4 we have $\theta_1\theta_j = -\kappa$. By this and $\theta_1 = -1$, we obtain $\theta_j = \kappa$. This implies $j = 0$, for a contradiction. Claim 5 is proved.
Claim 6. Each vertex i in Δ^R_E is adjacent at most two vertices in Δ^R_E.
Proof of Claim 6. By Claims 4, 5 we see that for each vertex j in Δ^R_E that is adjacent vertex i, the eigenvalue θ_j is a root of the polynomial

$$\mathcal{P}(\theta_i, \mu) = \theta_i^2 - \beta\theta_i\mu + \mu^2 - \gamma(\theta_i + \mu) - \varrho.$$

This polynomial is quadratic in μ, so it has at most two distinct roots. Claim 6 is proved.
We have shown that the graph Δ^R_E is a path. Consequently $\mathcal{X}$ is Q-polynomial with respect to E. $\square$

The balanced set condition is very useful. We illustrate with some applications.

Theorem 16.20. *Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then*

$$\frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{71}$$

is independent of i for $2 \leq i \leq d - 1$.

Proof. Fix an integer i $(2 \leq i \leq d-1)$. Pick $x \in X$ and $z \in \Gamma_{i+1}(x)$ and $y \in \Gamma_{i-2}(x) \cap \Gamma_3(z)$. By the balanced set condition,

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} \rho(w) - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} \rho(w) = r^3_{1,2}\big(\rho(y) - \rho(z)\big), \tag{72}$$

where

$$r^3_{1,2} = p^3_{1,2}\frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_3}. \qquad (73)$$

Take the inner product of $\rho(x)$ with each side of (72); this yields

$$p^3_{1,2}(\theta^*_{i-1} - \theta^*_i) = r^3_{1,2}(\theta^*_{i-2} - \theta^*_{i+1}).$$

Evaluating this using (73) and rearranging terms, we obtain

$$\frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} = \frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}.$$

The result follows. □

Definition 16.21. Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Define $\beta \in \mathbb{R}$ such that $\beta + 1$ is the common value of (71). We call β the *fundamental parameter* of E.

Corollary 16.22. *Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then the dual eigenvalues have the following closed forms.*

(i) *Assume $\beta \neq \pm 2$. Then*

$$\theta^*_i = a + bq^i + cq^{-i} \qquad (0 \leq i \leq d),$$

where $\beta = q + q^{-1}$.

(ii) *Assume $\beta = 2$. Then*

$$\theta^*_i = a + bi + ci^2 \qquad (0 \leq i \leq d).$$

(iii) *Assume $\beta = -2$. Then*

$$\theta^*_i = a + b(-1)^i + ci(-1)^i \qquad (0 \leq i \leq d).$$

Proof. The dual eigenvalues satisfy the three-term recurrence

$$\theta^*_{i+1} - (\beta+1)\theta^*_i + (\beta+1)\theta^*_{i-1} - \theta^*_{i-2} = 0 \qquad (2 \leq i \leq d-1).$$

For this recurrence the characteristic polynomial is

$$\lambda^3 - (\beta+1)\lambda^2 + (\beta+1)\lambda - 1.$$

This polynomial has roots $1, q, q^{-1}$ where $\beta = q+q^{-1}$. The result follows by linear algebra. □

Note 16.23. Under the assumptions of Corollary 16.22, the eigenvalues $\{\theta_i\}_{i=0}^d$ have similar closed forms.

Theorem 16.24. *Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then for $x \in X$ the subgraph of (X, R_1) induced on $\Gamma_{d-1}(x) \cup \Gamma_d(x)$ is connected.*

Proof. The graph $\Gamma = (X, R_1)$ is distance-regular; let ∂ denote its path-length distance function. A path $\{y_i\}_{i=0}^r$ in Γ will be called *geodesic* whenever $\partial(y_0, y_r) = r$. We will use a proof by contradiction, and assume the subgraph induced on $\Gamma_{d-1}(x)\cup\Gamma_d(x)$ is disconnected. Let C be the vertex set of a connected component of the subgraph induced on $\Gamma_{d-1}(x)\cup\Gamma_d(x)$. Let the set Δ consist of the vertices in X that lie on a geodesic from x to C. Note that $\Delta \neq X$, since $C \neq \Gamma_{d-1}(x) \cup \Gamma_d(x)$. We partition $\Delta = \cup_{j=0}^d \Delta_j$ where $\Delta_j = \Delta \cap \Gamma_j(x)$ for $0 \leq j \leq d$. Note that $C = \Delta_{d-1} \cup \Delta_d$. Each vertex in Δ_d is adjacent to c_d vertices in Δ_{d-1}. Each vertex in Δ_{d-1} is adjacent to b_{d-1} vertices in Δ_d. For $0 \leq j \leq d-1$, each vertex in Δ_j is adjacent to at least one vertex in Δ_{j+1}.

A vertex in Δ will be called a *border* whenever it is adjacent to a vertex in $X\backslash\Delta$. Since $\Delta \neq X$ and Γ is connected, Δ contains at least one border vertex. Let t denote the maximal integer j ($0 \leq j \leq d$) such that Δ_j contains a border vertex. By the construction $1 \leq t \leq d-2$. Pick a border vertex $w \in \Delta_t$. There exists $y \in \Delta_{t+2}$ such that $\partial(y, w) = 2$. Let $z \in X\backslash\Delta$ be adjacent to w. Define $\xi = \partial(x, z)$. By the triangle inequality $\xi \in \{t-1, t, t+1\}$. Note that $\xi \neq t-1$; otherwise z is on a geodesic from x to C passing through w, forcing $z \in \Delta$ for a contradiction. Therefore $\xi = t$ or $\xi = t+1$.

We next show that $\partial(y, z) = 3$. Because $\partial(y, w) = 2$ and $\partial(w, z) = 1$, the triangle inequality implies that $\partial(y, z) \leq 3$. By the maximality of t and since $y \in \Delta_{t+2}$, we see that y is not a border and not adjacent to a border. Therefore Δ contains all the vertices in X that are at distance at most 2 from y. The vertex z is not in Δ, so $\partial(y, z) \geq 3$. We have shown that $\partial(y, z) = 3$.

Note that $\Gamma(y) \cap \Gamma_2(z) \subseteq \Gamma_{t+1}(x)$ and $\Gamma_2(y) \cap \Gamma(z) \subseteq \Gamma_t(x)$. We apply the balanced set condition to y and z using $i = 1, j = 2, k = 3$ and then take the inner product of each side with $\rho(x)$; this gives

$$p^3_{1,2}(\theta^*_{t+1} - \theta^*_t) = p^3_{1,2}\frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_3}(\theta^*_{t+2} - \theta^*_\xi).$$

Rearranging terms, we obtain

$$\frac{\theta^*_\xi - \theta^*_{t+2}}{\theta^*_t - \theta^*_{t+1}} = \frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}. \tag{74}$$

By Theorem 16.20,

$$\frac{\theta^*_{t-1} - \theta^*_{t+2}}{\theta^*_t - \theta^*_{t+1}} = \frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}. \tag{75}$$

Comparing (74), (75) we obtain $\theta^*_\xi = \theta^*_{t-1}$. We have $\xi = t-1$ since $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are mutually distinct. We mentioned earlier that $\xi \neq t-1$, for a contradiction. We conclude that the subgraph induced on $\Gamma_{d-1}(x) \cup \Gamma_d(x)$ is connected. □

Lecture 24

Chapter 3: Codes and designs in association schemes

Throughout this chapter, $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme with eigenmatrices P and Q. We work over $\mathbb{R}$. Any scalar that we mention is understood to be in $\mathbb{R}$.

17 Linear programming approach to association schemes

In this section we introduce the linear programming approach. We motivate things with a problem.

Problem 17.1. Let Y denote a subset of X such that no two vertices in Y are R_1-related. How large can Y be?

We now attack the above problem. Recall the standard module $V = \mathbb{R}^X$. Define the vector $\psi_Y \in V$ by

$$\psi_Y = \sum_{y \in Y} \hat{y}.$$

For $0 \leq j \leq d$ the scalar $\|E_j \psi_Y\|^2$ is nonnegative. Let us compute this scalar. We have

$$\begin{aligned} \|E_j \psi_Y\|^2 &= \left\langle \sum_{y \in Y} E_j \hat{y}, \sum_{z \in Y} E_j \hat{z} \right\rangle \\ &= \sum_{y \in Y} \sum_{z \in Y} \langle E_j \hat{y}, E_j \hat{z} \rangle \\ &= \frac{|Y|}{|X|} \sum_{i=0}^{d} a_i Q_j(i), \end{aligned}$$

where

$$a_i = \frac{|(Y \times Y) \cap R_i|}{|Y|} \qquad (0 \leq i \leq d).$$

Of course $a_i \geq 0$ for $0 \leq i \leq d$. Moreover

$$a_0 = 1, \qquad a_1 = 0, \qquad |Y| = \sum_{i=0}^{d} a_i.$$

We can gain insight about Problem 17.1 by solving the following linear programming problem.

Problem 17.2. Maximize

$$g = \sum_{i=0}^{d} a_i$$

subject to the following constraints:

(i) $a_0 = 1$ and $a_1 = 0$ and $a_i \geq 0$ for $2 \leq i \leq d$;

(ii) $\sum_{i=0}^{d} a_i Q_j(i) \geq 0$ for $0 \leq j \leq d$.

Problems 17.1, 17.2 are related as follows. Let g_0 denote the maximal value of g in Problem 17.2. Then $|Y| \leq g_0$ for all subsets Y from Problem 17.1.

Example 17.3. Assume the association scheme $\mathcal{X}$ is the 3-cube $H(3,2)$. We can see at a glance that for Problem 17.1, the answer is 4. Let us find g_0. We have

$$Q = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}$$

We maximize $g = \sum_{i=0}^{3} a_i$ subject to

$$\begin{array}{llll} a_0 = 1, & a_1 = 0, & a_2 \geq 0, & a_3 \geq 0, \\ a_2 + 3a_3 \leq 3, & a_2 - 3a_3 \leq 3, & a_3 - a_2 \leq 1. & \end{array}$$

A graph of the inequalities reveals that g is maximized at $(a_2, a_3) = (3, 0)$. Therefore $g_0 = 1 + 0 + 3 + 0 = 4$.

We have some comments about Problem 17.2. Recall that $Q_0(i) = 1$ for $0 \leq i \leq d$. So in part (ii), the case $j = 0$ provides no information, and can be ignored. Since $a_1 = 0$ we can remove a_1 from the entire problem. Concerning part (i), sometimes it is convenient to drop the requirement that $a_0 = 1$. In this case, Problem 17.2 has type (C, M) below.

Fix an integer $d \geq 1$ and define the set $D = \{0, 1, \ldots, d\}$. Define a subset $M \subseteq D$ such that $0 \in M$. Define $D^\times = D \backslash \{0\}$ and $M^\times = M \backslash \{0\}$. Pick $C \in \mathrm{Mat}_{d+1}(\mathbb{R})$, with (i,j)-entry denoted $C_j(i)$ for $0 \leq i, j \leq d$. Assume that $C_0(i) = 1$ for $0 \leq i \leq d$.

Problem (C, M): Maximize

$$g = \sum_{i \in M} a_i C_0(i)$$

subject to

$$a_i \geq 0 \quad (i \in M^\times), \qquad\qquad \sum_{i \in M} a_i C_j(i) \geq 0 \quad (j \in D^\times). \tag{76}$$

Definition 17.4. A vector $\{a_i\}_{i \in M}$ is called a *program* for (C, M) whenever it satisfies (76) and $a_0 = 1$. A program $\{a_i\}_{i \in M}$ for (C, M) is called *maximal* whenever it gives the maximal value of g. This maximal value of g is denoted g_0. Problem (C, M) is called *feasible* whenever there exists a program for (C, M).

The following problem is related to Problem (C, M).

Problem $(C, M)'$: Minimize

$$\gamma = \sum_{j \in D} \alpha_j C_j(0)$$

subject to

$$\alpha_j \geq 0 \quad (j \in D^\times), \qquad \sum_{j \in D} \alpha_j C_j(i) \leq 0 \quad (i \in M^\times). \tag{77}$$

Definition 17.5. A vector $\{\alpha_j\}_{j \in D}$ is called a *program* for $(C, M)'$ whenever it satisfies (77) and $\alpha_0 = 1$. A program $\{\alpha_j\}_{j \in D}$ for $(C, M)'$ is called *minimal* whenever it gives the minimal value of γ. This minimal value of γ is denoted γ_0. Problem $(C, M)'$ is called *feasible* whenever there exists a program for $(C, M)'$.

Problem (C, M) and Problem $(C, M)'$ are related as follows.

Lemma 17.6. *Let $\{a_i\}_{i \in M}$ and $\{\alpha_j\}_{j \in D}$ denote programs for (C, M) and $(C, M)'$ respectively. Then $g \leq \gamma$.*

Proof. We show that

$$g \leq \sum_{i \in M} \sum_{j \in D} a_i \alpha_j C_j(i) \leq \gamma.$$

We have

$$\begin{aligned}
g &= \alpha_0 g \\
&= \alpha_0 \sum_{i \in M} a_i C_0(i) \\
&\leq \alpha_0 \sum_{i \in M} a_i C_0(i) + \sum_{j \in D^\times} \alpha_j \left(\sum_{i \in M} a_i C_j(i) \right) \\
&= \sum_{j \in D} \sum_{i \in M} \alpha_j a_i C_j(i) \\
&= \sum_{i \in M} \sum_{j \in D} a_i \alpha_j C_j(i).
\end{aligned}$$

We also have

$$\begin{aligned}\gamma &= a_0\gamma \\ &= a_0 \sum_{j\in D} \alpha_j C_j(0) \\ &\geq a_0 \sum_{j\in D} \alpha_j C_j(0) + \sum_{i\in M^\times} a_i \left(\sum_{j\in D} \alpha_j C_j(i)\right) \\ &= \sum_{i\in M}\sum_{j\in D} a_i \alpha_j C_j(i).\end{aligned}$$

□

We now state the duality theorem for linear programming.

Theorem 17.7. *Assume that Problems (C, M) and $(C, M)'$ are feasible. Then there exists a program $\{a_i\}_{i\in M}$ for (C, M) and a program $\{\alpha_j\}_{j\in D}$ for $(C, M)'$ such that $g = \gamma$. Moreoever $g_0 = g = \gamma = \gamma_0$.*

The proof of Theorem 17.7 can be found in the textbook, pages 110–112.

Next, we consider how to find the programs $\{a_i\}_{i\in M}$ and $\{\alpha_j\}_{j\in D}$ in Theorem 17.7.

Lemma 17.8. *Given a program $\{a_i\}_{i\in M}$ for (C, M) and a program $\{\alpha_j\}_{j\in D}$ for $(C, M)'$. Then $g = \gamma$ if and only if the following* (i), (ii) *hold:*

(i) *for $i \in M^\times$,*

$$a_i \sum_{j\in D} \alpha_j C_j(i) = 0.$$

(ii) *for $j \in D^\times$,*

$$\alpha_j \sum_{i\in M} a_i C_j(i) = 0.$$

Proof. Immediate from the proof of Lemma 17.6. □

Lecture 25

Example 17.9. For even $d = 2t$, the *orthogonality graph* Ω_d has the same vertex set as the hypercube $H(d, 2)$; vertices y, z are adjacent in Ω_d whenever $(y, z) \in R_t$ in $H(d, 2)$. A set of vertices Y for Ω_d is called independent whenever no two vertices in Y are adjacent in Ω_d. Our problem is to find the maximal size of an independent set in Ω_d. First assume that t is odd. Recall that $H(d, 2)$ is bipartite, and note that either half of the bipartition is an independent set in Ω_d. This independent set has cardinality 2^{d-1}, which is maximal. Next assume that t is even. In this case, the problem is open. The above linear programming technique gives an upper bound of $g_0 = 2^n/n$ for the size of an independent subset. Thus for $H(4,2)$ we have $g_0 = 16/4 = 4$. For $H(4,2)$ the linear programming details are shown below. We have $d = 4$. We have $D = \{0,1,2,3,4\}$ and $M = \{0,1,3,4\}$. We take $C = Q$ where

$$Q = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Problem (C, M): Maximize

$$g = a_0 + a_1 + a_3 + a_4$$

subject to

$$\begin{aligned} &a_1 \geq 0, \quad a_3 \geq 0, \quad a_4 \geq 0, \qquad 4a_0 + 2a_1 - 2a_3 - 4a_4 \geq 0, \\ &6a_0 + 6a_4 \geq 0, \qquad 4a_0 - 2a_1 + 2a_3 - 4a_4 \geq 0, \qquad a_0 - a_1 - a_3 + a_4 \geq 0. \end{aligned}$$

Problem $(C, M)'$: Minimize

$$\gamma = \alpha_0 + 4\alpha_1 + 6\alpha_2 + 4\alpha_3 + \alpha_4$$

subject to

$$\begin{aligned} &\alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad \alpha_3 \geq 0, \quad \alpha_4 \geq 0, \qquad \alpha_0 + 2\alpha_1 - 2\alpha_3 - \alpha_4 \leq 0, \\ &\alpha_0 - 2\alpha_1 + 2\alpha_3 - \alpha_4 \leq 0, \qquad \alpha_0 - 4\alpha_1 + 6\alpha_2 - 4\alpha_3 + \alpha_4 \leq 0. \end{aligned}$$

Suppose we are given a program $\{a_i\}_{i \in M}$ for (C, M) and a program $\{\alpha_j\}_{j \in D}$ for $(C, M)'$ such that $g = \gamma$. Then $a_0 = 1$ and

$$\begin{aligned} &\alpha_1(4a_0 + 2a_1 - 2a_3 - 4a_4) = 0, \qquad \alpha_2(6a_0 + 6a_4) = 0, \\ &\alpha_3(4a_0 - 2a_1 + 2a_3 - 4a_4) = 0, \qquad \alpha_4(a_0 - a_1 - a_3 + a_4) = 0. \end{aligned}$$

Moreover, $\alpha_0 = 1$ and

$$\begin{aligned} &a_1(\alpha_0 + 2\alpha_1 - 2\alpha_3 - \alpha_4) = 0, \qquad a_3(\alpha_0 - 2\alpha_1 + 2\alpha_3 - \alpha_4) = 0, \\ &a_4(\alpha_0 - 4\alpha_1 + 6\alpha_2 - 4\alpha_3 + \alpha_4) = 0. \end{aligned}$$

For the above 9 equations, there are 13 solutions (found using Maple). Among these solutions, only one satisfies the inequalities in Problem (C, M) and Problem $(C, M)'$. This unique solution is

$$\begin{aligned} &a_0 = 1, \quad a_1 = 1, \quad a_3 = 1, \quad a_4 = 1, \\ &\alpha_0 = 1, \quad \alpha_1 = 1/4, \quad \alpha_2 = 0, \quad \alpha_3 = 1/4, \quad \alpha_4 = 1. \end{aligned}$$

For this solution $g = 4 = \gamma$. Therefore $g_0 = 4 = \gamma_0$.

For more information see

E. de Klerk, D. V. Pasechnik.
A note on the stability number of an orthogonality graph. arXiv:math/0505038.

Ferdinand Ihringer, Hajime Tanaka.
The Independence Number of the Orthogonality Graph in Dimension 2^k.
arXiv:1901.04860.

Note 17.10. For $d = 8$ the solution is

$$
\begin{aligned}
&a_0 = 1, \quad a_1 = 9/2, \quad a_2 = 7, \quad a_3 = 7/2, \quad a_5 = 7/2, \quad a_6 = 7, \quad a_8 = 9/2, \quad a_9 = 1,\\
&\alpha_0 = 1, \quad \alpha_1 = 1/8, \quad \alpha_2 = 0, \quad \alpha_3 = 1/8, \quad \alpha_4 = 1/5,\\
&\alpha_5 = 1/8, \quad \alpha_6 = 0, \quad \alpha_7 = 1/8, \quad \alpha_8 = 1.
\end{aligned}
$$

For these values $g = 32 = \gamma$ and

$$
\begin{aligned}
&a_2^* = 224, \quad a_1^* = a_3^* = a_4^* = a_5^* = a_6^* = a_7^* = a_8^* = 0,\\
&\alpha_1^* = \alpha_2^* = \alpha_3^* = \alpha_5^* = \alpha_6^* = \alpha_7^* = \alpha_8^* = 0.
\end{aligned}
$$

For general $d = 2t$ (t even) the solution is described as follows. Recall $\theta_i = d - 2i$ for $0 \le i \le d$. Note that

$$
P_2(i) = \frac{\theta_i^2 - d}{2} \qquad P_i(2) = \frac{\binom{d}{i}}{\binom{d}{2}} P_2(i) \qquad (0 \le i \le d).
$$

Also for $0 \le i \le d$, if i is odd then $P_i(t) = P_t(i) = 0$, and if $i = 2\ell$ is even then

$$
P_i(t) = (-1)^\ell \binom{t}{\ell}, \qquad P_t(i) = (-1)^\ell \binom{d}{t} \frac{(2\ell-1)(2\ell-3)\cdots 3\cdot 1}{(d-1)(d-3)\cdots(d-2\ell+1)}
$$

We have

$$
\begin{aligned}
a_i &= \frac{1}{d}\binom{d}{i} + \frac{d-1}{d} P_i(2) = \binom{d}{i}\frac{\theta_i^2}{d^2} \qquad (0 \le i \le d),\\
\alpha_i &= \frac{1}{d} + \frac{d-1}{d}\frac{P_t(i)}{\binom{d}{t}} = \begin{cases} 1/d & \text{if } i \text{ is odd;}\\ 1/d + (-1)^\ell \frac{d-1}{d}\,\frac{(2\ell-1)(2\ell-3)\cdots 3\cdot 1}{(d-1)(d-3)\cdots(d-2\ell+1)} & \text{if } i = 2\ell \text{ is even} \end{cases}
\end{aligned}
$$

For these values $g = 2^d/d = \gamma$ and

$$
\begin{aligned}
&a_2^* = \frac{(d-1)2^d}{d}, \qquad a_i^* = 0 \qquad (1 \le i \le d, i \ne 2).\\
&\alpha_i^* = 0 \qquad (1 \le i \le d, i \ne t).
\end{aligned}
$$

18 Subsets of an association scheme

In this section, we investigate the linear programming approach in more detail. Recall the symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ with eigenmatrices P and Q.

Let Y denote a nonempty subset of X.

Definition 18.1. The *inner distribution of* Y is the row vector $\{a_i\}_{i=0}^d$ where

$$a_i = \frac{|(Y \times Y) \cap R_i|}{|Y|} \qquad (0 \le i \le d).$$

We sometimes let a_Y denote the inner distribution of Y.

The *dual distribution of* Y is the row vector $\{a_j^*\}_{j=0}^d$ where

$$a_j^* = \sum_{i=0}^d a_i Q_j(i) \qquad (0 \le j \le d).$$

We sometimes let a_Y^* denote the dual distribution of Y.

Observe that

$$a_Y^* = a_Y Q, \qquad a_Y = |X|^{-1} a_Y^* P.$$

Moreover

$$a_j = |X|^{-1} \sum_{i=0}^d a_i^* P_j(i) \qquad (0 \le j \le d).$$

Lemma 18.2. *With reference to Definition 18.1,*

(i) $a_0 = 1$;

(ii) $a_0^* = \sum_{i=0}^d a_i = |Y|$.

Proof. Recall that $Q_0(i) = 1$ for $0 \le i \le d$. □

Definition 18.3. By the *characteristic vector of* Y we mean the vector

$$\psi_Y = \sum_{y \in X} \hat{y}.$$

Lemma 18.4. *For* $0 \le i \le d$ *we have*

$$a_i = \frac{\psi_Y^t A_i \psi_Y}{|Y|}, \qquad a_i^* = \frac{|X|}{|Y|} \psi_Y^t E_i \psi_Y.$$

Proof. We have

$$\psi_Y^t A_i \psi_Y = \sum_{y,z \in Y} (A_i)_{y,z} = |(Y \times Y) \cap R_i| = |Y| a_i.$$

We also have

$$a_i^* = \sum_{\ell=0}^d a_\ell Q_i(\ell) = |Y|^{-1} \sum_{\ell=0}^d \psi_Y^t A_\ell \psi_Y Q_i(\ell) = |Y|^{-1} \psi_Y^t \left(\sum_{\ell=0}^d Q_i(\ell) A_\ell \right) \psi_Y = \frac{|X|}{|Y|} \psi_Y^t E_i \psi_Y.$$

□

Corollary 18.5. *For $0 \le i \le d$,*

$$\|E_i \psi_Y\|^2 = \frac{|Y|}{|X|}\, a_i^*.$$

Moreover $a_i^* \ge 0$.

Proof. Use Lemma 18.4 and

$$\|E_i \psi_Y\|^2 = (E_i \psi_Y)^t E_i \psi_Y = \psi_Y^t E_i^2 \psi_Y = \psi_Y^t E_i \psi_Y.$$

□

Recall the set $D = \{0, 1, \dots, d\}$.

Definition 18.6. We define a matrix B_Y with entries indexed by $X \times D$. For $x \in X$ and $i \in D$ the (x, i)-entry of B_Y is

$$B_Y(x, i) = |Y \cap \Gamma_i(x)|.$$

We call B_Y the *outer distribution of* Y.

Lemma 18.7. *We have*

$$a_Y = \frac{\psi_Y^t B_Y}{|Y|}.$$

Proof. For $0 \le i \le d$ the i^{th} entry of either side is equal to

$$|Y|^{-1} \sum_{x \in Y} |Y \cap \Gamma_i(x)|.$$

□

Lemma 18.8. *The following hold for $0 \le i \le d$:*

(i) *the vector $A_i \psi_Y$ is equal to column i of B_Y;*

(ii) *the vector $|X| E_i \psi_Y$ is equal to column i of $B_Y Q$.*

Proof. (i) For $x \in X$ the x-coordinate of $A_i \psi_Y$ is equal to

$$\sum_{y \in X} (A_i)_{x,y} (\psi_Y)_y = \sum_{y \in Y} (A_i)_{x,y} = \sum_{y \in Y \cap \Gamma_i(x)} 1 = |Y \cap \Gamma_i(x)| = B_Y(x, i).$$

(ii) Use (i) and $E_i = |X|^{-1} \sum_{j=0}^{d} Q_i(j) A_j$. □

Our next general goal is to compute the rank of the matrix B_Y.

We bring in some notation. For any vector $u = (u_1, u_2, \dots, u_r)$ let Δ_u denote the diagonal matrix with diagonal entries $u_1, u_2, \dots, u_r$.

Theorem 18.9. *With the above notation,*

$$B_Y^t B_Y = \frac{|Y|}{|X|} P^t \Delta_{a_Y^*} P.$$

Proof. For $0 \le i, j \le d$ we show that the (i, j)-entry of each side is equal to

$$|Y| \sum_{k=0}^{d} p_{i,j}^k a_k.$$

Using Lemma 18.8 the (i, j)-entry of $B_Y^t B_Y$ is equal to

$$(A_i \psi_Y)^t A_j \psi_Y = \psi_Y^t A_i A_j \psi_Y = \sum_{k=0}^{d} p_{i,j}^k \psi_Y^t A_k \psi_Y = |Y| \sum_{k=0}^{d} p_{i,j}^k a_k.$$

The (i, j)-entry of $P^t \Delta_{a_Y^*} P$ is equal to

$$\begin{aligned}
\sum_{\ell=0}^{d} P_{i,\ell}^t a_\ell^* P_{\ell,j} &= \sum_{\ell=0}^{d} P_i(\ell) a_\ell^* P_j(\ell) \\
&= \sum_{\ell=0}^{d} a_\ell^* \sum_{k=0}^{d} p_{i,j}^k P_k(\ell) \\
&= \sum_{k=0}^{d} p_{i,j}^k \sum_{\ell=0}^{d} a_\ell^* P_k(\ell) \\
&= |X| \sum_{k=0}^{d} p_{i,j}^k a_k.
\end{aligned}$$

□

Corollary 18.10. *The rank of the matrix B_Y is equal to the number of nonzero scalars among $\{a_i^*\}_{i=0}^d$.*

Proof. By Theorem 18.9 and since P is invertible. □

Lecture 26

Let us clarify what it means for some a_i^* to be zero.

Lemma 18.11. *For $0 \le i \le d$ the following are equivalent:*

(i) $a_i^* = 0$;

(ii) $E_i \psi_Y = 0$;

(iii) *column i of $B_Y Q$ is zero.*

Proof. By Corollary 18.5 and Lemma 18.8(ii). □

Definition 18.12. We define some parameters as follows.

(i) Define

$$\delta = \min\{i|1 \le i \le d,\ a_i \ne 0\}, \qquad \delta^* = \min\{i|1 \le i \le d,\ a_i^* \ne 0\}.$$

We call δ (resp. δ^*) the *minimum distance* (resp. *dual minimum distance*) of Y.

(ii) Define

$$s = \big|\{i|1 \le i \le d,\ a_i \ne 0\}\big|, \qquad s^* = \big|\{i|1 \le i \le d,\ a_i^* \ne 0\}\big|.$$

We call s (resp. s^*) the *degree* (resp. *dual degree*) of Y.

(iii) Define

$$t = \max\{i|1 \le i \le d,\ a_1^* = a_2^* = \cdots = a_i^* = 0\} = \delta^* - 1.$$

We call t the *strength* of Y.

The above definitions depend on the given orderings $\{R_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$.

Our next general goal is to explain the MacWilliams inequality. This inequality will play an important role for codes and designs.

We bring in some notation. For any row vector $u = (u_0, u_1, \dots, u_d)$ such that $u_0 \ne 0$, define

$$s(u) = \big|\{i|1 \le i \le d,\ u_i \ne 0\}\big|.$$

We also define

$$t(u) = \max\{i|1 \le i \le d,\ u_1 = u_2 = \cdots = u_i = 0\}.$$

If $u_1 \ne 0$ then we define $t(u) = 0$.

Theorem 18.13. (MacWilliams inequality). *Consider a row vector $u = (u_0, u_1, \dots, u_d)$ such that $u_0 \ne 0$.*

(i) *Assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Then*

$$s(uQ) \ge \left\lfloor \frac{t(u)}{2} \right\rfloor.$$

(ii) *Assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Then*

$$s(uP) \ge \left\lfloor \frac{t(u)}{2} \right\rfloor.$$

Proof. (i) Write $\theta_i = P_1(i)$ for $0 \le i \le d$. For $0 \le k \le d$ there exists a polynomial $v_k(z)$ that has degree k and $P_k(i) = v_k(\theta_i)$ for $0 \le i \le d$. Abbreviate $t = t(u)$. By assumption $u_i = 0$ for $1 \le i \le t$. Abbreviate $s = s(uQ)$. We assume $s < \lfloor t/2 \rfloor$ and get a contradiction. Note that $2s + 2 \le t$. Define the set $S = \{i | 1 \le i \le d, \sum_{\ell=0}^{d} u_\ell Q_i(\ell) \neq 0\}$. So $s = |S|$. Define a polynomial

$$f(z) = (z - \theta_0) \prod_{j \in S} (z - \theta_j).$$

The degree of $f(z)$ is $s+1$. For $0 \le i \le d$,

$$f(\theta_i) = 0 \qquad \text{if and only if} \quad i \in S \cup \{0\}.$$

The degree of $f(z)^2$ is $2s+2$. Therefore $f(z)^2$ is a linear combination of $\{v_i(z)\}_{i=0}^{2s+2}$. Write

$$f(z)^2 = \sum_{i=0}^{2s+2} b_i v_i(z) \qquad\qquad b_i \in \mathbb{R}.$$

Define

$$b_i = 0 \qquad\qquad (2s+3 \le i \le d).$$

For $1 \le i \le d$ we have $u_i b_i = 0$, because $u_i = 0$ for $1 \le i \le t$ and $b_i = 0$ for $t+1 \le i \le d$. Consider the row vector

$$b = (b_0, b_1, \dots, b_d).$$

Observe that

$$uQ(bP^t)^t = uQPb^t = |X|ub^t = |X| \sum_{i=0}^{d} u_i b_i = |X| u_0 b_0.$$

Observe that for $j \in S \cup \{0\}$,

$$(bP^t)(j) = \sum_{k=0}^{d} b_k P_k(j) = \sum_{k=0}^{d} b_k v_k(\theta_j) = f(\theta_j)^2 = 0.$$

Therefore

$$uQ(bP^t)^t = 0.$$

We have $u_0 b_0 = 0$ and $u_0 \neq 0$, so $b_0 = 0$. Observe that

$$\sum_{i=0}^{d} f(\theta_i)^2 Q_i(0) = \sum_{i=0}^{d} \sum_{j=0}^{d} b_j P_j(i) Q_i(0) = \sum_{j=0}^{d} b_j \sum_{i=0}^{d} Q_i(0) P_j(i) = \sum_{j=0}^{d} b_j (QP)_{0,j} = |X| b_0 = 0.$$

This is impossible because $f(\theta_i)^2 > 0$ and $Q_i(0) = m_i > 0$ for $i \notin S \cup \{0\}$. □

Corollary 18.14. *Assume that the ordering $\{E_i\}_{i=0}^{d}$ is Q-polynomial. Let Y denote a nonempty subset of X that has degree s and strength t. Then*

$$s \ge \lfloor t/2 \rfloor.$$

Proof. Apply Theorem 18.13(ii) with $u = a_Y Q$. □

19 Codes in a P-polynomial association scheme

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Throughout this section we assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Recall that the graph $\Gamma = (X, R_1)$ is distance-regular. Let ∂ denote the path-length distance function for Γ. Recall that for $y, z \in X$ and $0 \le i \le d$, $\partial(y, z) = i$ if and only if $(y, z) \in R_i$. Abbreviate $A = A_1$ and $\theta_i = P_1(i)$ for $0 \le i \le d$. Recall the valencies $\{k_i\}_{i=0}^d$. Abbreviate $k = k_1$.

For $x \in X$ and $0 \le e \le d$ we consider the set of vertices

$$\{y \in X | \partial(x, y) \le e\}.$$

This set has a partition

$$\{y \in X | \partial(x, y) \le e\} = \cup_{i=0}^e \Gamma_i(x).$$

Therefore

$$\big|\{y \in X | \partial(x, y) \le e\}\big| = \sum_{i=0}^{e} |\Gamma_i(x)| = \sum_{i=0}^{e} k_i.$$

Let Y denote a subset of X with $|Y| \ge 2$. We call Y a *code*. We consider the minimum distance $\delta = \min\{\partial(y, z) | y, z \in Y,\ y \ne z\}$. We have $1 \le \delta \le d$.

Definition 19.1. For $0 \le e \le d$, the code Y is e *error correcting* whenever $\partial(y, z) \ge 2e + 1$ for distinct $y, z \in Y$.

Lemma 19.2. *The code Y is e error correcting for $0 \le e \le \lfloor (\delta - 1)/2 \rfloor$.*

Proof. Clear. □

Lemma 19.3. *Write $e = \lfloor (\delta - 1)/2 \rfloor$. Then the dual degree s^* of Y satisfies*

$$s^* \ge e.$$

Proof. Apply the first MacWilliams inequality to the inner distribution of Y. □

Lemma 19.4. *Write $e = \lfloor (\delta - 1)/2 \rfloor$.*

(i) *The following subsets are mutually disjoint:*

$$\{z \in X | \partial(y, z) \le e\} \qquad y \in Y. \tag{78}$$

(ii) *We have*

$$|Y|(k_0 + k_1 + \cdots + k_e) \le |X|. \tag{79}$$

(iii) *Equality holds in (79) if and only if the subsets (78) partition X.*

(iv) *Assume that equality holds in (79). Then $\delta = 2e + 1$ is odd.*

Proof. (i)–(iii) By our above comments.
(iv) Suppose not. Then $\delta = 2e+2$ is even. Pick $y \in Y$. Pick $z \in X$ such that $\partial(y,z) = e+1$. The vertex z is not contained in any of the subsets (78). Therefore the subsets (78) do not partition X, a contradiction. $\square$

Definition 19.5. The code Y is called *perfect* whenever equality holds in (79).

Recall the characteristic vector ψ_Y. Recall the Bose-Mesner algebra $\mathcal{M}$ and the vector $\mathbf{1} = \sum_{x \in X} \hat{x}$.

Theorem 19.6. (Lloyd type theorem, I). *Assume that Y is perfect, and write $\delta = 2e+1$.*

(i) *The vectors $\{A_i\psi_Y\}_{i=0}^{e}$ are linearly independent;*

(ii) $(A_0 + A_1 + \cdots + A_e)\psi_Y = \mathbf{1}$*;*

(iii) $(A - kI)(A_0 + A_1 + \cdots + A_e)\psi_Y = 0$*;*

(iv) *the vectors $\{A_i\psi_Y\}_{i=0}^{e}$ span $\mathcal{M}\psi_Y$;*

(v) $\dim \mathcal{M}\psi_Y = 1 + e$*;*

(vi) *the dual degree $s^* = e$.*

Proof. (i) For these vectors the nonzero coordinates are in disjoint locations.
(ii) Because X is partitioned by the subsets

$$\{z \in X | \partial(y,z) \le e\} \qquad\qquad y \in Y.$$

(iii) The graph Γ is regular with valency k, so $A\mathbf{1} = k\mathbf{1}$. The result follows by this and (ii).
(iv) Let W denote the span of $\{A_i\psi_Y\}_{i=0}^{e}$. Note that $\psi_Y \in W$. We show that $W = \mathcal{M}\psi_Y$. To do this, it suffices to show that $AW \subseteq W$. For $0 \le i \le e-1$ we have $AA_i\psi_Y \in W$, because AA_i is a linear combination of A_{i-1}, A_i, A_{i+1}. Also, $AA_e\psi_Y \in W$ in view of (iii).
(v) By (i), (iv) above.
(vi) We show that the dimension of $\mathcal{M}\psi_Y$ is equal to $1+s^*$. The vector space $\mathcal{M}$ has a basis $\{E_i\}_{i=0}^{d}$. The vector space $\mathcal{M}\psi_Y$ has a basis consisting of the nonzero vectors among $\{E_i\psi_Y\}_{i=0}^{d}$. The cardinality of this basis is $1+s^*$, in view of Lemma 18.11. So the dimension of $\mathcal{M}\psi_Y$ is equal to $1+s^*$. The result follows by this and (v). $\square$

Lecture 27

Theorem 19.7. (Lloyd type theorem, II). *Assume that Y is perfect, and write $\delta = 2e+1$. Define a polynomial*

$$\Psi_e(z) = \sum_{i=0}^{e} v_i(z),$$

where the polynomial $v_i(z)$ has degree i and $A_i = v_i(A)$. Then the roots of $\Psi_e(z)$ are

$$\{\theta_i | 1 \le i \le d,\ E_i\psi_Y \ne 0\}.$$

Proof. The polynomial $\Psi_e(z)$ has degree e. There are e many integers i $(1 \le i \le d)$ such that $E_i\psi_Y \neq 0$. For these i we show that $\Psi_e(\theta_i) = 0$. We have

$$\begin{aligned} 0 &= (A - kI)(A_0 + A_1 + \cdots + A_e)\psi_Y \\ &= (A - kI)\Psi_e(A)\psi_Y. \end{aligned}$$

For $1 \le i \le d$,

$$\begin{aligned} 0 &= E_i(A - kI)\Psi_e(A)\psi_Y \\ &= (\theta_i - k)\Psi_e(\theta_i)E_i\psi_Y. \end{aligned}$$

If $E_i\psi_Y \neq 0$ then $\Psi_e(\theta_i) = 0$, by the above equation and $\theta_i \neq k$. The result follows. □

The polynomial $\Psi_e(z)$ in Theorem 19.7 is called the *Lloyd polynomial*. This polynomial is determined by e and the intersection numbers of the scheme.

Next we consider Lloyd I, II using linear programming. Let Y denote a code with minimum distance δ, and write $e = \lfloor (\delta - 1)/2 \rfloor$. Note that $\delta = 2e+1$ or $\delta = 2e+2$. Let us take $M = \{0, 2e+1, 2e+2, \ldots, d\}$ and $C = Q$. Recall the inner distribution vector $\{a_i\}_{i=0}^d$ of Y. We have $a_0 = 1$ and $a_i = 0$ for $1 \le i \le 2e$. The vector $\{a_i\}_{i \in M}$ is a program for (Q, M). Next we display a program $\{\alpha_j\}_{j=0}^d$ for $(Q, M)'$. Define

$$K_e = \sum_{i=0}^{e} k_i.$$

Note that

$$K_e = \sum_{i=0}^{e} P_i(0) = \sum_{i=0}^{e} v_i(\theta_0) = \Psi_e(\theta_0).$$

Define

$$\alpha_j = \left(\frac{\Psi_e(\theta_j)}{K_e}\right)^2 \qquad (0 \le j \le d).$$

By construction, $\alpha_0 = 1$ and $\alpha_j \ge 0$ for $1 \le j \le d$. Recall that

$$Q_j(i) = \frac{P_i(j)m_j}{k_i} = \frac{v_i(\theta_j)m_j}{k_i} \qquad (0 \le i, j \le d).$$

So for $i \in M^\times$,

$$\begin{aligned} \sum_{j=0}^{d} \alpha_j Q_j(i) &= \sum_{j=0}^{d} \left(\frac{\Psi_e(\theta_j)}{K_e}\right)^2 \frac{v_i(\theta_j)m_j}{k_i} \\ &= \frac{1}{K_e^2 k_i} \sum_{j=0}^{d} \big(\Psi_e(\theta_j)\big)^2 v_i(\theta_j)m_j. \end{aligned}$$

By construction, the polynomial $\Psi_e(z)$ has degree e. Write

$$\big(\Psi_e(z)\big)^2 = \sum_{\ell=0}^{2e} c_\ell v_\ell(z). \qquad\qquad c_\ell \in \mathbb{R}.$$

We recall the orthogonality relation for P. For $0 \le i, j \le d$ we have

$$\delta_{i,j}|X|k_i = \sum_{\ell=0}^{d} P_i(\ell)P_j(\ell)m_\ell = \sum_{\ell=0}^{d} v_i(\theta_\ell)v_j(\theta_\ell)m_\ell.$$

For $i \in M^\times$,

$$\begin{aligned}
\sum_{j=0}^{d} \alpha_j Q_j(i) &= \frac{1}{K_e^2 k_i}\sum_{j=0}^{d}\big(\Psi_e(\theta_j)\big)^2 v_i(\theta_j)m_j \\
&= \frac{1}{K_e^2 k_i}\sum_{j=0}^{d}\sum_{\ell=0}^{2e} c_\ell v_\ell(\theta_j)v_i(\theta_j)m_j \\
&= \frac{1}{K_e^2 k_i}\sum_{\ell=0}^{2e} c_\ell \sum_{j=0}^{d} v_\ell(\theta_j)v_i(\theta_j)m_j \\
&= \frac{1}{K_e^2 k_i}\sum_{\ell=0}^{2e} c_\ell |X|\delta_{\ell,i}k_i \\
&= 0.
\end{aligned}$$

We have shown that the vector $\{\alpha_j\}_{j=0}^d$ is a program for $(Q,M)'$. Next, we compute the objective function γ for the program $\{\alpha_j\}_{j=0}^d$. We have

$$\begin{aligned}
\gamma &= \sum_{\ell=0}^{d} \alpha_\ell Q_\ell(0) = \frac{1}{K_e^2}\sum_{\ell=0}^{d}\big(\Psi_e(\theta_\ell)\big)^2 m_\ell \\
&= \frac{1}{K_e^2}\sum_{\ell=0}^{d}\sum_{i=0}^{e}\sum_{j=0}^{e} v_i(\theta_\ell)v_j(\theta_\ell)m_\ell = \frac{1}{K_e^2}\sum_{i=0}^{e}\sum_{j=0}^{e}\sum_{\ell=0}^{d} v_i(\theta_\ell)v_j(\theta_\ell)m_\ell \\
&= \frac{1}{K_e^2}\sum_{i=0}^{e}\sum_{j=0}^{e}|X|\delta_{i,j}k_i = \frac{|X|}{K_e^2}\sum_{i=0}^{e} k_i = \frac{|X|}{K_e}.
\end{aligned}$$

Using the linear programming bound,

$$|Y| = \sum_{i\in M} a_i = g \le \gamma = \frac{|X|}{K_e}.$$

In other words,

$$|Y|(k_0 + k_1 + \cdots + k_e) \le |X|. \tag{80}$$

Equality is attained in (80) if and only if $g = \gamma$ if and only if $\{a_i\}_{i\in M}$ is a maximal program for (Q, M) and $\{\alpha_j\}_{j=0}^d$ is a minimal program for $(Q, M)'$. Assume this is the case. The code Y is perfect, so $\delta = 2e + 1$ is odd. We have

$$\alpha_j \left(\sum_{i\in M} a_i Q_j(i) \right) = 0 \qquad\qquad (1 \le j \le d).$$

Thus for $1 \le j \le d$ such that $\sum_{i\in M} a_i Q_j(i) \ne 0$, we have $\alpha_j = 0$ and hence $\Psi_e(\theta_j) = 0$. The number of such j is equal to the dual degree s^* of Y, and the degree of $\Psi_e(z)$ is equal to e, so $s^* \le e$. We have $s^* \ge e$ by the MacWilliams inequality, so $s^* = e$. For the polynomial $\Psi_e(z)$ the set of roots is $\{\theta_j | 1 \le j \le d,\ \sum_{i\in M} a_i Q_j(i) \ne 0\}$. Recall that for $1 \le j \le d$, $E_j \psi_Y \ne 0$ if and only if $\sum_{i\in M} a_i Q_j(i) \ne 0$. So for $\Psi_e(z)$ the set of roots is $\{\theta_j | 1 \le j \le d,\ E_j \psi_Y \ne 0\}$.

Lecture 28

20 Designs in a Q-polynomial association scheme

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Throughout this section we assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Abbreviate $\theta_i^* = Q_1(i)$ for $0 \le i \le d$.

Let Y denote a nonempty subset of X. Recall the inner distribution $\{a_i\}_{i=0}^d$ and dual distribution $\{a_i^*\}_{i=0}^d$ of Y.

Definition 20.1. For an integer t $(0 \le t \le d)$, we call Y a *t-design* whenever $a_i^* = 0$ for $1 \le i \le t$.

Assume that Y is a t-design, and write $e = \lfloor t/2 \rfloor$. Recall the degree

$$s = \big|\{i | 1 \le i \le d,\ a_i \ne 0\}\big|.$$

By Corollary 18.14,

$$s \ge e.$$

Recall the multiplicities $\{m_i\}_{i=0}^d$ of $\mathcal{X}$. The Lloyd theorem has the following dual.

Theorem 20.2. *Assume that Y is a t-design, and write $e = \lfloor t/2 \rfloor$.*

(i) *We have*

$$|Y| \ge m_0 + m_1 + \cdots + m_e. \tag{81}$$

(ii) *Assume that equality holds in* (81)*. Then $t = 2e$ is even.*

(iii) *Assume that equality holds in* (81)*. Then $s = e$.*

(iv) *Assume that equality holds in* (81). *Define a polynomial*

$$\Psi_e^*(z) = \sum_{i=0}^{e} v_i^*(z),$$

where the polynomial $v_i^*(z)$ *has degree* i *and* $Q_i(j) = v_i^*(\theta_j^*)$ *for* $0 \le j \le d$. *Then the roots of* $\Psi_e^*(z)$ *are*

$$\{\theta_j^* | 1 \le j \le d, \ a_j \ne 0\}.$$

Proof. We use the linear programming method. Let us take $M = \{0, 2e+1, 2e+2, \dots, d\}$ and $C = P$. Recall the dual distribution vector $\{a_i^*\}_{i=0}^d$ of Y. We have $a_0^* = |Y|$ and $a_i^* = 0$ for $1 \le i \le 2e$. For $0 \le j \le d$ we have

$$a_j = |X|^{-1} \sum_{i=0}^{d} a_i^* P_j(i).$$

Define

$$b_i = \frac{a_i^*}{|Y|} \qquad (0 \le i \le d).$$

We have $b_0 = 1$ and $b_i = 0$ for $1 \le i \le 2e$. Moreover for $0 \le j \le d$,

$$\sum_{i=0}^{d} b_i P_j(i) = \frac{|X|}{|Y|} a_j \ge 0.$$

By these comments, the vector $\{b_i\}_{i \in M}$ is a program for (P, M). Next we display a program $\{\beta_j\}_{j=0}^d$ for $(P, M)'$. Define

$$K_e^* = \sum_{i=0}^{e} m_i.$$

Note that

$$K_e^* = \sum_{i=0}^{e} Q_i(0) = \sum_{i=0}^{e} v_i^*(\theta_0^*) = \Psi_e^*(\theta_0^*).$$

Define

$$\beta_j = \left(\frac{\Psi_e^*(\theta_j^*)}{K_e^*} \right)^2 \qquad (0 \le j \le d).$$

By construction, $\beta_0 = 1$ and $\beta_j \ge 0$ for $1 \le j \le d$. Recall that

$$P_j(i) = \frac{Q_i(j) k_j}{m_i} = \frac{v_i^*(\theta_j^*) k_j}{m_i} \qquad (0 \le i, j \le d).$$

So for $i \in M^\times$,

$$\sum_{j=0}^{d} \beta_j P_j(i) = \sum_{j=0}^{d} \left(\frac{\Psi_e^*(\theta_j^*)}{K_e^*} \right)^2 \frac{v_i^*(\theta_j^*) k_j}{m_i}$$
$$= \frac{1}{(K_e^*)^2 m_i} \sum_{j=0}^{d} \big(\Psi_e^*(\theta_j^*)\big)^2 v_i^*(\theta_j^*) k_j.$$

By construction, the polynomial $\Psi_e^*(z)$ has degree e. Write

$$\big(\Psi_e^*(z)\big)^2 = \sum_{\ell=0}^{2e} c_\ell^* v_\ell^*(z). \qquad\qquad c_\ell^* \in \mathbb{R}.$$

We state the orthogonality relation for Q. For $0 \le i, j \le d$ we have

$$\delta_{i,j}|X|m_i = \sum_{\ell=0}^{d} Q_i(\ell)Q_j(\ell)k_\ell = \sum_{\ell=0}^{d} v_i^*(\theta_\ell^*)v_j^*(\theta_\ell^*)k_\ell.$$

For $i \in M^\times$,

$$\sum_{j=0}^{d} \beta_j P_j(i) = \frac{1}{(K_e^*)^2 m_i} \sum_{j=0}^{d} \big(\Psi_e^*(\theta_j^*)\big)^2 v_i^*(\theta_j^*) k_j$$
$$= \frac{1}{(K_e^*)^2 m_i} \sum_{j=0}^{d} \sum_{\ell=0}^{2e} c_\ell^* v_\ell^*(\theta_j^*) v_i^*(\theta_j^*) k_j$$
$$= \frac{1}{(K_e^*)^2 m_i} \sum_{\ell=0}^{2e} c_\ell^* \sum_{j=0}^{d} v_\ell^*(\theta_j^*) v_i^*(\theta_j^*) k_j$$
$$= \frac{1}{(K_e^*)^2 m_i} \sum_{\ell=0}^{2e} c_\ell^* |X| \delta_{\ell,i} m_i$$
$$= 0.$$

We have shown that the vector $\{\beta_j\}_{j=0}^d$ is a program for $(P, M)'$. Next, we compute the objective function γ for the program $\{\beta_j\}_{j=0}^d$. We have

$$\gamma = \sum_{\ell=0}^{d} \beta_\ell P_\ell(0) = \frac{1}{(K_e^*)^2} \sum_{\ell=0}^{d} \big(\Psi_e^*(\theta_\ell^*)\big)^2 k_\ell$$
$$= \frac{1}{(K_e^*)^2} \sum_{\ell=0}^{d} \sum_{i=0}^{e} \sum_{j=0}^{e} v_i^*(\theta_\ell^*) v_j^*(\theta_\ell^*) k_\ell = \frac{1}{(K_e^*)^2} \sum_{i=0}^{e} \sum_{j=0}^{e} \sum_{\ell=0}^{d} v_i^*(\theta_\ell^*) v_j^*(\theta_\ell^*) k_\ell$$
$$= \frac{1}{(K_e^*)^2} \sum_{i=0}^{e} \sum_{j=0}^{e} |X| \delta_{i,j} m_i = \frac{|X|}{(K_e^*)^2} \sum_{i=0}^{e} m_i = \frac{|X|}{K_e^*}.$$

Using the linear programming bound,

$$\frac{|X|}{|Y|} = \sum_{i \in M} b_i = g \leq \gamma = \frac{|X|}{K_e^*}.$$

In other words,

$$|Y| \geq m_0 + m_1 + \cdots + m_e. \tag{82}$$

Equality is attained in (82) if and only if $g = \gamma$ if and only if $\{b_i\}_{i \in M}$ is a maximal program for (P, M) and $\{\beta_j\}_{j=0}^d$ is a minimal program for $(P, M)'$. Assume this is the case. We have

$$\beta_j \left(\sum_{i \in M} b_i P_j(i) \right) = 0 \qquad\qquad (1 \leq j \leq d).$$

Thus for $1 \leq j \leq d$ such that $\sum_{i \in M} b_i P_j(i) \neq 0$, we have $\beta_j = 0$ and hence $\Psi_e^*(\theta_j^*) = 0$. The number of such j is equal to the degree s of Y, and the degree of $\Psi_e^*(z)$ is equal to e, so $s \leq e$. We mentioned earlier that $s \geq e$, so $s = e$. For the polynomial $\Psi_e^*(z)$ the set of roots is equal to $\{\theta_j^* | 1 \leq j \leq d, \ \sum_{i \in M} b_i P_j(i) \neq 0\}$, which is equal to $\{\theta_j^* | 1 \leq j \leq d, \ a_j \neq 0\}$.

It remains to show that $t = 2e$ is even. We suppose not, and get a contradiction. We have $t = 2e + 1$. For the rest of this proof, fix a vertex $x \in Y$ and consider the subconstituent algebra $T = T(x)$. For $0 \leq i \leq d$, T contains the dual primitive idempotent $E_i^* = E_i^*(x)$ and the dual associate matrix $A_i^* = A_i^*(x)$. Recall that $A_i^* = v_i^*(A^*)$, where $A^* = A_1^*$ is the dual adjacency matrix. Recall the characteristic vector ψ_Y. We have

$$\psi_Y = I\psi_Y = \sum_{j=0}^{d} E_j^* \psi_Y = \hat{x} + \sum_{j=1}^{d} E_j^* \psi_Y.$$

Therefore

$$\Psi_e^*(A^*)\psi_Y = \Psi_e^*(\theta_0^*)\hat{x} + \sum_{j=1}^{d} \Psi_e^*(\theta_j^*) E_j^* \psi_Y.$$

Recall that $\Psi_e^*(\theta_0^*) = K_e^*$. Also for $1 \leq j \leq d$ we have $\Psi_e^*(\theta_j^*) E_j^* \psi_Y = 0$, because $E_j^* \psi_Y = 0$ (if $a_j = 0$) and $\Psi_e^*(\theta_j^*) = 0$ (if $a_j \neq 0$). By these comments,

$$\Psi_e^*(A^*)\psi_Y = K_e^* \hat{x}.$$

We consider the vector

$$E_{e+1} \Psi_e^*(A^*) \psi_Y$$

from two points of view. On one hand,

$$E_{e+1} \Psi_e^*(A^*) \psi_Y = K_e^* E_{e+1} \hat{x}.$$

The vectors $\{E_j\hat{x}\}_{j=0}^d$ form a basis for the primary T-module, so $E_{e+1}\hat{x} \neq 0$. Also $K_e^* \neq 0$. Therefore

$$E_{e+1}\Psi_e^*(A^*)\psi_Y \neq 0.$$

On the other hand,

$$\begin{aligned}
E_{e+1}\Psi_e^*(A^*)\psi_Y &= E_{e+1}\Psi_e^*(A^*)I\psi_Y \\
&= E_{e+1}\Psi_e^*(A^*)\left(\sum_{j=0}^{d} E_j\right)\psi_Y \\
&= E_{e+1}\Psi_e^*(A^*)E_0\psi_Y + \sum_{j=t+1}^{d} E_{e+1}\Psi_e^*(A^*)E_j\psi_Y.
\end{aligned}$$

The polynomial $\Psi_e^*(z)$ has degree e, so by the triple product relations,

$$E_i\Psi_e^*(A^*)E_j = 0 \quad \text{if} \quad |i-j| > e \qquad\qquad (0 \leq i, j \leq d).$$

Consequently

$$E_{e+1}\Psi_e^*(A^*)E_0 = 0.$$

Also for $t+1 \leq j \leq d$,

$$E_{e+1}\Psi_e^*(A^*)E_j = 0$$

because

$$j - e - 1 \geq t + 1 - e - 1 = t - e = e + 1.$$

By these comments

$$E_{e+1}\Psi_e^*(A^*)\psi_Y = 0.$$

This is a contradiction, so $t \neq 2e+1$. We have shown that $t = 2e$ is even. □

Lecture 29

Until further notice, we assume that our association scheme $\mathcal{X}$ is the Hamming scheme $H(d,q)$. Let Y denote a nonempty subset of X. Recall the inner distribution $\{a_i\}_{i=0}^d$ and the dual distribution $\{a_j^*\}_{j=0}^d$. Recall that

$$a_j^* = \sum_{i=0}^{d} a_i Q_j(i) \qquad\qquad (0 \leq j \leq d). \tag{83}$$

We bring in some generating functions. Let x, y denote commuting indeterminates. Define

$$W(x,y) = \sum_{i=0}^{d} a_i x^{d-i} y^i, \qquad\qquad W^*(x,y) = \sum_{i=0}^{d} a_i^* x^{d-i} y^i.$$

We are going to show that

$$W^*(x,y) = W\big(x+(q-1)y, x-y\big).$$

As we will see, this identity depends only on (83). So let us state the result a bit abstractly. Consider any vectors $\{\alpha_i\}_{i=0}^d$ and $\{\alpha_j^*\}_{j=0}^d$ such that

$$\alpha_j^* = \sum_{i=0}^{d} \alpha_i Q_j(i) \qquad\qquad (0 \le j \le d). \tag{84}$$

Define

$$W(x,y) = \sum_{i=0}^{d} \alpha_i x^{d-i} y^i, \qquad\qquad W^*(x,y) = \sum_{i=0}^{d} \alpha_i^* x^{d-i} y^i. \tag{85}$$

Proposition 20.3. *Referring to* (84), (85) *we have*

$$W^*(x,y) = W\big(x+(q-1)y, x-y\big).$$

Proof. Recall the Krawtchouk polynomials $\{K_i(z)\}_{i=0}^d$. Recall that $Q_i(j) = K_i(j)$ for $0 \le i, j \le d$. Recall that for $0 \le j \le d$,

$$\sum_{i=0}^{d} K_i(j) z^i = (1-z)^j \big(1+(q-1)z\big)^{d-j}.$$

Observe that

$$\begin{aligned}
W^*(x,y) &= \sum_{j=0}^{d} \alpha_j^* x^{d-j} y^j \\
&= \sum_{j=0}^{d} x^{d-j} y^j \sum_{i=0}^{d} \alpha_i K_j(i) \\
&= \sum_{i=0}^{d} \alpha_i \sum_{j=0}^{d} K_j(i) x^{d-j} y^j \\
&= x^d \sum_{i=0}^{d} \alpha_i \sum_{j=0}^{d} K_j(i) z^j \qquad\qquad z = y/x \\
&= x^d \sum_{i=0}^{d} \alpha_i (1-z)^i \big(1+(q-1)z\big)^{d-i} \\
&= \sum_{i=0}^{d} \alpha_i (x-y)^i \big(x+(q-1)y\big)^{d-i} \\
&= W\big(x+(q-1)y, x-y\big).
\end{aligned}$$

□

21 Some constraints on the inner distribution and dual distribution

In this section, we discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$.

Let Y denote a nonempty subset of X. Recall the inner distribution vector $\{a_i\}_{i=0}^d$ of Y and the dual distribution vector $\{a_i^*\}_{i=0}^d$ of Y.

Theorem 21.1. *Assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Then in the vector $\{a_i^*\}_{i=0}^d$, the number of consecutive zero entries is at most $2s$, where s is the degree of Y. In particular, the strength t of Y is at most $2s$.*

Proof. We assume that there exists an integer ξ $(0 \leq \xi \leq d - 2s)$ such that $a_j^* = 0$ for $\xi \leq j \leq \xi + 2s$. We will get a contradiction. For Y we have the inner distribution vector $\{a_i\}_{i=0}^d$. Define the set

$$S = \{i | 1 \leq i \leq d,\ a_i \neq 0\}.$$

Note that $s = |S|$. Abbreviate $\theta_i^* = Q_1(i)$ for $0 \leq i \leq d$. Define a polynomial

$$f(z) = \prod_{i \in S} (z - \theta_i^*).$$

The polynomial $f(z)$ has degree s. Note that

$$f(\theta_0^*) = \prod_{i \in S} (\theta_0^* - \theta_i^*) \neq 0.$$

Fix a vertex $x \in Y$ and consider the subconstituent algebra $T = T(x)$. Recall the dual adjacency matrix $A^* = A_1^*(x)$. Recall the characteristic vector ψ_Y. We have

$$\psi_Y = I\psi_Y = \sum_{j=0}^d E_j^* \psi_Y = \hat{x} + \sum_{j=1}^d E_j^* \psi_Y.$$

Therefore

$$f(A^*)\psi_Y = f(\theta_0^*)\hat{x} + \sum_{j=1}^d f(\theta_j^*) E_j^* \psi_Y.$$

For $1 \leq j \leq d$ we have $f(\theta_j^*) E_j^* \psi_Y = 0$, because $E_j^* \psi_Y = 0$ (if $a_j = 0$) and $f(\theta_j^*) = 0$ (if $a_j \neq 0$). By these comments,

$$f(A^*)\psi_Y = f(\theta_0^*)\hat{x}.$$

We consider the vector

$$E_{\xi+s} f(A^*) \psi_Y$$

from two points of view. On one hand,

$$E_{\xi+s}f(A^*)\psi_Y = f(\theta_0^*)E_{\xi+s}\hat{x}.$$

The vectors $\{E_j\hat{x}\}_{j=0}^d$ form a basis for the primary T-module, so $E_{\xi+s}\hat{x} \neq 0$. Also $f(\theta_0^*) \neq 0$. Therefore

$$E_{\xi+s}f(A^*)\psi_Y \neq 0.$$

On the other hand, for $\xi \leq j \leq \xi + 2s$ we have $E_j\psi_Y = 0$ because $a_j^* = 0$. Therefore,

$$\begin{aligned} E_{\xi+s}f(A^*)\psi_Y &= E_{\xi+s}f(A^*)I\psi_Y \\ &= E_{\xi+s}f(A^*)\left(\sum_{j=0}^{d} E_j\right)\psi_Y \\ &= \sum_{j=0}^{\xi-1} E_{\xi+s}f(A^*)E_j\psi_Y + \sum_{j=\xi+2s+1}^{d} E_{\xi+s}f(A^*)E_j\psi_Y. \end{aligned}$$

The polynomial $f(z)$ has degree s, so by the triple product relations,

$$E_if(A^*)E_j = 0 \quad \text{if} \quad |i-j| > s \qquad\qquad (0 \leq i, j \leq d).$$

For $0 \leq j \leq \xi - 1$ we have

$$E_{\xi+s}f(A^*)E_j = 0$$

because

$$\xi + s - j \geq \xi + s - (\xi - 1) = s + 1.$$

For $\xi + 2s + 1 \leq j \leq d$ we have

$$E_{\xi+s}f(A^*)E_j = 0$$

because

$$j - \xi - s \geq \xi + 2s + 1 - \xi - s = s + 1.$$

By these comments

$$E_{\xi+s}f(A^*)\psi_Y = 0.$$

This is a contradiction, and the result follows. □

Theorem 21.2. *Assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Then in the vector $\{a_i\}_{i=0}^d$, the number of consecutive zero entries is at most $2s^*$, where s^* is the dual degree of Y. In particular, the minimum distance of Y is at most $2s^* + 1$.*

Proof. We assume that there exists an integer ξ $(0 \le \xi \le d - 2s^*)$ such that $a_i = 0$ for $\xi \le i \le \xi + 2s^*$. We will get a contradiction. For Y we have the dual distribution vector $\{a_j^*\}_{j=0}^d$. Define the set

$$S^* = \{j | 1 \le j \le d,\ a_j^* \ne 0\}.$$

Note that $s^* = |S^*|$. Abbreviate $\theta_i = P_1(i)$ for $0 \le i \le d$. Define a polynomial

$$f^*(z) = \prod_{j \in S^*} (z - \theta_j).$$

The polynomial $f^*(z)$ has degree s^*. Note that

$$f^*(\theta_0) = \prod_{j \in S^*} (\theta_0 - \theta_j) \ne 0.$$

Recall the adjacency matrix $A = A_1$. Recall the characteristic vector ψ_Y and the vector $\mathbf{1} = \sum_{y \in X} \hat{y}$. Note that

$$E_0 \psi_Y = |X|^{-1} J \psi_Y = \frac{|Y|}{|X|}\mathbf{1}.$$

We have

$$\psi_Y = I\psi_Y = \sum_{j=0}^{d} E_j \psi_Y = \frac{|Y|}{|X|}\mathbf{1} + \sum_{j=1}^{d} E_j \psi_Y.$$

Therefore

$$f^*(A)\psi_Y = \frac{|Y|}{|X|} f^*(\theta_0)\mathbf{1} + \sum_{j=1}^{d} f^*(\theta_j) E_j \psi_Y.$$

For $1 \le j \le d$ we have $f^*(\theta_j)E_j\psi_Y = 0$, because $E_j\psi_Y = 0$ (if $a_j^* = 0$) and $f^*(\theta_j) = 0$ (if $a_j^* \ne 0$). By these comments,

$$f^*(A)\psi_Y = \frac{|Y|}{|X|} f^*(\theta_0)\mathbf{1}.$$

Fix a vertex $x \in Y$ and consider the subconstituent algebra $T = T(x)$. We consider the vector

$$E^*_{\xi+s^*} f^*(A)\psi_Y$$

from two points of view. On one hand,

$$E^*_{\xi+s^*} f^*(A)\psi_Y = \frac{|Y|}{|X|} f^*(\theta_0) E^*_{\xi+s^*}\mathbf{1}.$$

The vectors $\{E_i^*\mathbf{1}\}_{i=0}^d$ form a basis for the primary T-module, so $E^*_{\xi+s^*}\mathbf{1} \neq 0$. Also $f^*(\theta_0) \neq 0$. Therefore

$$E^*_{\xi+s^*} f^*(A)\psi_Y \neq 0.$$

On the other hand, for $\xi \leq i \leq \xi + 2s^*$ we have $E_i^*\psi_Y = 0$ because $a_i = 0$. Therefore,

$$\begin{aligned} E^*_{\xi+s^*} f^*(A)\psi_Y &= E^*_{\xi+s^*} f^*(A) I \psi_Y \\ &= E^*_{\xi+s^*} f^*(A) \left(\sum_{i=0}^{d} E_i^* \right) \psi_Y \\ &= \sum_{i=0}^{\xi-1} E^*_{\xi+s^*} f^*(A) E_i^* \psi_Y + \sum_{i=\xi+2s^*+1}^{d} E^*_{\xi+s^*} f^*(A) E_i^* \psi_Y. \end{aligned}$$

The polynomial $f^*(z)$ has degree s^*, so by the triple product relations,

$$E_i^* f^*(A) E_j^* = 0 \quad \text{if} \quad |i-j| > s^* \qquad\qquad (0 \leq i, j \leq d).$$

For $0 \leq i \leq \xi - 1$ we have

$$E^*_{\xi+s^*} f^*(A) E_i^* = 0$$

because

$$\xi + s^* - i \geq \xi + s^* - (\xi - 1) = s^* + 1.$$

For $\xi + 2s^* + 1 \leq i \leq d$ we have

$$E^*_{\xi+s^*} f^*(A) E_i^* = 0$$

because

$$i - \xi - s^* \geq \xi + 2s^* + 1 - \xi - s^* = s^* + 1.$$

By these comments

$$E^*_{\xi+s^*} f^*(A)\psi_Y = 0.$$

This is a contradiction, and the result follows. □

The two theorems in this section can be found in

Sho Suda. New parameters of subsets in polynomial schemes.
`arXiv:1008.0189`.

Lecture 30

22 On the strength and degree of a Q-polynomial association scheme

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Throughout this section we assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Abbreviate $\theta_i^* = Q_1(i)$ for $0 \le i \le d$. Recall that $\theta_0^* = m_1$ and

$$E_1 = |X|^{-1} \sum_{i=0}^d \theta_i^* A_i.$$

Let Y denote a nonempty subset of X. Recall the inner distribution $\{a_i\}_{i=0}^d$ and dual distribution $\{a_i^*\}_{i=0}^d$. Recall the degreee

$$s = \big|\{i | 1 \le i \le d,\ a_i \ne 0\}\big|$$

and the strength

$$t = \max\{i | 1 \le i \le d,\ a_1^* = a_2^* = \cdots = a_i^* = 0\}.$$

Abbreviate $e = \lfloor t/2 \rfloor$. We saw earlier that $s \ge e$ and

$$|Y| \ge m_0 + m_1 + \cdots + m_e. \tag{86}$$

Our next goal is to show that

$$|Y| \le m_0 + m_1 + \cdots + m_s. \tag{87}$$

We will show that the following are equivalent: (i) equality holds in (86); (ii) equality holds in (87); (iii) $s = e$.

As a warmup, first assume that $s = 0$. Then $|Y| = 1 = m_0$. In this case $a_i^* = m_i \ne 0$ for $0 \le i \le d$. So $t = 0$ and $e = 0$.

To continue the warmup, assume that $s = 1$. Recall $t \le 2s = 2$, so $t \in \{0, 1, 2\}$. We have $|Y| \ge 2$. There exists a unique integer i $(1 \le i \le d)$ such that $a_i \ne 0$. For distinct $y, z \in Y$ we have $(y, z) \in R_i$. We have $a_i = |Y| - 1$. Let M denote $|X|$ times the inner product matrix for $\{E_1 \hat{y}\}_{y \in Y}$. The entries of M are

$$M_{y,z} = \begin{cases} \theta_0^* & \text{if } \ y = z; \\ \theta_i^* & \text{if } \ y \ne z \end{cases} \qquad\qquad y, z \in Y.$$

The matrix M is positive semidefinite, so its eigenvalues are nonnegative. These eigenvalues are $\theta_0^* + (|Y| - 1)\theta_i^*$ (with multiplicity 1) and $\theta_0^* - \theta_i^*$ (with multiplicity $|Y| - 1$). We have $\theta_0^* > \theta_i^*$. We have $\theta_0^* + (|Y| - 1)\theta_i^* = a_1^* \ge 0$, with equality if and only if $t \ge 1$. Assume for the moment that $t = 0$. The matrix M has all eigenvalues positive, so M is invertible. The vectors $\{E_1\hat{y}\}_{y\in Y}$ are linearly independent. Therefore $|Y| \leq m_1$, so $|Y| < m_0 + m_1$.

Next assume that $t \geq 1$. We have $a_1^* = 0$. The matrix M has rank $|Y| - 1$. We have $|Y| - 1 \leq m_1$ so $|Y| \leq m_0 + m_1$. We have $\theta_0^* + (|Y| - 1)\theta_i^* = 0$ so

$$\theta_i^* = \frac{-\theta_0^*}{|Y| - 1}.$$

We now consider $a_2^* = Q_2(0) + (|Y| - 1)Q_2(i)$. Recall that $Q_2(j) = v_2^*(\theta_j^*)$ for $0 \leq j \leq d$, where

$$v_2^*(z) = \frac{z^2 - q_{1,1}^1 z - \theta_0^*}{q_{1,1}^2}.$$

We have

$$\begin{aligned} q_{1,1}^2 a_2^* &= (\theta_0^*)^2 - q_{1,1}^1\theta_0^* - \theta_0^* + \big(|Y| - 1\big)\big((\theta_i^*)^2 - q_{1,1}^1\theta_i^* - \theta_0^*\big) \\ &= (\theta_0^*)^2 - \theta_0^* + \big(|Y| - 1\big)\big((\theta_i^*)^2 - \theta_0^*\big) \\ &= (\theta_0^*)^2 + \big(|Y| - 1\big)(\theta_i^*)^2 - |Y|\theta_0^* \end{aligned}$$

so

$$\begin{aligned} \big(|Y| - 1\big)q_{1,1}^2 a_2^* &= \big(|Y| - 1\big)(\theta_0^*)^2 + \big(|Y| - 1\big)^2(\theta_i^*)^2 - \big(|Y| - 1\big)|Y|\theta_0^* \\ &= \big(|Y| - 1\big)(\theta_0^*)^2 + (\theta_0^*)^2 - \big(|Y| - 1\big)|Y|\theta_0^* \\ &= |Y|(\theta_0^*)^2 - \big(|Y| - 1\big)|Y|\theta_0^* \\ &= |Y|\theta_0^*\big(\theta_0^* + 1 - |Y|\big) \\ &= |Y|\theta_0^*\big(m_0 + m_1 - |Y|\big). \end{aligned}$$

Therefore $|Y| \leq m_0 + m_1$, with equality if and only if $a_2^* = 0$ if and only if $t = 2$.

In summary, for $s = 1$ we have $|Y| \leq m_0 + m_1$, with equality if and only if $t = 2$.

We now consider the case of general s.

Theorem 22.1. *Let Y denote a nonempty subset of X with degree s. Then*

$$|Y| \leq m_0 + m_1 + \cdots + m_s. \tag{88}$$

Proof. Recall the standard module $V = \mathbb{R}^X$. The subspace $\sum_{i=0}^s E_iV$ has dimension $\sum_{i=0}^s m_i$. Define

$$E = \sum_{i=0}^s E_i.$$

For $y \in Y$ we have

$$E\hat{y} \in \sum_{i=0}^s E_iV.$$

It suffices to show that the vectors $\{E\hat{y}\}_{y\in Y}$ are linearly independent.

Define the set

$$S = \{i | 1 \le i \le d,\ a_i \ne 0\}.$$

We have $s = |S|$. Define the polynomial

$$f(z) = \prod_{i\in S} \frac{z-\theta_i^*}{\theta_0^*-\theta_i^*}.$$

We have $f(\theta_0^*) = 1$. Also for $1 \le j \le d$, $f(\theta_j^*) = 0$ if and only if $j \in S$. The polynomial $f(z)$ has degree s. Write

$$f(z) = \sum_{i=0}^{s} \gamma_i v_i^*(z) \qquad\qquad \gamma_i \in \mathbb{R},$$

where each $v_i^*(z)$ has degree i and $Q_i(j) = v_i^*(\theta_j^*)$ for $0 \le j \le d$. Define

$$F = |X| \sum_{i=0}^{s} \gamma_i E_i.$$

For $y \in Y$ we have

$$F\hat{y} \in \sum_{i=0}^{s} E_i V.$$

For $y, z \in Y$ we show that

$$\langle E\hat{y}, F\hat{z}\rangle = \delta_{y,z}. \tag{89}$$

Write $(y, z) \in R_k$. We have

$$\begin{aligned}\langle E\hat{y}, F\hat{z}\rangle &= |X| \sum_{i=0}^{s}\sum_{j=0}^{s} \gamma_j \langle E_i\hat{y}, E_j\hat{z}\rangle = |X| \sum_{i=0}^{s} \gamma_i \langle E_i\hat{y}, E_i\hat{z}\rangle \\ &= \sum_{i=0}^{s} \gamma_i Q_i(k) = \sum_{i=0}^{s} \gamma_i v_i^*(\theta_k^*) = f(\theta_k^*).\end{aligned}$$

If $y = z$, then $k = 0$ and $f(\theta_0^*) = 1$. If $y \ne z$, then $k \in S$ and $f(\theta_k^*) = 0$. By these comments we obtain (89).

We can now easily show that $\{E\hat{y}\}_{y\in Y}$ are linearly independent. Suppose we are given real numbers $\{\alpha_y\}_{y\in Y}$ such that

$$0 = \sum_{y\in Y} \alpha_y E\hat{y}.$$

We show that $\alpha_y = 0$ for $y \in Y$. For $y \in Y$,

$$0 = \sum_{z\in Y} \alpha_z \langle E\hat{z}, F\hat{y}\rangle = \sum_{z\in Y} \alpha_z \delta_{y,z} = \alpha_y.$$

We have shown the vectors $\{E\hat{y}\}_{y\in Y}$ are linearly independent. This implies the inequality (88). □

Lecture 31

(Adjusting the notes from Lecture 30 by inserting a few results) Our next general goal is to treat the case of equality in Theorem 22.1. We will make use of the following concepts.

Recall the standard module V.

Definition 22.2. We turn the vector space V into a commutative, associative, $\mathbb{R}$-algebra with multiplication $\circ$ defined as follows:

$$\hat{y} \circ \hat{z} = \delta_{y,z} \hat{y} \qquad\qquad y, z \in X. \tag{90}$$

The algebra V is isomorphic to the algebra of functions $X \to \mathbb{R}$. Motivated by this, we call the algebra V the *function algebra*.

In order to illustrate the multiplication $\circ$, let $v, w \in V$ and write

$$v = \sum_{y \in X} v_y \hat{y}, \qquad\qquad w = \sum_{y \in X} w_y \hat{y} \qquad\qquad v_y, w_y \in \mathbb{R}.$$

Then

$$v \circ w = \sum_{y \in X} v_y w_y \hat{y}.$$

Lemma 22.3. *For the function algebra V, the multiplicative identity is $\mathbf{1} = \sum_{y \in X} \hat{y}$.*

Proof. Routine. □

Lemma 22.4. *Fix $x \in X$ and write $T = T(x)$. For $v \in V$ and $0 \leq i \leq d$,*

$$A_i^* v = |X| E_i \hat{x} \circ v.$$

Proof. Write $v = \sum_{y \in X} v_y \hat{y}$. Pick $y \in X$. The y-coordinate of $A_i^* v$ is

$$(A_i^* v)_y = (A_i^*)_{y,y} v_y = |X| (E_i)_{x,y} v_y.$$

The y-coordinate of $E_i \hat{x} \circ v$ is

$$\big(E_i \hat{x} \circ v\big)_y = (E_i \hat{x})_y v_y = (E_i)_{y,x} v_y = (E_i)_{x,y} v_y.$$

The result follows. □

Proposition 22.5. *For $0 \leq i, j \leq d$,*

$$\operatorname{Span}\big(E_i V \circ E_j V\big) = \sum_{\substack{0 \leq k \leq d \\ q_{i,j}^k \neq 0}} E_k V. \tag{91}$$

Proof. We first establish the inclusion $\subseteq$. By construction $E_iV = \text{Span}\{E_i\hat{y}|y \in X\}$. We show that for $x \in X$,

$$E_i\hat{x} \circ E_jV \subseteq \sum_{\substack{0 \le k \le d \\ q^k_{i,j} \ne 0}} E_kV.$$

Write $T = T(x)$. We have

$$E_i\hat{x} \circ E_jV = A^*_i E_jV \subseteq \sum_{\substack{0 \le k \le d \\ q^k_{i,j} \ne 0}} E_kV.$$

Next we establish the inclusion $\supseteq$. For $0 \le k \le d$ such that $q^k_{i,j} \ne 0$, we show that $\text{Span}\big(E_iV \circ E_jV\big) \supseteq E_kV$. We have

$$\begin{aligned}
\text{Span}\big(E_iV \circ E_jV\big) &= \text{Span}\{E_i\hat{y} \circ E_j\hat{z}|y, z \in X\} \\
&\supseteq \text{Span}\{E_i\hat{y} \circ E_j\hat{y}|y \in X\} \\
&= \text{Span}\{(E_i \circ E_j)\hat{y}|y \in X\} \\
&= (E_i \circ E_j)V \\
&\supseteq (E_i \circ E_j)E_kV \\
&= \left(|X|^{-1} \sum_{\ell=0}^{d} q^\ell_{i,j} E_\ell\right) E_kV \\
&= |X|^{-1} q^k_{i,j} E_kV \\
&= E_kV.
\end{aligned}$$

□

We now consider the case of equality in Theorem 22.1.

Theorem 22.6. *Let Y denote a nonempty subset of X with degree s and stength t. Write $e = \lfloor t/2 \rfloor$. Then the following are equivalent:*

(i) $|Y| = \sum_{i=0}^{e} m_i$;

(ii) $|Y| = \sum_{i=0}^{s} m_i$;

(iii) $s = e$.

Assume (i)–(iii) *holds, and write $E = \sum_{i=0}^{s} E_i$. Then the vectors $\{E\hat{y}\}_{y \in Y}$ form a basis for $\sum_{i=0}^{s} E_iV$ such that*

$$\langle E\hat{y}, E\hat{z}\rangle = \delta_{y,z}\frac{|Y|}{|X|} \qquad (y, z \in Y).$$

Proof. (i) $\Rightarrow$ (iii) This is Theorem 20.2(iii).
(iii) $\Rightarrow$ (i), (ii) We have

$$m_0 + m_1 + \cdots + m_e \le |Y| \le m_0 + m_1 + \cdots + m_e$$

and hence equality thoughout.
(ii) $\Rightarrow$ (iii) It suffices to show that $t = 2s$. We adopt the notation from the proof of Theorem 22.1.

By the proof of Theorem 22.1, the vectors $\{E\hat{y}\}_{y\in Y}$ are linearly independent and contained in $\sum_{i=0}^{s} E_i V$. The subspace $\sum_{i=0}^{s} E_i V$ has dimension $\sum_{i=0}^{s} m_i = |Y|$. Therefore $\{E\hat{y}\}_{y\in Y}$ is a basis for $\sum_{i=0}^{s} E_i V$. Recall that

$$\langle E\hat{y}, F\hat{z}\rangle = \delta_{y,z} \qquad (y, z \in Y).$$

Therefore, the vectors $\{F\hat{y}\}_{y\in Y}$ form a basis for $\sum_{i=0}^{s} E_i V$ that is dual to the basis $\{E\hat{y}\}_{y\in Y}$. Let $H \in M_Y(\mathbb{R})$ denote the transition matrix from the basis $\{E\hat{y}\}_{y\in Y}$ to the basis $\{F\hat{y}\}_{y\in Y}$. For $z \in Y$,

$$F\hat{z} = \sum_{y\in Y} H_{y,z} E\hat{y}.$$

For $y, z \in Y$ we compute the (y, z)-entry of H. Write $(y, z) \in R_k$. We have

$$\langle F\hat{y}, F\hat{z}\rangle = \sum_{w\in Y} H_{w,z}\langle F\hat{y}, E\hat{w}\rangle = \sum_{w\in Y} H_{w,z}\delta_{y,w} = H_{y,z}.$$

Therefore

$$\begin{aligned} H_{y,z} = \langle F\hat{y}, F\hat{z}\rangle &= |X|^2 \sum_{i=0}^{s}\sum_{j=0}^{s} \gamma_i\gamma_j \langle E_i\hat{y}, E_j\hat{z}\rangle \\ &= |X|^2 \sum_{i=0}^{s} \gamma_i^2 \langle E_i\hat{y}, E_i\hat{z}\rangle = |X| \sum_{i=0}^{s} \gamma_i^2 Q_i(k) = |X| \sum_{i=0}^{s} \gamma_i^2 v_i^*(\theta_k^*). \end{aligned}$$

The matrix H^{-1} is the transition matrix from the basis $\{F\hat{y}\}_{y\in Y}$ to the basis $\{E\hat{y}\}_{y\in Y}$. For $z \in Y$,

$$E\hat{z} = \sum_{y\in Y} (H^{-1})_{y,z} F\hat{y}.$$

For $y, z \in Y$ we compute the (y, z)-entry of H^{-1}. Write $(y, z) \in R_k$. We have

$$\langle E\hat{y}, E\hat{z}\rangle = \sum_{w\in Y} (H^{-1})_{w,z}\langle E\hat{y}, F\hat{w}\rangle = \sum_{w\in Y} (H^{-1})_{w,z}\delta_{y,w} = (H^{-1})_{y,z}.$$

Therefore

$$\begin{aligned} (H^{-1})_{y,z} = \langle E\hat{y}, E\hat{z}\rangle &= \sum_{i=0}^{s}\sum_{j=0}^{s} \langle E_i\hat{y}, E_j\hat{z}\rangle \\ &= \sum_{i=0}^{s} \langle E_i\hat{y}, E_i\hat{z}\rangle = |X|^{-1} \sum_{i=0}^{s} Q_i(k) = |X|^{-1} \sum_{i=0}^{s} v_i^*(\theta_k^*). \end{aligned}$$

Recall the vector ψ_Y. Note that

$$\psi_Y = \sum_{i=0}^{d} E_i\psi_Y = E_0\psi_Y + \sum_{i=1}^{d} E_i\psi_Y = \frac{|Y|}{|X|}\mathbf{1} + \sum_{i=1}^{d} E_i\psi_Y.$$

Our goal is to show that $t = 2s$. It suffices to show that $a_i^* = 0$ for $1 \leq i \leq 2s$. It suffices to show that $E_i\psi_Y = 0$ for $1 \leq i \leq 2s$. It suffices to show that

$$\left\langle \psi_Y - \frac{|Y|}{|X|}\mathbf{1}, E_0V + E_1V + \cdots + E_{2s}V \right\rangle = 0. \tag{92}$$

By Proposition 22.5 and since the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial,

$$\begin{aligned} E_0V + E_1 + \cdots + E_{2s}V &= \text{Span}\big\{u \circ v | u, v \in E_0V + E_1V + \cdots + E_sV\big\} \\ &= \text{Span}\big\{E\hat{y} \circ F\hat{z} | y, z \in Y\big\}. \end{aligned}$$

By these comments, the requirement (92) becomes

$$\left\langle \psi_Y - \frac{|Y|}{|X|}\mathbf{1}, E\hat{y} \circ F\hat{z} \right\rangle = 0, \qquad\qquad y, z \in Y. \tag{93}$$

We will show (93). For $y, z \in Y$ we have

$$\langle \mathbf{1}, E\hat{y} \circ F\hat{z} \rangle = \langle E\hat{y}, F\hat{z} \rangle = \delta_{y,z}.$$

For $y, z \in Y$ we also have

$$\begin{aligned} \langle \psi_Y, E\hat{y} \circ F\hat{z} \rangle &= \sum_{w \in Y} \langle \hat{w}, E\hat{y} \circ F\hat{z} \rangle = \sum_{w \in Y} \langle E\hat{y}, \hat{w} \rangle \langle F\hat{z}, \hat{w} \rangle \\ &= \sum_{w \in Y} \langle E\hat{y}, E\hat{w} \rangle \langle F\hat{z}, E\hat{w} \rangle = \sum_{w \in Y} \langle E\hat{y}, E\hat{w} \rangle \delta_{z,w} = \langle E\hat{y}, E\hat{z} \rangle \\ &= |X|^{-1} \sum_{i=0}^{s} v_i^*(\theta_k^*), \qquad\qquad (y, z) \in R_k. \end{aligned}$$

With these comments in mind, we verify (93). Pick $y, z \in Y$. First assume that $y = z$. Then y, z satisfy (93) because

$$\langle \psi_Y, E\hat{y} \circ F\hat{z} \rangle = |X|^{-1} \sum_{i=0}^{s} v_i^*(\theta_0^*) = |X|^{-1} \sum_{i=0}^{s} m_i = \frac{|Y|}{|X|} = \frac{|Y|}{|X|} \big\langle \mathbf{1}, E\hat{y} \circ F\hat{z} \big\rangle.$$

Next assume that $y \neq z$. Write $(y, z) \in R_k$. In equation (93), the left-hand side is equal to

$$\langle \psi_Y, E\hat{y} \circ F\hat{z} \rangle = |X|^{-1} \sum_{i=0}^{s} v_i^*(\theta_k^*).$$

Therefore, the vertices y, z satisfy (93) provided that θ^*_k is a root of the polynomial $\sum_{i=0}^{s} v^*_i(z)$. By these comments, $t = 2s$ provided that θ^*_k is a root of $\sum_{i=0}^{s} v^*_i(z)$ for all $k \in S$. The polynomial $\sum_{i=0}^{s} v^*_i(z)$ has degree $s = |S|$. Recall that the polynomial $f(z) = \sum_{i=0}^{s} \gamma_i v^*_i(z)$ has roots $\{\theta^*_k\}_{k \in S}$. Therefore, $t = 2s$ provided that $f(z)$ and $\sum_{i=0}^{s} v^*_i(z)$ agree up to a scalar factor. In other words, $t = 2s$ provided that γ_i is independent of i for $0 \le i \le s$. We now compute $\{\gamma_i\}_{i=0}^{s}$. We have

$$1 = \sum_{i=0}^{s} \gamma_i m_i \tag{94}$$

because

$$1 = f(\theta^*_0) = \sum_{i=0}^{s} \gamma_i v^*_i(\theta^*_0) = \sum_{i=0}^{s} \gamma_i m_i.$$

If γ_i is independent of i for $0 \le i \le s$, then the common value must be $|Y|^{-1}$. Our next goal is to show

$$\gamma_i = |Y|^{-1} \qquad\qquad (0 \le i \le s). \tag{95}$$

For $0 \le i \le s$ we have $\gamma_i \ge 0$; otherwise the vectors $\{E'\hat{y}\}_{y \in Y}$ remain linearly independent, where $E' = E - E_i$. But these vectors are contained in the subspace $E_0V + \cdots + E_{i-1}V + E_{i+1}V + \cdots + E_sV$ of dimension $|Y| - m_i$, a contradiction. For notational convenience, define $\gamma_i = 0$ for $s + 1 \le i \le d$. For $0 \le i \le s$ we have

$$m_i(1 - |Y|\gamma_i) = \sum_{k=1}^{d} \sum_{j=0}^{d} \gamma_j q^k_{i,j} a^*_k, \tag{96}$$

which is obtained using the definition of the a^*_k. By (96) we obtain $\gamma_i \le |Y|^{-1}$ for $0 \le i \le s$. Combining this with (94) we obtain (95). We have shown (95), so $t = 2s$. We have

$$F = |X| \sum_{i=0}^{s} \gamma_i E_i = \frac{|X|}{|Y|} \sum_{i=0}^{s} E_i = \frac{|X|}{|Y|} E.$$

Recall that $\langle E\hat{y}, F\hat{z} \rangle = \delta_{y,z}$ for $y, z \in Y$. By these comments

$$\langle E\hat{y}, E\hat{z} \rangle = \delta_{y,z} \frac{|Y|}{|X|} \qquad\qquad (y, z \in Y).$$

□

Lecture 32

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ that is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents.

Until further notice, Y denotes a nonempty subset of X that has degree s and strength $t = 2s$.

Recall the inner distribution $\{a_i\}_{i=0}^d$ of Y. Recall the set $S = \{i | 1 \leq i \leq d,\ a_i \neq 0\}$. Recall that $|S| = s$. Reindexing the relations $\{R_i\}_{i=0}^d$ if necessary, we may assume without loss that

$$S = \{1, 2, \ldots, s\}.$$

Definition 22.7. For $0 \leq i \leq d$ let R_i^Y denote the restriction of R_i to $Y \times Y$. By construction, R_i^Y is nonempty if and only if $0 \leq i \leq s$.

Lemma 22.8. *The following hold:*

(i) $R_0^Y = \{(y, y) | y \in Y\}$*;*

(ii) *the relations* $\{R_i^Y\}_{i=0}^s$ *partition* $Y \times Y$*;*

(iii) R_i^Y *is symmetric for* $0 \leq i \leq s$*.*

Proof. By the construction. □

Our next general goal is to show that $\mathcal{Y} = (Y, \{R_i^Y\}_{i=0}^s)$ is a symmetric association scheme.

Definition 22.9. Let the matrix $Q^Y \in M_{s+1}(\mathbb{R})$ be the submatrix of Q associated with rows $0, 1, \ldots, s$ and columns $0, 1, \ldots, s$. So Q^Y has (i, j)-entry

$$Q_{i,j}^Y = Q_j(i) = v_j^*(\theta_i^*) \qquad\qquad (0 \leq i, j \leq s).$$

Lemma 22.10. *The matrix* Q^Y *is invertible.*

Proof. The polynomial $v_i^*(z)$ has degree i for $0 \leq i \leq s$. The scalars $\{\theta_j^*\}_{j=0}^s$ are mutually distinct. By these comments the matrix Q^Y is essentially Vandermonde, and hence invertible. □

The following example should clarify what is meant by essentially Vandermonde.

Example 22.11. Assume that $s = 2$. Then

$$Q^Y = \begin{pmatrix} 1 & \theta_0^* & \frac{(\theta_0^*)^2 - q_{1,1}^1 \theta_0^* - q_{1,1}^0}{q_{1,1}^2} \\ 1 & \theta_1^* & \frac{(\theta_1^*)^2 - q_{1,1}^1 \theta_1^* - q_{1,1}^0}{q_{1,1}^2} \\ 1 & \theta_2^* & \frac{(\theta_2^*)^2 - q_{1,1}^1 \theta_2^* - q_{1,1}^0}{q_{1,1}^2} \end{pmatrix}.$$

Via elementary column operations, we can reduce Q^Y to the Vandermonde matrix

$$\begin{pmatrix} 1 & \theta_0^* & (\theta_0^*)^2 \\ 1 & \theta_1^* & (\theta_1^*)^2 \\ 1 & \theta_2^* & (\theta_2^*)^2 \end{pmatrix}.$$

The above Vandermonde matrix is invertible. An elementary column operation changes the determinant by a nonzero scalar factor. Therefore Q^Y is invertible.

Definition 22.12. Define a matrix $P^Y \in M_{s+1}(\mathbb{R})$ such that

$$P^Y Q^Y = |Y| I.$$

For $0 \leq i, j \leq s$ the (i,j)-entry of P^Y is denoted $P_j^Y(i)$.

Lemma 22.13. *For $x \in Y$ and $0 \leq i \leq s$,*

$$|\Gamma_i(x) \cap Y| = a_i.$$

Proof. By definition, a_i is the average value of $|\Gamma_i(x) \cap Y|$, where the average is over all $x \in Y$. Therefore, It suffices to show that $|\Gamma_i(x) \cap Y|$ does not depend on the choice of x. Recall the vector ψ_Y. Recall that $E_j \psi_Y = 0$ for $1 \leq j \leq 2s$ and

$$E_0 \psi_Y = \frac{|Y|}{|X|} \mathbf{1}.$$

For $0 \leq j \leq 2s$ we have

$$\begin{aligned}
\delta_{0,j} \frac{|Y|}{|X|} &= \langle E_j \hat{x}, E_j \psi_Y \rangle = \sum_{y \in Y} \langle E_j \hat{x}, E_j \hat{y} \rangle = \sum_{k=0}^{s} \sum_{y \in \Gamma_k(x) \cap Y} \langle E_j \hat{x}, E_j \hat{y} \rangle \\
&= |X|^{-1} \sum_{k=0}^{s} \sum_{y \in \Gamma_k(x) \cap Y} Q_j(k) = |X|^{-1} \sum_{k=0}^{s} \sum_{y \in \Gamma_k(x) \cap Y} v_j^*(\theta_k^*) \\
&= |X|^{-1} \sum_{k=0}^{s} \big| \Gamma_k(x) \cap Y \big| v_j^*(\theta_k^*).
\end{aligned}$$

For notational convenience, define

$$z_k = \big| \Gamma_k(x) \cap Y \big| \qquad (0 \leq k \leq s).$$

By the above equations,

$$\big(z_0, z_1, \ldots, z_s \big) Q^Y = \big(|Y|, 0, \ldots, 0 \big).$$

Therefore

$$\big(z_0, z_1, \ldots, z_s \big) = \big(1, 0, \ldots, 0 \big) P^Y. \tag{97}$$

This shows that for $0 \leq i \leq s$ the number z_i does not depend on the choice of x. Therefore $z_i = a_i$ for $0 \leq i \leq s$. The result follows. □

We have some comments about P^Y.

Lemma 22.14. *The following hold for $0 \leq i \leq s$:*

(i) $P_0^Y(i) = 1$;

(ii) $P_i^Y(0) = a_i$.

Proof. (i) In the previous lecture we saw that

$$\frac{(z-\theta_1^*)(z-\theta_2^*)\cdots(z-\theta_s^*)}{(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)\cdots(\theta_0^*-\theta_s^*)} = \frac{v_0^*(z)+v_1^*(z)+\cdots+v_s^*(z)}{|Y|}.$$

Taking $z \in \{\theta_0^*, \theta_1^*, \ldots, \theta_s^*\}$ we obtain

$$Q^Y \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} |Y| \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = P^Y \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(ii) In the proof of Lemma 22.13 we obtained

$$\big(a_0, a_1, \ldots, a_s\big) = \big(1, 0, \ldots, 0\big) P^Y.$$

□

The following result will help us understand the combinatorial regularity of $\mathcal{Y}$.

Lemma 22.15. *Let $0 \le i, j, k \le s$ and $x, y \in Y$ with $(x, y) \in R_k$. Then*

$$\big\langle E_i\hat{x} \circ E_j\hat{y}, \psi_Y \big\rangle = \delta_{i,j} \frac{|Y|}{|X|^2} v_i^*(\theta_k^*).$$

Proof. We have

$$\begin{aligned} \big\langle E_i\hat{x} \circ E_j\hat{y}, \psi_Y \big\rangle &= \sum_{\ell=0}^{d} \sum_{h=0}^{d} \big\langle E_\ell(E_i\hat{x} \circ E_j\hat{y}), E_h\psi_Y \big\rangle \\ &= \sum_{\ell=0}^{d} \big\langle E_\ell(E_i\hat{x} \circ E_j\hat{y}), E_\ell\psi_Y \big\rangle. \end{aligned}$$

In the above sum, the ℓ-summand is zero for $1 \le \ell \le d$, because

$$\begin{aligned} &E_\ell(E_i\hat{x} \circ E_j\hat{y}) = 0 \qquad\qquad (2s+1 \le \ell \le d), \\ &E_\ell\psi_Y = 0 \qquad\qquad (1 \le \ell \le 2s). \end{aligned}$$

By these comments

$$\begin{aligned} \big\langle E_i\hat{x} \circ E_j\hat{y}, \psi_Y \big\rangle = \big\langle E_0(E_i\hat{x} \circ E_j\hat{y}), E_0\psi_Y \big\rangle &= \frac{|Y|}{|X|} \big\langle E_0(E_i\hat{x} \circ E_j\hat{y}), \mathbf{1} \big\rangle \\ = \frac{|Y|}{|X|} \big\langle E_i\hat{x} \circ E_j\hat{y}, E_0\mathbf{1} \big\rangle &= \frac{|Y|}{|X|} \big\langle E_i\hat{x} \circ E_j\hat{y}, \mathbf{1} \big\rangle \\ = \frac{|Y|}{|X|} \big\langle E_i\hat{x}, E_j\hat{y} \big\rangle = \delta_{i,j} \frac{|Y|}{|X|^2} Q_i(k) &= \delta_{i,j} \frac{|Y|}{|X|^2} v_i^*(\theta_k^*). \end{aligned}$$

□

Recall the Bose-Mesner algebra $\mathcal{M}$.

Definition 22.16. For $M \in \mathcal{M}$ let M^Y denote the restriction of M to $Y \times Y$. Define

$$\mathcal{M}^Y = \text{Span}\{M^Y | M \in \mathcal{M}\}.$$

By construction $\mathcal{M}^Y$ is a subspace of $M_Y(\mathbb{R})$. It will turn out that $\mathcal{M}^Y$ is a subalgebra of $M_Y(\mathbb{R})$.

We make an observation. For $0 \le i \le d$, $A_i^Y \neq 0$ if and only if $0 \le i \le s$.

Lemma 22.17. *Each of the following is a basis for the vector space* $\mathcal{M}^Y$*:*

$$\{A_i^Y\}_{i=0}^s, \qquad \{E_i^Y\}_{i=0}^s.$$

Moreover the following hold for $0 \le i \le s$*:*

$$E_i^Y = |X|^{-1} \sum_{j=0}^s Q_i(j) A_j^Y, \qquad A_i^Y = \frac{|X|}{|Y|} \sum_{j=0}^s P_i^Y(j) E_j^Y.$$

Proof. The matrices $\{A_i\}_{i=0}^d$ form a basis for $\mathcal{M}$, so the matrices $\{A_i^Y\}_{i=0}^d$ span $\mathcal{M}^Y$. We have $A_i^Y = 0$ for $s+1 \le i \le d$, so $\{A_i^Y\}_{i=0}^s$ span $\mathcal{M}^Y$. The matrices $\{A_i^Y\}_{i=0}^s$ are linearly independent, since their nonzero entries are in disjoint locations. Therefore $\{A_i^Y\}_{i=0}^s$ is a basis for $\mathcal{M}^Y$. The remaining assertions follow from the construction and Definition 22.12. □

Theorem 22.18. $\mathcal{Y} = (Y, \{R_i^Y\}_{i=0}^s)$ *is a symmetric association scheme.*

Proof. Recall Lemma 22.8. It remains to show that for $0 \le i, j, k \le s$ and $x, y \in Y$ with $(x, y) \in R_k$, the number

$$r_{i,j}^k = \left|\Gamma_i(x) \cap \Gamma_j(y) \cap Y\right|$$

is independent of the choice of x, y. We have

$$|\Gamma_i(x) \cap \Gamma_j(y) \cap Y| = \langle A_i \hat{x} \circ A_j \hat{y}, \psi_Y \rangle = \langle A_i^Y \hat{x} \circ A_j^Y \hat{y}, \psi_Y \rangle$$

$$= \frac{|X|^2}{|Y|^2} \sum_{\ell=0}^s \sum_{h=0}^s P_i^Y(\ell) P_j^Y(h) \langle E_\ell^Y \hat{x} \circ E_h^Y \hat{y}, \psi_Y \rangle$$

$$= \frac{|X|^2}{|Y|^2} \sum_{\ell=0}^s \sum_{h=0}^s P_i^Y(\ell) P_j^Y(h) \langle E_\ell \hat{x} \circ E_h \hat{y}, \psi_Y \rangle$$

$$= |Y|^{-1} \sum_{\ell=0}^s P_i^Y(\ell) P_j^Y(\ell) v_\ell^*(\theta_k^*).$$

The result follows. □

Corollary 22.19. *The algebra $\mathcal{M}^Y$ is the Bose-Mesner algebra of the association scheme $\mathcal{Y}$. Moreover*

$$A_i^Y A_j^Y = \sum_{k=0}^{s} r_{i,j}^k A_k^Y \qquad\qquad (0 \le i, j \le s).$$

Proof. By construction $\{A_i^Y\}_{i=0}^s$ are the associate matrices of $\mathcal{Y}$, and the $r_{i,j}^k$ are the intersection numbers of $\mathcal{Y}$. □

Note 22.20. It is known that the association scheme $\mathcal{Y}$ is Q-polynomial. See the paper

Sho Suda. New parameters of subsets in polynomial schemes.
`arXiv:1008.0189`.

Lecture 33

23 Relative t-designs

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ that is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents.

Until further notice, Y denotes a nonempty subset of X.

Recall the inner distribution $\{a_i\}_{i=0}^d$ of Y and the dual distribution $\{a_i^*\}_{i=0}^d$ of Y. Recall that for $0 \le t \le d$, the set Y is a t-design whenever the strength of Y is at least t. This occurs if and only if $a_i^* = 0$ for $1 \le i \le t$, if and only if $E_i \psi_Y = 0$ for $1 \le i \le t$.

We now introduce the notion of a relative t-design.

Until further notice, fix $x \in X$ and write $T = T(x)$.

Definition 23.1. For $0 \le t \le d$, we call Y a *relative t-design* with respect to x, whenever the vectors $E_i \psi_Y$ and $E_i \hat{x}$ are linearly dependent for $1 \le i \le t$.

Note 23.2. The subset Y is always a relative 0-design with respect to x, because $E_0 \psi_Y$ and $E_0 \hat{x}$ are both scalar multiples of $\mathbf{1}$.

We now investigate the linear dependencies in Definition 23.1. Recall that

$$\|E_i \psi_Y\|^2 = \frac{|Y|}{|X|} a_i^*, \qquad \|E_i \hat{x}\|^2 = |X|^{-1} m_i \qquad\qquad (0 \le i \le d).$$

Lemma 23.3. *For $0 \le i \le d$,*

$$\langle E_i \psi_Y, E_i \hat{x} \rangle = |X|^{-1} \sum_{\ell=0}^{d} Q_i(\ell) |\Gamma_\ell(x) \cap Y|.$$

Proof. We have

$$\langle E_i\psi_Y, E_i\hat{x}\rangle = \langle \psi_Y, E_i^2\hat{x}\rangle = \langle \psi_Y, E_i\hat{x}\rangle$$
$$= |X|^{-1}\sum_{\ell=0}^{d} Q_i(\ell)\langle \psi_Y, A_\ell\hat{x}\rangle = |X|^{-1}\sum_{\ell=0}^{d} Q_i(\ell)|\Gamma_\ell(x)\cap Y|.$$

□

Lemma 23.4. *For $0 \le i \le d$ we have*

$$a_i^* \ge \frac{\left(\sum_{\ell=0}^{d} Q_i(\ell)|\Gamma_\ell(x)\cap Y|\right)^2}{|Y|m_i}, \tag{98}$$

with equality if and only if $E_i\psi_Y$, $E_i\hat{x}$ are linearly dependent. In this case

$$E_i\psi_Y = \frac{\langle E_i\psi_Y, E_i\hat{x}\rangle}{\|E_i\hat{x}\|^2}E_i\hat{x}. \tag{99}$$

Proof. Compute the inner product matrix

$$\begin{pmatrix} \|E_i\psi_Y\|^2 & \langle E_i\psi_Y, E_i\hat{x}\rangle \\ \langle E_i\hat{x}, E_i\psi_Y\rangle & \|E_i\hat{x}\|^2 \end{pmatrix}$$

using Lemma 23.3 and the comments above it. The inner product matrix is positive semidefinite, so the determinate is nonnegative. The determinate is zero if and only if $E_i\psi_Y$, $E_i\hat{x}$ are linearly dependent. In this case, the dependency (99) is readily computed. □

Corollary 23.5. *For $1 \le t \le d$ the following are equivalent:*

(i) *Y is a relative t-design with respect to x;*

(ii) *equality holds in* (98) *for $1 \le i \le t$.*

Proof. By Definition 23.1 and Lemma 23.4. □

Theorem 23.6. *Define*

$$s = \left|\{k|1 \le k \le d,\ \Gamma_k(x)\cap Y \ne \emptyset\}\right|.$$

Assume that Y is a t-design with $t+1 \ge s$. Then each nonempty $\Gamma_k(x)\cap Y$ is a relative $(t+1-s)$-design with respect to x.

Proof. Define the set

$$S = \{k|1 \le k \le d,\ \Gamma_k(x)\cap Y \ne \emptyset\}.$$

Note that $s = |S|$. For $0 \le k \le d$,

$$E_k^*\psi_Y = \sum_{y\in\Gamma_k(x)\cap Y}\hat{y}.$$

For $1 \le k \le d$, $E_k^*\psi_Y \ne 0$ if and only if $k \in S$.

It suffices to show that for $k \in S$ the vectors

$$E_i E_k^* \psi_Y, \qquad E_i \hat{x}$$

are linearly dependent for $1 \le i \le t+1-s$.

We will be discussing the vector $E_0^*\psi_Y$. Note that $E_0^*\psi_Y = \hat{x}$ if $x \in Y$, and $E_0^*\psi_Y = 0$ if $x \notin Y$. Consider the following two sets of vectors:

(i) $\{E_k^*\psi_Y\}_{k\in S} \cup \{E_0^*\psi_Y\}$;

(ii) $\{E_i\hat{x} \circ \psi_Y\}_{i=0}^{s-1} \cup \{E_0^*\psi_Y\}$.

We claim that the above sets (i), (ii) have the same span. To prove the claim, note that for $0 \le i \le s-1$,

$$\begin{aligned} E_i\hat{x} \circ \psi_Y &= |X|^{-1} \sum_{k=0}^{d} Q_i(k) A_k \hat{x} \circ \psi_Y = |X|^{-1} \sum_{k=0}^{d} Q_i(k) E_k^* \psi_Y \\ &= |X|^{-1} \left(m_i E_0^* \psi_Y + \sum_{k\in S} Q_i(k) E_k^* \psi_Y \right). \end{aligned}$$

Consider the $s \times s$ submatrix of Q that has rows indexed by S and columns indexed by $\{0, 1, 2, \ldots, s-1\}$. This submatrix is essentially Vandermonde, and hence invertible. The claim follows.

For $k \in S$,

$$E_k^*\psi_Y \in \mathrm{Span}\{E_i\hat{x} \circ \psi_Y\}_{i=0}^{s-1} + \mathrm{Span}\{\hat{x}\}.$$

Also for $0 \le i \le s-1$,

$$\begin{aligned} E_i\hat{x} \circ \psi_Y &= E_i\hat{x} \circ \left(\psi_Y - \frac{|Y|}{|X|}\mathbf{1} \right) + \frac{|Y|}{|X|} E_i\hat{x} \circ \mathbf{1} \\ &= E_i\hat{x} \circ \left(\psi_Y - \frac{|Y|}{|X|}\mathbf{1} \right) + \frac{|Y|}{|X|} E_i\hat{x}. \end{aligned}$$

We saw earlier that

$$\psi_Y - \frac{|Y|}{|X|}\mathbf{1} \in E_{t+1}V + \cdots + E_dV.$$

So for $0 \le i \le s-1$,

$$\begin{aligned} E_i\hat{x} \circ \left(\psi_Y - \frac{|Y|}{|X|}\mathbf{1} \right) &\in E_{t+1-i}V + \cdots + E_dV \\ &\subseteq E_{t+2-s}V + \cdots + E_dV. \end{aligned}$$

The Bose-Mesner algebra $\mathcal{M}$ has basis $\{E_i\}_{i=0}^d$. The primary T-module $\mathcal{M}\hat{x}$ has basis $\{E_i\hat{x}\}_{i=0}^d$. Let $k \in S$. By the above comments,

$$E_k^* \psi_Y \in E_{t+2-s}V + \cdots + E_d V + \mathcal{M}\hat{x}.$$

So for $1 \le i \le t+1-s$,

$$E_i E_k^* \psi_Y \in E_i \mathcal{M}\hat{x} = \text{Span}\{E_i \hat{x}\}.$$

Therefore, the vectors

$$E_i E_k^* \psi_Y, \qquad E_i \hat{x}$$

are linearly dependent. The result follows. $\square$

Lemma 23.7. *Let s denote the degree of Y. Assume that Y is a t-design with $t+1 \ge s$. Then for $x \in Y$ and $0 \le k \le d$,*

$$\left|\Gamma_k(x) \cap Y\right| = a_k.$$

Proof. In Lemma 22.13 we proved this under the assumption that $t = 2s$. However we did not use the full strength of that assumption. We only used $t+1 \ge s$. $\square$

Recall the norm $\|A\|^2 = \text{tr}(A^t A)$ for $A \in M_X(\mathbb{R})$.

Lemma 23.8. *For $0 \le k, \ell \le d$ we have*

$$\Big\| |X| E_k \Delta_Y E_\ell - |Y| \delta_{k,\ell} E_k \Big\|^2 = |Y| \sum_{j=1}^d q_{k,\ell}^j a_j^*, \tag{100}$$

where Δ_Y is the diagonal matrix in $M_X(\mathbb{R})$ with (y,y)-entry 1 if $y \in Y$ and 0 if $y \notin Y$ $(y \in X)$.

Proof. Routine using

$$\psi_Y^t E_j \psi_Y = \frac{|Y|}{|X|} a_j^* \qquad (0 \le j \le d)$$

and

$$q_{k,\ell}^0 = \delta_{k,\ell} m_\ell$$

and

$$(E_k)_{y,y} = |X|^{-1} m_k \qquad (y \in X).$$

Write $\Delta = \Delta_Y$. First assume that $k \neq \ell$. We have

$$\begin{aligned}
\Big\| |X| E_k \Delta E_\ell \Big\|^2 &= |X|^2 \mathrm{tr}\Big(\big(E_k \Delta E_\ell\big)^t E_k \Delta E_\ell \Big) \\
&= |X|^2 \mathrm{tr}\big(E_\ell \Delta E_k^2 \Delta E_\ell \big) \\
&= |X|^2 \mathrm{tr}\big(\Delta E_k \Delta E_\ell \big) \\
&= |X|^2 \sum_{y \in X} \sum_{z \in X} \Delta_{y,y} (E_k)_{y,z} \Delta_{z,z} (E_\ell)_{z,y} \\
&= |X|^2 \sum_{y \in Y} \sum_{z \in Y} (E_k)_{y,z} (E_\ell)_{z,y} \\
&= |X|^2 \sum_{y \in Y} \sum_{z \in Y} (E_k)_{y,z} (E_\ell)_{y,z} \\
&= |X|^2 \sum_{y \in Y} \sum_{z \in Y} (E_k \circ E_\ell)_{y,z} \\
&= |X|^2 \psi_Y^t \big(E_k \circ E_\ell \big) \psi_Y \\
&= |X| \sum_{j=1}^{d} q_{k,\ell}^j \psi_Y^t E_j \psi_Y \\
&= |Y| \sum_{j=1}^{d} q_{k,\ell}^j a_j^*.
\end{aligned}$$

The result is proved for $k \neq \ell$. The proof for $k = \ell$ is similar. □

Corollary 23.9. *For* $0 \leq t \leq d$ *the following are equivalent:*

(i) Y *is a* t*-design;*

(ii) *for* $k, \ell \geq 0$ *such that* $k + \ell \leq t$,

$$|X| E_k \Delta_Y E_\ell = |Y| \delta_{k,\ell} E_k.$$

Proof. Routine using Lemma 23.8. □

Let s denote the degree of Y. For notational convenience, assume that $a_i \neq 0$ for $1 \leq i \leq s$.

Theorem 23.10. *Assume that* Y *is a* t*-design with* $t \geq 2s - 2$. *Then the following* (i)–(iv) *hold.*

(i) $\mathcal{Y} = (Y, \{R_i^Y\}_{i=0}^s)$ *is a symmetric association scheme.*

(ii) $\mathcal{Y}$ *has primitive idempotents* cE_i^Y $(0 \leq i \leq s-1)$ *and* $I - c\sum_{i=0}^{s-1} E_i^Y$, *where* $c = |X|/|Y|$.

(iii) $\mathcal{Y}$ *is* Q*-polynomial with respect to the above ordering of the primitive idempotents.*

(iv) *Assume that* $t = 2s$. *Then* $I - c\sum_{i=0}^{s-1} E_i^Y = cE_s^Y$.

Proof. (i), (ii) By Corollary 23.9 we find that cE_i^Y $(0 \le i \le s-1)$ and $I - c\sum_{i=0}^{s-1} E_i^Y$ are mutually orthogonal idempotents. These are linearly independent and contained in $\mathcal{M}^Y$. They must form a basis for $\mathcal{M}^Y$, because $\mathcal{M}^Y$ has dimension $s+1$. By these comments the subspace $\mathcal{M}^Y$ is closed under matrix multiplication. Therefore $\mathcal{Y}$ is a symmetric association scheme.
(iii) By the construction and since $\mathcal{X}$ is Q-polynomial with respect to $\{E_i\}_{i=0}^d$.
(iv) We saw earlier that $c\sum_{i=0}^{s} E_i^Y = I^Y$. □

Lecture 34

24 Linear programming in the hypercube

We return our attention to the hypercube $H(d, 2)$. When we first introduced linear programming, we considered an example involving the orthogonality graph Ω_d. For $d = 4$ we worked out the solution by brute force. In this section we give the solution for all d.

Let X denote the vertex set of $H(d, 2)$. Recall that $|X| = 2^d$. Recall the bipartition $X = X^+ \cup X^-$. Note that each of $X^\pm$ has size $|X|/2 = 2^{d-1}$.

We now recall the orthogonality graph.

Definition 24.1. For even $d = 2t \ge 2$, the *orthogonality graph* Ω_d has vertex set X; vertices y, z are adjacent in Ω_d whenever $(y, z) \in R_t$ in $H(d, 2)$. A set of vertices $Y \subseteq X$ is called *independent* in Ω_d whenever no two vertices in Y are adjacent in Ω_d.

Problem 24.2. Find the maximal size of an independent set in Ω_d.

The above problem is easily solved for t odd, and much harder for t even. Let us first dispense with the case of t odd.

Lemma 24.3. *Assume that t is odd.*

(i) 2^{d-1} *is the maximum size of an independent set in* Ω_d*;*

(ii) *each of* $X^\pm$ *is an independent set of size* 2^{d-1}*;*

(iii) *there is no other independent set in* Ω_d *of size* 2^{d-1}*.*

Proof. The sets $X^\pm$ are independent in Ω_d, because t is odd and for $x, y \in X^\pm$ we have $(x, y) \in R_k$ with k even. Assume $Y \subseteq X$ is independent in Ω_d. We show $|Y| \le 2^{d-1}$, with equality if and only if $Y = X^\pm$. Let $\overline{Y} = X \backslash Y$. Note that $Y = X^\pm$ if and only if $\overline{Y}$ is independent in Ω_d. The graph Ω_d is regular; let κ denote the valency. We count the edges in Ω_d between Y and $\overline{Y}$. Since Y is independent, every vertex in Y is adjacent to exactly κ vertices in $\overline{Y}$. Therefore the edge count is $|Y|\kappa$. Each vertex in $\overline{Y}$ is adjacent to at most κ vertices in Y. Therefore the edge count is at most $|\overline{Y}|\kappa$, with equality iff $\overline{Y}$ is independent in Ω_d. By these comments $|Y|\kappa \le |\overline{Y}|\kappa$, with equality iff $\overline{Y}$ is independent in Ω_d. Therefore $|Y| \le |\overline{Y}|$, with equality iff $\overline{Y}$ is independent in Ω_d. Therefore $|Y| \le 2^{d-1}$, with equality iff $\overline{Y}$ is independent in Ω_d. The result follows. □

For the rest of this section, we assume that t is even. In this case, the above Problem 24.2 is open, so we consider the following related problem.

Problem 24.4. Use linear programming to find an upper bound on the size of an independent set in Ω_d.

We will prove the following result.

Theorem 24.5. *For t even, the linear programming upper bound is $2^d/d$ for the size of an independent set in Ω_d.*

We recall some facts about $H(d, 2)$. The intersection numbers are

$$c_i = i, \qquad b_i = d - i \qquad (0 \le i \le d). \tag{101}$$

The valencies are

$$k_i = \binom{d}{i} \qquad (0 \le i \le d). \tag{102}$$

The eigenvalues and dual eigenvalues are

$$\theta_i = d - 2i, \qquad \theta^*_i = d - 2i \qquad (0 \le i \le d). \tag{103}$$

The eigenmatrices P and Q satisfy $P = Q$. Their entries are given by

$$P_i(j) = Q_i(j) = K_i(j) \qquad (0 \le i, j \le d), \tag{104}$$

where $\{K_i\}_{i=0}^d$ are the Krawtchouk polynomials. For $0 \le j \le d$ we have

$$K_0(j) = 1, \qquad K_1(j) = d - 2j, \qquad K_2(j) = \frac{(d - 2j)^2 - d}{2}. \tag{105}$$

The Krawtchouk polynomial generating function is

$$\sum_{i=0}^{d} K_i(j) z^i = (1 - z)^j (1 + z)^{d-j} \qquad (0 \le j \le d). \tag{106}$$

The Krawtchouk polynomials satisfy

$$\frac{K_i(j)}{k_i} = \frac{K_j(i)}{k_j} \qquad (0 \le i, j \le d). \tag{107}$$

Lemma 24.6. *We have*

$$K_2(i) = \frac{\theta_i^2 - d}{2} \qquad K_i(2) = \frac{\binom{d}{i}}{\binom{d}{2}} K_2(i) \qquad (0 \le i \le d).$$

Proof. By (105) and (107). □

Recall $d = 2t$ with t even.

Lemma 24.7. *The following hold for $0 \leq i \leq d$.*

(i) *Assume i is odd. Then $K_i(t) = K_t(i) = 0$.*

(ii) *Assume $i = 2\ell$ is even. Then*

$$K_i(t) = (-1)^\ell \binom{t}{\ell}, \qquad K_t(i) = (-1)^\ell \binom{d}{t} \frac{(2\ell-1)(2\ell-3)\cdots 3 \cdot 1}{(d-1)(d-3)\cdots(d-2\ell+1)}$$

Proof. To obtain $K_i(t)$ we use the generating function. We have

$$\sum_{i=0}^{d} K_i(t) z^i = (1-z)^t (1+z)^t = (1-z^2)^t = \sum_{\ell=0}^{t} (-1)^\ell \binom{t}{\ell} z^{2\ell}.$$

To obtain $K_t(i)$ from $K_i(t)$ we use (107). $\square$

We are now ready to apply linear programming with

$$D = \{0, 1, \ldots, d\}, \qquad M = D \backslash \{t\}, \qquad C = Q.$$

Lemma 24.8. *The following is a program for Problem (Q, M):*

$$a_i = \frac{1}{d}\binom{d}{i} + \frac{d-1}{d} K_i(2) = \binom{d}{i} \frac{\theta_i^2}{d^2} \qquad (i \in M). \tag{108}$$

For this program the objective function is $g = 2^d/d$. Moreover

$$a_2^* = \frac{(d-1)2^d}{d}, \qquad a_i^* = 0 \qquad (1 \leq i \leq d, i \neq 2),$$

where

$$a_j^* = \sum_{i \in M} a_i Q_j(i) \qquad (1 \leq j \leq d). \tag{109}$$

Proof. By construction, $a_0 = 1$ and $a_i \geq 0$ for $i \in M^\times$. For notational convenience, define $a_t = 0$. Note that (108) holds at $i = t$ because $\theta_t = 0$. Define

$$\big(a_0^*, a_1^*, \ldots, a_d^*\big) = \big(a_0, a_1, \ldots, a_d\big) Q.$$

Note that

$$\big(a_0, a_1, \ldots, a_d\big) = \frac{1}{d}\Big(\text{row 0 of } P\Big) + \frac{d-1}{d}\Big(\text{row 2 of } P\Big).$$

Therefore

$$\begin{aligned}
\big(a_0^*, a_1^*, \ldots, a_d^*\big) &= \big(a_0, a_1, \ldots, a_d\big) Q \\
&= \frac{1}{d}\Big(\text{row 0 of } P\Big) Q + \frac{d-1}{d}\Big(\text{row 2 of } P\Big) Q \\
&= \frac{1}{d}\Big(\text{row 0 of } PQ\Big) + \frac{d-1}{d}\Big(\text{row 2 of } PQ\Big) \\
&= \frac{|X|}{d}\Big(\text{row 0 of } I\Big) + \frac{(d-1)|X|}{d}\Big(\text{row 2 of } I\Big) \\
&= \Big(\frac{2^d}{d}, 0, \frac{(d-1)2^d}{d}, 0, 0, \ldots, 0\Big).
\end{aligned}$$

Note that $a_j^* \geq 0$ for $1 \leq j \leq d$. By these comments $\{a_i\}_{i \in M}$ is a program for Problem (Q, M). Note that $g = a_0^* = 2^d/d$. The result follows. $\square$

Lemma 24.9. *The following is a program for Problem $(Q, M)'$: for $0 \leq i \leq d$,*

$$\alpha_i = \frac{1}{d} + \frac{d-1}{d} \frac{K_t(i)}{\binom{d}{t}} = \begin{cases} 1/d & \text{if } i \text{ is odd;} \\ 1/d + (-1)^\ell \frac{d-1}{d} \frac{(2\ell-1)(2\ell-3)\cdots 3 \cdot 1}{(d-1)(d-3)\cdots(d-2\ell+1)} & \text{if } i = 2\ell \text{ is even} \end{cases} \tag{110}$$

For this program the objective function is $\gamma = 2^d/d$. Moreover

$$\alpha_j^* = 0 \qquad (j \in M^\times),$$

where

$$\alpha_j^* = \sum_{i \in D} \alpha_i Q_i(j) \qquad (j \in M^\times). \tag{111}$$

Proof. Using (110), we find $\alpha_0 = 1$ and $\alpha_i \geq 0$ for $1 \leq i \leq d$. Define

$$\big(\alpha_0^*, \alpha_1^*, \ldots, \alpha_d^*\big) = \big(\alpha_0, \alpha_1, \ldots, \alpha_d\big) Q^t.$$

By (110),

$$\big(\alpha_0, \alpha_1, \ldots, \alpha_d\big) = \frac{1}{d}\Big(\text{row } 0 \text{ of } P^t\Big) + \frac{d-1}{d} \frac{1}{\binom{d}{t}} \Big(\text{row } t \text{ of } P^t\Big).$$

Therefore

$$\begin{aligned}
\big(\alpha_0^*, \alpha_1^*, \ldots, \alpha_d^*\big) &= \big(\alpha_0, \alpha_1, \ldots, \alpha_d\big) Q^t \\
&= \frac{1}{d}\Big(\text{row } 0 \text{ of } P^t\Big) Q^t + \frac{d-1}{d} \frac{1}{\binom{d}{t}} \Big(\text{row } t \text{ of } P^t\Big) Q^t \\
&= \frac{1}{d}\Big(\text{row } 0 \text{ of } P^t Q^t\Big) + \frac{d-1}{d} \frac{1}{\binom{d}{t}} \Big(\text{row } t \text{ of } P^t Q^t\Big) \\
&= \frac{|X|}{d}\Big(\text{row } 0 \text{ of } I\Big) + \frac{(d-1)|X|}{d\binom{d}{t}} \Big(\text{row } t \text{ of } I\Big) \\
&= \Big(\frac{2^d}{d}, 0, \ldots, 0, \frac{(d-1)2^d}{d\binom{d}{t}}, 0, \ldots, 0\Big).
\end{aligned}$$

This shows that $\alpha_j^* = 0$ unless $j = t$ $(1 \leq j \leq d)$. Therefore $\alpha_j^* = 0$ for $j \in M^\times$. Consequently $\alpha_j^* \leq 0$ for $j \in M^\times$. By these comments $\{\alpha_i\}_{i=0}^d$ is a program for Problem $(Q, M)'$. Note that $\gamma = \alpha_0^* = 2^d/d$. The result follows. $\square$

We displayed a program for Problem (Q, M) and a program for Problem $(Q, M)'$ such that $g = 2^d/d = \gamma$. Therefore, every program for Problem (Q, M) has objective function at most $2^d/d$. Consequently, an independent set in Ω_d has cardinality at most $2^d/d$. Theorem 24.5 is proved.

Lecture 35

25 Some open problems

In this section we give some open problems related to association schemes and graph theory in general. These problems are at the research level; an elegant solution or substantial progress is surely publishable. The problems are in a raw form; feel free to adjust any given problem into a more elegant or suitable form.

All the graphs discussed in this section are assumed to be finite, undirected, and connected, without loops or multiple edges. Fix a finite set X with $|X| \geq 2$.

We motivate the first problem with some comments about association schemes. Let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denote a symmetric association scheme with primitive idempotents $\{E_i\}_{i=0}^d$. Recall the standard module $V = \mathbb{R}^X$ and the function algebra product $\circ : V \times V \to V$. Recall that for $0 \leq i, j \leq d$,

$$\text{Span}\big(E_i V \circ E_j V\big) = \sum_{\substack{0 \leq k \leq d \\ q^k_{i,j} \neq 0}} E_k V.$$

Assume that $\mathcal{X}$ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$. Then $E_1 V$ generates V in the function algebra. Moreover

$$E_1 V \circ E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V \qquad (0 \leq i \leq d),$$

where $E_{-1} = 0$ and $E_{d+1} = 0$. For $0 \leq i \leq d$ define

$$(E_1 V)^{\circ i} = \text{Span}\big(E_1 V \circ E_1 V \circ \cdots \circ E_1 V\big) \qquad (i \text{ copies}).$$

We interpret $(E_1 V)^{\circ 0} = E_0 V = \text{Span}(\mathbf{1})$, where $\mathbf{1} = \sum_{y \in X} \hat{y}$. We have

$$\sum_{\ell=0}^{i} E_\ell V = \sum_{\ell=0}^{i} (E_1 V)^{\circ \ell} \qquad (0 \leq i \leq d).$$

We are done with the motivation. Now let $\Gamma = (X, \mathcal{R})$ denote any graph with vertex set X and adjacency relation $\mathcal{R}$. Let $A \in M_X(\mathbb{R})$ denote the adjacency matrix of Γ. We assume that Γ is regular with valency k; thus each vertex in X is adjacent to exactly k vertices in X. In this case k is the maximal eigenvalue of A, and the corresponding eigenspace is spanned by $\mathbf{1}$. We denote this eigenspace by V_0 and call it trivial. Let $\{V_i\}_{i=1}^D$ denote an ordering of the nontrivial eigenspaces of A.

Definition 25.1. The above ordering $\{V_i\}_{i=0}^D$ is called *Q-polynomial* whenever

$$\sum_{\ell=0}^{i} V_\ell = \sum_{\ell=0}^{i} (V_1)^{\circ \ell} \qquad (0 \leq i \leq D).$$

Definition 25.2. A graph Γ is said to be *Q-polynomial* whenever Γ is regular, and there exists at least one Q-polynomial ordering of its eigenspaces.

Example 25.3. Assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a Q-polynomial association scheme. Let S denote a subset of $\{1, 2, \ldots, d\}$ such that $A = \sum_{i \in S} A_i$ generates the Bose-Mesner algebra of $\mathcal{X}$. Then for the relation $\mathcal{R} = \cup_{i \in S} R_i$ the graph $(X, \mathcal{R})$ is Q-polynomial.

Conjecture 25.4. Let Γ denote a Q-polynomial graph. Let $\mathcal{M}$ denote the subalgebra of $M_X(\mathbb{R})$ generated by the adjacency matrix A of Γ. Then $\mathcal{M}$ is the Bose-Mesner algebra of a Q-polynomial association scheme.

We now describe the Q-polynomial property from another point of view.

Definition 25.5. Let $\Gamma = (X, \mathcal{R})$ denote a regular graph. Let U denote a nontrivial eigenspace of Γ. We say that Γ is *Q-polynomial with respect to U* whenever there exists a Q-polynomial ordering $\{V_i\}_{i=0}^D$ of the eigenspaces of Γ such that $V_1 = U$.

Let $\Gamma = (X, \mathcal{R})$ denote a graph. Recall the bilinear form $\langle\, , \rangle : V \times V \to \mathbb{R}$ such that $\langle u, v \rangle = u^t v$ for all $u, v \in V$.

Definition 25.6. Assume that Γ is regular, and let U denote a nontrivial eigenspace of Γ that generates V in the function algebra. Define the integer

$$D = \min\Big\{ i \Big| i \geq 0,\ \sum_{\ell=0}^{i} U^{\circ \ell} = V \Big\}.$$

Next, we recursively define some subspaces $\{U^{(i)}\}_{i=0}^D$ of V. Define $U^{(0)} = V_0$ and $U^{(1)} = U$. For $2 \leq i \leq D$ define $U^{(i)}$ to be the orthogonal complement of $U^{(0)} + U^{(1)} + \cdots + U^{(i-1)}$ in $\sum_{\ell=0}^{i} U^{\circ \ell}$.

Referring to Definition 25.6, for $0 \leq i \leq D$ we have

$$\sum_{\ell=0}^{i} U^{(\ell)} = \sum_{\ell=0}^{i} U^{\circ \ell},$$

with the sum on the left being orthogonal and direct. In particular

$$V = U^{(0)} + U^{(1)} + \cdots + U^{(D)} \qquad \text{(orthogonal direct sum)}.$$

Lemma 25.7. *Let U denote a nontrivial eigenspace of a regular graph $\Gamma = (X, \mathcal{R})$. Then the following are equivalent:*

(i) Γ *is Q-polynomial with respect to U;*

(ii) U *generates V in the function algebra, and $U^{(i)}$ is an eigenspace of* Γ *for* $0 \leq i \leq D$.

Proof. By the construction above the lemma, and since the eigenspaces of Γ are mutually orthogonal. $\square$

In order to clarify things, let us consider a special case. Referring to Lemma 25.7, assume that each eigenspace of Γ has dimension one. Let the vector $u = \sum_{x \in X} u_x \hat{x}$ be a basis for U. Consider the vectors

$$\sum_{x \in X} (u_x)^i \hat{x} \qquad (0 \le i \le D). \tag{112}$$

Applying the Gram-Schmidt orthogonalization process to the vectors (112), we obtain a sequence of polynomials $f_i \in \mathbb{R}[\lambda]$ $(0 \le i \le D)$ such that f_i has degree i and the vector $\sum_{x \in X} f_i(u_x)\hat{x}$ is a basis for $U^{(i)}$ $(0 \le i \le D)$. The polynomials $\{f_i\}_{i=0}^D$ are orthogonal; the orthogonality is

$$\sum_{x \in X} f_i(u_x) f_j(u_x) = \begin{cases} 0 & \text{if } i \ne j; \\ \ne 0 & \text{if } i = j \end{cases} \qquad (0 \le i, j \le D).$$

By the theory of orthogonal polynomials,

$$\lambda f_i \in \text{Span}\{f_{i-1}, f_i, f_{i+1}\} \qquad (0 \le i \le D),$$

where $f_{-1} = 0$ and $f_{D+1} = \prod_{x \in X}(\lambda - u_x)$. In summary we have the following result.

Lemma 25.8. *Assume that Γ is regular, and every eigenspace of Γ has dimension one. Let U denote a nontrivial eigenspace of Γ, with basis $u = \sum_{x \in X} u_x \hat{x}$. Then the following are equivalent:*

(i) Γ *is Q-polynomial with respect to U;*

(ii) *there exists a sequence of orthogonal polynomials $\{f_i\}_{i=0}^D$ such that*

$$\sum_{x \in X} f_i(u_x)\hat{x}$$

is an eigenvector of Γ for $0 \le i \le D$.

Problem 25.9. Hunt for some Q-polynomial graphs that have all eigenspaces of dimension one.

We now consider a related problem. For this problem we view $V = \mathbb{C}^X$.

Definition 25.10. Let $\Gamma = (X, \mathcal{R})$ denote a graph. Let $\Phi \in \mathbb{C}[\lambda, \mu]$ denote a polynomial in two variables. By a *Φ-hyper-eigenvector* for Γ we mean a vector $u = \sum_{x \in X} u_x \hat{x} \in V$ such that for all $x \in X$ the multiset of scalars $\{u_y\}_{y \in \Gamma(x)}$ gives all the roots of the polynomial $\Phi(\lambda, u_x)$ and all the roots of the polynomial $\Phi(u_x, \mu)$. We call Φ the *hyper-eigenvalue* of u.

Referring to Definition 25.10, if $u = \sum_{x \in X} u_x \hat{x}$ is a Φ-hyper-eigenvector then $\Phi(u_x, u_y) = 0$ for all pairs of adjacent vertices $x, y \in X$.

Referring to Definition 25.10, suppose that $u = \sum_{x \in X} u_x \hat{x}$ is a Φ-hyper-eigenvector. The scalars $\{u_x\}_{x \in X}$ might not be distinct. For $x, y \in X$ write $x \sim y$ whenever $u_x = u_y$. Note that $\sim$ is an equivalence relation. Partition X into the equivalence classes for $\sim$. This partition is equitable.

Example 25.11. Assume that Γ is regular with valency k. Recall that the vector $\mathbf{1} = \sum_{y \in X} \hat{y}$ is an eigenvector for Γ with eigenvalue k. The vector $\mathbf{1}$ is a Φ-hyper-eigenvector for Γ with hyper-eigenvalue $\Phi(\lambda, \mu) = (\lambda - \mu)^k$.

Example 25.12. Assume that Γ is the 3-cube $H(3, 2)$. Let $\uparrow, \downarrow$ denote a basis for a 2-dimensional vector space W. We interpret $V = W \otimes W \otimes W$. We interpret $\{\hat{x}\}_{x \in X}$ to be the set of vectors

$$r \otimes s \otimes t \qquad\qquad r, s, t \in \{\uparrow, \downarrow\}.$$

We now give the hyper-eigenvalues and their hyper-eigenvectors.

(i) For $\Phi = (\lambda - \mu)^3$;

$$(\uparrow + \downarrow) \otimes (\uparrow + \downarrow) \otimes (\uparrow + \downarrow).$$

(ii) For $\Phi = (\lambda - \mu)^2(\lambda + \mu)$:

$$(\uparrow - \downarrow) \otimes (\uparrow + \downarrow) \otimes (\uparrow + \downarrow),$$
$$(\uparrow + \downarrow) \otimes (\uparrow - \downarrow) \otimes (\uparrow + \downarrow),$$
$$(\uparrow + \downarrow) \otimes (\uparrow + \downarrow) \otimes (\uparrow - \downarrow).$$

(iii) For $\Phi = (\lambda - \mu)(\lambda + \mu)^2$:

$$(\uparrow - \downarrow) \otimes (\uparrow - \downarrow) \otimes (\uparrow + \downarrow),$$
$$(\uparrow - \downarrow) \otimes (\uparrow + \downarrow) \otimes (\uparrow - \downarrow),$$
$$(\uparrow + \downarrow) \otimes (\uparrow - \downarrow) \otimes (\uparrow - \downarrow).$$

(iv) For $\Phi = (\lambda + \mu)^3$:

$$(\uparrow - \downarrow) \otimes (\uparrow - \downarrow) \otimes (\uparrow - \downarrow).$$

Problem 25.13. Find the hyper-eigenvalues and hyper-eigenvectors for your favorite graph.

Definition 25.14. Let A and B denote square matrices of the same size. We say that A and B are *linked* whenever there exists an invertible matrix P such that $P^{-1}AP$ is diagonal and PBP^{-1} is diagonal.

Every linked pair of matrices over $\mathbb{C}$ appears in the following way.

Example 25.15. Let V denote a finite-dimensional vector space over $\mathbb{C}$. Let $S : V \to V$ and $T : V \to V$ denote diagonalizable linear transformations.

(i) Pick an eigenbasis for S. Let A denote the matrix that represents T in this basis.

(ii) Pick an eigenbasis for T. Let B denote the matrix that represents S in this basis.

(iii) Let P denote the transition matrix from the basis in (i) to the basis in (ii).

The matrices $P^{-1}AP$ and PBP^{-1} are diagonal, so A and B are linked.

Problem 25.16. What pairs of matrices are linked?

Definition 25.17. Let Γ and Γ' denote graphs with the same number of vertices. We say that Γ and Γ' are *linked* whenever their adjacency matrices are linked.

Problem 25.18. What pairs of graphs are linked?

Problem 25.19. What matrix is linked to itself?

Problem 25.20. What graph is linked to itself?

Example 25.21. It is routine to check that the complete graph K_n is linked to itself, and any hypercube $H(d, 2)$ is linked to itself. It is true but not obvious that the path P_n of length n is linked to itself.