Next, we consider how the algebra T acts on the standard module V. By a *T-module* we mean a subspace $W \subseteq V$ such that $TW \subseteq W$. A T-module W is *irreducible* whenever W is nonzero, and W does not contain a T-module besides 0 and W.

Lemma 6.19. *Let W denote a T-module. Then the orthogonal complement $W^{\perp}$ is a T-module.*

Proof. For $A \in T$ we have $\overline{A}^t \in T$. Also

$$\langle Au, v\rangle = \langle u, \overline{A}^t v\rangle \qquad u, v \in V.$$

By these comments we obtain the result. □

Corollary 6.20. *The standard module V is an orthogonal direct sum of irreducible T-modules.*

Proof. Use Lemma 6.19. □

Next, we describe a particular irreducible T-module called the primary T-module. Recall the vector $\mathbf{1} = \sum_{y\in X} \hat{y}$. For $0 \leq i \leq d$ define the vector

$$\mathbf{1}_i = \sum_{y\in\Gamma_i(x)} \hat{y}.$$

Observe that

$$E_i^*\mathbf{1} = \mathbf{1}_i = A_{i'}\hat{x} \qquad (0 \leq i \leq d).$$

Consequently

$$\mathcal{M}^* E_0 V = \mathcal{M} E_0^* V. \tag{28}$$

Lemma 6.21. *The vector space $\mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$ is an irreducible T-module.*

Proof. Define $\mathcal{V} = \mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$. We have $\mathcal{M}\mathcal{V} \subseteq \mathcal{V}$ since $\mathcal{V} = \mathcal{M}E_0^*V$. We have $\mathcal{M}^*\mathcal{V} \subseteq \mathcal{V}$ since $\mathcal{V} = \mathcal{M}^*E_0V$. Therefore $T\mathcal{V} \subseteq \mathcal{V}$, so $\mathcal{V}$ is a T-module. We show that the T-module $\mathcal{V}$ is irreducible. The standard T-module V is a direct sum of irreducible T-modules. There exists an irreducible T-module that is not orthogonal to $\hat{x}$. This T-module is closed under E_0^*, so it contains $\hat{x}$ and also $\mathcal{M}\hat{x} = \mathcal{V}$. This T-module must equal $\mathcal{V}$ by irreducibility. □

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Definition 6.22. Define $\mathcal{V} = \mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$. The T-module $\mathcal{V}$ is called *primary.*

Lemma 6.23. *For $0 \leq i \leq d$ we have*

$$|X| E_i \hat{x} = A_i^* \mathbf{1}. \tag{29}$$

Proof. Both vectors in (29) have y-coordinate $|X|(E_i)_{y,x}$ for $y \in X$. $\square$

Definition 6.24. For $0 \le i \le d$ let $\mathbf{1}_i^*$ denote the common vector in (29).

We clarify the definitions. Note that $\mathbf{1}_0 = \hat{x}$ and $\mathbf{1}_0^* = \mathbf{1}$. Moreover

$$\mathbf{1}_0^* = \sum_{i=0}^{d} \mathbf{1}_i, \qquad \mathbf{1}_0 = |X|^{-1} \sum_{i=0}^{d} \mathbf{1}_i^*.$$

The following result is routinely verified.

Lemma 6.25. *For the primary T-module $\mathcal{V}$,*

(i) $\mathbf{1}_i$ *is a basis for* $E_i^*\mathcal{V}$ $(0 \le i \le d)$;

(ii) $\{\mathbf{1}_i\}_{i=0}^d$ *is a basis for* $\mathcal{V}$;

(iii) $\mathbf{1}_i^*$ *is a basis for* $E_i\mathcal{V}$ $(0 \le i \le d)$;

(iv) $\{\mathbf{1}_i^*\}_{i=0}^d$ *is a basis for* $\mathcal{V}$.

Next, we describe how the bases $\{\mathbf{1}_i\}_{i=0}^d$ and $\{\mathbf{1}_i^*\}_{i=0}^d$ are related.

Lemma 6.26. *For $0 \le j \le d$ we have*

(i) $\mathbf{1}_j = |X|^{-1} \sum_{i=0}^{d} \overline{P_j(i)} \mathbf{1}_i^*$;

(ii) $\mathbf{1}_j^* = \sum_{i=0}^{d} \overline{Q_j(i)} \mathbf{1}_i$.

Proof. (i) Observe

$$\begin{aligned}\mathbf{1}_j = E_j^*\mathbf{1} = |X|^{-1} \sum_{i=0}^{d} P_j(i) A_i^* \mathbf{1} &= |X|^{-1} \sum_{i=0}^{d} P_j(i) \mathbf{1}_{\hat{i}}^* \\ = |X|^{-1} \sum_{i=0}^{d} P_j(\hat{i}) \mathbf{1}_i^* &= |X|^{-1} \sum_{i=0}^{d} \overline{P_j(i)} \mathbf{1}_i^*.\end{aligned}$$

(ii) Observe

$$\mathbf{1}_j^* = |X| E_j \hat{x} = \sum_{i=0}^{d} Q_j(i) A_i \hat{x} = \sum_{i=0}^{d} Q_j(i) \mathbf{1}_{i'} = \sum_{i=0}^{d} Q_j(i') \mathbf{1}_i = \sum_{i=0}^{d} \overline{Q_j(i)} \mathbf{1}_i.$$

$\square$

Next we describe how the algebra T acts on the bases $\{\mathbf{1}_i\}_{i=0}^d$ and $\{\mathbf{1}_i^*\}_{i=0}^d$.

Lemma 6.27. *For $0 \le i, j \le d$ we have*

(i) $E_i^* \mathbf{1}_j = \delta_{i,j} \mathbf{1}_j$;

(ii) $A_i^* \mathbf{1}_j = Q_i(j) \mathbf{1}_j$;

(iii) $E_i\mathbf{1}_j = |X|^{-1}\overline{P_j(i)}\sum_{h=0}^d \overline{Q_i(h)}\mathbf{1}_h$;

(iv) $A_i\mathbf{1}_j = \sum_{k=0}^d p^k_{i',j}\mathbf{1}_k$.

Proof. (i) Clear.
(ii) Observe

$$A^*_i\mathbf{1}_j = A^*_iE^*_j\mathbf{1} = Q_i(j)E^*_j\mathbf{1} = Q_i(j)\mathbf{1}_j.$$

(iii) Observe

$$\begin{aligned} E_i\mathbf{1}_j &= E_iA_{j'}\hat{x} = A_{j'}E_i\hat{x} = P_{j'}(i)E_i\hat{x} = \overline{P_j(i)}E_i\hat{x} \\ &= |X|^{-1}\overline{P_j(i)}\mathbf{1}^*_i = |X|^{-1}\overline{P_j(i)}\sum_{h=0}^d \overline{Q_i(h)}\mathbf{1}_h. \end{aligned}$$

(iv) Observe

$$A_i\mathbf{1}_j = A_iA_{j'}\hat{x} = \sum_{k=0}^d p^k_{i,j'}A_k\hat{x} = \sum_{k=0}^d p^k_{i,j'}\mathbf{1}_{k'} = \sum_{k=0}^d p^{k'}_{i,j'}\mathbf{1}_k = \sum_{k=0}^d p^k_{i',j}\mathbf{1}_k.$$

$\square$

Lemma 6.28. *For $0 \le i, j \le d$ we have*

(i) $E_i\mathbf{1}^*_j = \delta_{i,j}\mathbf{1}^*_j$;

(ii) $A_i\mathbf{1}^*_j = P_i(j)\mathbf{1}^*_j$;

(iii) $E^*_i\mathbf{1}^*_j = |X|^{-1}\overline{Q_j(i)}\sum_{h=0}^d \overline{P_i(h)}\mathbf{1}^*_h$;

(iv) $A^*_i\mathbf{1}^*_j = \sum_{k=0}^D q^k_{\hat{i},j}\mathbf{1}^*_k$.

Proof. Similar to the proof of Lemma 6.27. (i) Clear.
(ii) Observe

$$A_i\mathbf{1}^*_j = |X|A_iE_j\hat{x} = |X|P_i(j)E_j\hat{x} = P_i(j)\mathbf{1}^*_j.$$

(iii) Observe

$$E^*_i\mathbf{1}^*_j = E^*_iA^*_{\hat{j}}\mathbf{1} = A^*_{\hat{j}}E^*_i\mathbf{1} = A^*_{\hat{j}}\mathbf{1}_i = Q_{\hat{j}}(i)\mathbf{1}_i = \overline{Q_j(i)}\mathbf{1}_i = |X|^{-1}\overline{Q_j(i)}\sum_{h=0}^d \overline{P_i(h)}\mathbf{1}^*_h.$$

(iv) Observe

$$A^*_i\mathbf{1}^*_j = A^*_iA^*_{\hat{j}}\mathbf{1} = \sum_{k=0}^d q^k_{i,\hat{j}}A^*_k\mathbf{1} = \sum_{k=0}^d q^k_{i,\hat{j}}\mathbf{1}^*_{\hat{k}} = \sum_{k=0}^d q^{\hat{k}}_{i,\hat{j}}\mathbf{1}^*_k = \sum_{k=0}^d q^k_{\hat{i},j}\mathbf{1}^*_k.$$

$\square$

Next we bring in the bilinear form.

Lemma 6.29. *For $0 \le i, j \le d$ we have*

(i) $\langle \mathbf{1}_i, \mathbf{1}_j \rangle = \delta_{i,j} k_i$;

(ii) $\langle \mathbf{1}_i^*, \mathbf{1}_j^* \rangle = \delta_{i,j} |X| m_i$;

(iii) $\langle \mathbf{1}_i, \mathbf{1}_j^* \rangle = \overline{P_i(j)} m_j = Q_j(i) k_i$.

Proof. (i) Routine.
(ii) Observe

$$\langle \mathbf{1}_i^*, \mathbf{1}_j^* \rangle = |X|^2 \langle E_i \hat{x}, E_j \hat{x} \rangle = |X|^2 \langle \hat{x}, E_i E_j \hat{x} \rangle = \delta_{i,j} |X|^2 \langle \hat{x}, E_i \hat{x} \rangle = \delta_{i,j} |X| m_i.$$

(iii) Observe

$$\begin{aligned} \langle \mathbf{1}_i, \mathbf{1}_j^* \rangle &= |X| \langle A_{i'} \hat{x}, E_j \hat{x} \rangle = |X| \langle \hat{x}, \overline{(A_{i'})}^t E_j \hat{x} \rangle = |X| \langle \hat{x}, A_i E_j \hat{x} \rangle \\ &= |X| \overline{P_i(j)} \langle \hat{x}, E_j \hat{x} \rangle = \overline{P_i(j)} m_j = Q_j(i) k_i. \end{aligned}$$

□

7 Duality for commutative association schemes

In this section we discuss the concept of duality for commutative association schemes. To motivate things, we start with a small example.

Consider the group $G = \mathbb{Z}/3\mathbb{Z}$ with three elements. Of course G is abelian, so each conjugacy class contains one element. Consider the conjugacy-class association scheme $\mathcal{X}$ for G. The associate matrices of $\mathcal{X}$ are

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We have $A_2 = A_1^2$ and $A_1^3 = I$. Let $\omega \in \mathbb{C}$ denote a primitive third root of unity. Note that

$$\overline{\omega} = \omega^2 = \omega^{-1}, \qquad 1 + \omega + \omega^2 = 0.$$

The primitive idempotents of $\mathcal{X}$ are

$$E_0 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad E_1 = \frac{1}{3} \begin{pmatrix} 1 & w^2 & w \\ w & 1 & w^2 \\ w^2 & w & 1 \end{pmatrix}, \qquad E_2 = \frac{1}{3} \begin{pmatrix} 1 & w & w^2 \\ w^2 & 1 & w \\ w & w^2 & 1 \end{pmatrix}.$$

The first and second eigenmatrices of $\mathcal{X}$ are

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

Note that

$$P = \overline{Q}. \tag{30}$$

We will interpret (30) using duality.

For the rest of this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

Definition 7.1. A *duality* of $\mathcal{X}$ is a $\mathbb{C}$-linear bijection $\Psi : \mathcal{M} \to \mathcal{M}$ that satisfies (i), (ii) below:

(i) $\Psi(AB) = \Psi(A) \circ \Psi(B)$ for all $A, B \in \mathcal{M}$;

(ii) $\Psi(\Psi(A)) = |X| A^t$ for all $A \in \mathcal{M}$.

We say that $\mathcal{X}$ is *self-dual* whenever $\mathcal{X}$ has a duality.

Lemma 7.2. *Assume that $\mathcal{X}$ has a duality Ψ. Then*

(i) $\Psi(A \circ B) = |X|^{-1} \Psi(A)\Psi(B)$ *for all* $A, B \in \mathcal{M}$;

(ii) $\Psi(A^t) = \big(\Psi(A)\big)^t$ *for all* $A \in \mathcal{M}$.

Proof. (i) Each side is equal to $|X|\Psi^{-1}(A^t \circ B^t)$.
(ii) Each side is equal to $|X|^{-1}\Psi^3(A)$. ☐

Lemma 7.3. *Assume that $\mathcal{X}$ has a duality Ψ. Then*

(i) $\Psi(I) = J$;

(ii) $\Psi(J) = |X| I$.

Proof. (i) For $A \in \mathcal{M}$,

$$\Psi(A) = \Psi(AI) = \Psi(A) \circ \Psi(I).$$

The result follows.
(ii) For $A \in \mathcal{M}$,

$$\Psi(A) = \Psi(A \circ J) = |X|^{-1}\Psi(A)\Psi(J).$$

So $|X|^{-1}\Psi(J) = I$. The result follows. ☐

Lemma 7.4. *Assume that $\mathcal{X}$ has a duality Ψ. Then there exits an ordering $\{R_i\}_{i=0}^d$ of the relations such that*

$$\Psi(E_i) = A_i \qquad (0 \le i \le d).$$

Proof. For $0 \le i, j \le d$ we have $E_iE_j = \delta_{i,j}E_i$. In this equation we apply Ψ to each side; this yields

$$\delta_{i,j}\Psi(E_i) = \Psi(E_iE_j) = \Psi(E_i) \circ \Psi(E_j).$$

By these comments, the sequence $\{\Psi(E_i)\}_{i=0}^d$ is a permutation of the sequence $\{A_i\}_{i=0}^d$. The result follows. □

Lemma 7.5. *Assume that $\mathcal{X}$ has a duality Ψ such that $\Psi(E_i) = A_i$ for $0 \le i \le d$. Then* (i)–(iv) *hold below:*

(i) $\Psi(A_i) = |X|E_i^t \qquad (0 \le i \le d)$;

(ii) $i' = \hat{i} \qquad (0 \le i \le d)$;

(iii) $P = \overline{Q}$;

(iv) $p_{i,j}^k = q_{i,j}^k \qquad (0 \le i, j, k \le d)$.

Proof. (i) We have

$$|X|E_i^t = \Psi(\Psi(E_i)) = \Psi(A_i).$$

(ii) We have

$$A_{\hat{i}} = \Psi(E_{\hat{i}}) = \Psi(E_i^t) = \left(\Psi(E_i)\right)^t = A_i^t = A_{i'}.$$

(iii) For $0 \le i \le d$ we have $A_i = \sum_{j=0}^d P_i(j)E_j$. In this equation we apply Ψ to each side; this yields

$$|X|E_i^t = \sum_{j=0}^d P_i(j)A_j.$$

We may now argue

$$\sum_{j=0}^d P_i(j)A_j = |X|E_i^t = |X|\overline{E_i} = \sum_{j=0}^d \overline{Q_i(j)}A_j$$

Therefore $P_i(j) = \overline{Q_i(j)}$ for $0 \le i, j \le d$. Consequently $P = \overline{Q}$.
(iv) We have

$$E_i \circ E_j = |X|^{-1}\sum_{k=0}^d q_{i,j}^k E_k \qquad (0 \le i, j \le d).$$

In this equation, we apply Ψ to each side and evaluate the result; this yields

$$A_iA_j = \sum_{k=0}^d q_{i,j}^k A_k \qquad (0 \le i, j \le d).$$