(iii) We have

$$\overline{A_i^*} = |X|\overline{(E_i)}^\natural = |X|(\overline{E_i})^\natural = |X|(E_{\hat{i}})^\natural = A_{\hat{i}}^*.$$

(iv) Apply $\natural$ to each side of

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{i,j}^k E_k.$$

□

Next, we consider how $\mathcal{M}$ and $\mathcal{M}^*$ are related.

Definition 6.13. Let $T = T(x)$ denote the subalgebra of $M_X(\mathbb{C})$ generated by $\mathcal{M}$ and $\mathcal{M}^*$. We call T the *subconstituent algebra of $\mathcal{X}$ with respect to x.*

We have some comments. By construction, the algebra T is finite-dimensional. Moreover T is noncommutative in general. The algebra T is closed under both the transpose map and complex-conjugation, because $\mathcal{M}$ and $\mathcal{M}^*$ are closed under both the transpose map and complex-conjugation.

Lecture 8

We are going to show that for $0 \le \alpha, \beta, \gamma \le d$,

$$E_\alpha^* A_\beta E_\gamma^* = 0 \text{ iff } p_{\alpha,\beta}^\gamma = 0;$$
$$E_\alpha A_\beta^* E_\gamma = 0 \text{ iff } q_{\alpha,\beta}^\gamma = 0.$$

The above equations are called the *triple product relations.*

To obtain the triple product relations, we endow the vector space $M_X(\mathbb{C})$ with a bilinear form $(\,,\,)$ such that $(A, B) = \text{tr}(A^t\overline{B})$ for all $A, B \in M_X(\mathbb{C})$. Abbreviate $\|A\|^2 = (A, A)$. For $A, B, C \in M_X(\mathbb{C})$ and $\alpha \in \mathbb{C}$, we have

$$(B, A) = \overline{(A, B)}, \qquad (\alpha A, B) = \alpha(A, B),$$
$$(A + B, C) = (A, C) + (B, C), \qquad \|A\|^2 \in \mathbb{R},$$
$$\|A\|^2 \ge 0, \qquad \|A\|^2 = 0 \text{ iff } A = 0,$$
$$(AB, C) = (B, \overline{A}^t C) = (A, C\overline{B}^t).$$

Lemma 6.14. *For $0 \le \alpha, \beta, \gamma, i, j, k \le d$ we have*

(i) $(E_\alpha^* A_\beta E_\gamma^*, E_i^* A_j E_k^*) = \delta_{\alpha,i}\delta_{\beta,j}\delta_{\gamma,k} k_\gamma p_{\alpha,\beta}^\gamma;$

(ii) $(E_\alpha A_\beta^* E_\gamma, E_i A_j^* E_k) = \delta_{\alpha,i}\delta_{\beta,j}\delta_{\gamma,k} m_\gamma q_{\alpha,\beta}^\gamma.$

Proof. (i) Using $\mathrm{tr}(BC) = \mathrm{tr}(CB)$,

$$\begin{aligned}(E^*_\alpha A_\beta E^*_\gamma, E^*_i A_j E^*_k) &= \mathrm{tr}\big((E^*_\alpha A_\beta E^*_\gamma)^t \overline{E^*_i A_j E^*_k}\big)\\ &= \mathrm{tr}\big(E^*_\gamma A_{\beta'} E^*_\alpha E^*_i A_j E^*_k\big)\\ &= \delta_{\alpha,i}\delta_{\gamma,k}\mathrm{tr}\big(E^*_\gamma A_{\beta'} E^*_\alpha A_j\big)\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\big(E^*_\gamma A_{\beta'} E^*_\alpha A_j\big) &= \sum_{y\in X}\sum_{z\in X}(E^*_\gamma)_{y,y}(A_{\beta'})_{y,z}(E^*_\alpha)_{z,z}(A_j)_{z,y}\\ &= \sum_{y\in X}\sum_{z\in X}(E^*_\gamma)_{y,y}(A_{\beta'}\circ A_{j'})_{y,z}(E^*_\alpha)_{z,z}\\ &= \delta_{\beta,j}\sum_{y\in X}\sum_{z\in X}(E^*_\gamma)_{y,y}(A_{\beta'})_{y,z}(E^*_\alpha)_{z,z}\\ &= \delta_{\beta,j}\sum_{\substack{y\in\Gamma_\gamma(x),\\ z\in\Gamma_\alpha(x)\cap\Gamma_{\beta'}(y)}} 1\\ &= \delta_{\beta,j}k_\gamma p^\gamma_{\alpha,\beta}.\end{aligned}$$

(ii) We have

$$\begin{aligned}(E_\alpha A^*_\beta E_\gamma, E_i A^*_j E_k) &= \mathrm{tr}\big((E_\alpha A^*_\beta E_\gamma)^t \overline{E_i A^*_j E_k}\big)\\ &= \mathrm{tr}\big(E_{\hat\gamma} A^*_\beta E_{\hat\alpha} E_{\hat i} A^*_{\hat j} E_{\hat k}\big)\\ &= \delta_{\alpha,i}\delta_{\gamma,k}\mathrm{tr}\big(E_{\hat\gamma} A^*_\beta E_{\hat\alpha} A^*_{\hat j}\big)\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\big(E_{\hat\gamma} A^*_\beta E_{\hat\alpha} A^*_{\hat j}\big) &= \sum_{y\in X}\sum_{z\in X}(E_{\hat\gamma})_{y,z}(A^*_\beta)_{z,z}(E_{\hat\alpha})_{z,y}(A^*_{\hat j})_{y,y}\\ &= |X|^2\sum_{y\in X}\sum_{z\in X}(E_{\hat\gamma})_{y,z}(E_\beta)_{x,z}(E_{\hat\alpha})_{z,y}(E_{\hat j})_{x,y}\\ &= |X|^2\sum_{y\in X}\sum_{z\in X}(E_{\hat j})_{x,y}(E_{\hat\gamma}\circ E_\alpha)_{y,z}(E_{\hat\beta})_{z,x}\\ &= |X|^2\Big((x,x)\text{-entry of } E_{\hat j}(E_{\hat\gamma}\circ E_\alpha)E_{\hat\beta}\Big)\\ &= |X|\mathrm{tr}\big(E_{\hat j}(E_{\hat\gamma}\circ E_\alpha)E_{\hat\beta}\big)\\ &= |X|\mathrm{tr}\big((E_{\hat\gamma}\circ E_\alpha)E_{\hat\beta}E_{\hat j}\big)\\ &= \delta_{\beta,j}|X|\mathrm{tr}\big((E_{\hat\gamma}\circ E_\alpha)E_{\hat\beta}\big)\\ &= \delta_{\beta,j}m_{\hat\beta}q^{\hat\beta}_{\hat\gamma,\alpha}\\ &= \delta_{\beta,j}m_\gamma q^\gamma_{\alpha,\beta}.\end{aligned}$$

□

Corollary 6.15. *For $0 \leq \alpha, \beta, \gamma \leq d$ we have*

(i) $\|E^*_\alpha A_\beta E^*_\gamma\|^2 = k_\gamma p^\gamma_{\alpha,\beta}$;

(ii) $\|E_\alpha A^*_\beta E_\gamma\|^2 = m_\gamma q^\gamma_{\alpha,\beta}$.

Proof. Set $i = \alpha$, $j = \beta$, $k = \gamma$ in Lemma 6.14. □

Corollary 6.15(ii) gives a second proof of the Krein condition.

Theorem 6.16. (Triple product relations). *For $0 \leq \alpha, \beta, \gamma \leq d$ we have*

(i) $E^*_\alpha A_\beta E^*_\gamma = 0$ iff $p^\gamma_{\alpha,\beta} = 0$;

(ii) $E_\alpha A^*_\beta E_\gamma = 0$ iff $q^\gamma_{\alpha,\beta} = 0$.

Proof. By Corollary 6.15. □

We bring in some notation. For subspaces R, S of $M_X(\mathbb{C})$, define

$$RS = \text{Span}\{rs | r \in R,\ s \in S\}.$$

Theorem 6.17. *With the above notation,*

(i) *the vector space $\mathcal{M}^*\mathcal{M}\mathcal{M}^*$ has an orthogonal basis*

$$\{E^*_\alpha A_\beta E^*_\gamma | 0 \leq \alpha, \beta, \gamma \leq d,\ p^\gamma_{\alpha,\beta} \neq 0\};$$

(ii) *the vector space $\mathcal{M}\mathcal{M}^*\mathcal{M}$ has an orthogonal basis*

$$\{E_\alpha A^*_\beta E_\gamma | 0 \leq \alpha, \beta, \gamma \leq d,\ q^\gamma_{\alpha,\beta} \neq 0\}.$$

Proof. By Lemma 6.14 and Theorem 6.16. □

We mention a consequence of Theorem 6.16. Recall the standard module V.

Proposition 6.18. *For $0 \leq j, k \leq d$ we have*

$$A_j E^*_k V \subseteq \sum_{\substack{0 \leq i \leq d, \\ p^k_{i,j} \neq 0}} E^*_i V, \qquad A^*_j E_k V \subseteq \sum_{\substack{0 \leq i \leq d, \\ q^k_{i,j} \neq 0}} E_i V. \tag{27}$$

Proof. Concerning the containment on the left in (27),

$$A_j E^*_k V = I A_j E^*_k V = \sum_{i=0}^{d} E^*_i A_j E^*_k V = \sum_{\substack{0 \leq i \leq d, \\ p^k_{i,j} \neq 0}} E^*_i A_j E^*_k V \subseteq \sum_{\substack{0 \leq i \leq d, \\ p^k_{i,j} \neq 0}} E^*_i V.$$

The containment on the right in (27) is similarly obtained. □

Next, we consider how the algebra T acts on the standard module V. By a *T-module* we mean a subspace $W \subseteq V$ such that $TW \subseteq W$. A T-module W is *irreducible* whenever W is nonzero, and W does not contain a T-module besides 0 and W.

Lemma 6.19. *Let W denote a T-module. Then the orthogonal complement $W^\perp$ is a T-module.*

Proof. For $A \in T$ we have $\overline{A}^t \in T$. Also

$$\langle Au, v\rangle = \langle u, \overline{A}^t v\rangle \qquad\qquad u, v \in V.$$

By these comments we obtain the result. □

Corollary 6.20. *The standard module V is an orthogonal direct sum of irreducible T-modules.*

Proof. Use Lemma 6.19. □

Next, we describe a particular irreducible T-module called the primary T-module. Recall the vector $\mathbf{1} = \sum_{y\in X} \hat{y}$. For $0 \le i \le d$ define the vector

$$\mathbf{1}_i = \sum_{y\in\Gamma_i(x)} \hat{y}.$$

Observe that

$$E_i^*\mathbf{1} = \mathbf{1}_i = A_{i'}\hat{x} \qquad\qquad (0 \le i \le d).$$

Consequently

$$\mathcal{M}^* E_0 V = \mathcal{M} E_0^* V. \tag{28}$$

Lemma 6.21. *The vector space $\mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$ is an irreducible T-module.*

Proof. Define $\mathcal{V} = \mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$. We have $\mathcal{M}\mathcal{V} \subseteq \mathcal{V}$ since $\mathcal{V} = \mathcal{M} E_0^* V$. We have $\mathcal{M}^*\mathcal{V} \subseteq \mathcal{V}$ since $\mathcal{V} = \mathcal{M}^* E_0 V$. Therefore $T\mathcal{V} \subseteq \mathcal{V}$, so $\mathcal{V}$ is a T-module. We show that the T-module $\mathcal{V}$ is irreducible. The standard T-module V is a direct sum of irreducible T-modules. There exists an irreducible T-module that is not orthogonal to $\hat{x}$. This T-module is closed under E_0^*, so it contains $\hat{x}$ and also $\mathcal{M}\hat{x} = \mathcal{V}$. This T-module must equal $\mathcal{V}$ by irreducibility. □

Definition 6.22. Define $\mathcal{V} = \mathcal{M}^* E_0 V = \mathcal{M} E_0^* V$. The T-module $\mathcal{V}$ is called *primary*.

Lemma 6.23. *For $0 \le i \le d$ we have*

$$|X| E_i \hat{x} = A_i^* \mathbf{1}. \tag{29}$$

Proof. Both vectors in (29) have y-coordinate $|X|(E_i)_{y,x}$ for $y \in X$. □

Definition 6.24. For $0 \le i \le d$ let $\mathbf{1}_i^*$ denote the common vector in (29).