Definition 5.3. By Lemma 5.2 the matrices $\{B_i\}_{i=0}^d$ form a basis for a commutative subalgebra $\mathcal{B}$ of $M_{d+1}(\mathbb{C})$. We call $\mathcal{B}$ the *intersection algebra* of $\mathcal{X}$.

Theorem 5.4. *There exists an algebra isomorphism $\mathcal{M} \to \mathcal{B}$ that sends $A_i \mapsto B_i$ for $0 \leq i \leq d$.*

Proof. Clear from Lemma 5.2(iv). □

Lecture 7

Let us recall some linear algebra. For the moment, let W denote any finite-dimensional vector space over $\mathbb{C}$, and let $\{w_i\}_{i=1}^n$ denote a basis for W. Let $A : W \to W$ denote a $\mathbb{C}$-linear map. There exists a unique $n \times n$ matrix B such that

$$Aw_j = \sum_{i=1}^n B_{i,j} w_i \qquad (1 \leq j \leq n).$$

We say that B *represents* A *with respect to* $\{w_i\}_{i=1}^n$. Let $\{w'_i\}_{i=1}^n$ denote a second basis for W. There exists a unique $n \times n$ matrix S such that

$$w'_j = \sum_{i=1}^n S_{i,j} w_i \qquad (1 \leq j \leq n).$$

The matrix S is invertible. We call S the *transition matrix from* $\{w_i\}_{i=1}^n$ *to* $\{w'_i\}_{i=1}^n$. By linear algebra, the matrix $S^{-1}BS$ represents A with respect to $\{w'_i\}_{i=1}^n$.

We return our attention to $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. The Bose-Mesner algebra $\mathcal{M}$ has bases $\{A_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$. Recall the first and second eigenmatrices P, Q. Then P is the transition matix from $\{E_i\}_{i=0}^d$ to $\{A_i\}_{i=0}^d$. Moreover, $|X|^{-1}Q$ is the transition matrix from $\{A_i\}_{i=0}^d$ to $\{E_i\}_{i=0}^d$.

For $A \in \mathcal{M}$, there exists a $\mathbb{C}$-linear map $L_A : \mathcal{M} \to \mathcal{M}$ that sends $B \mapsto AB$ for all $B \in \mathcal{M}$.

Theorem 5.5. *With the above notation, the following* (i)–(iv) *hold for $0 \leq i \leq d$:*

(i) *B_i^t represents L_{A_i} with respect to the basis $\{A_\ell\}_{\ell=0}^d$;*

(ii) *the matrix* $\operatorname{diag}(P_i(0), P_i(1), \ldots, P_i(d))$ *represents L_{A_i} with respect to $\{E_\ell\}_{\ell=0}^d$;*

(iii) $PB_i^tP^{-1} = \operatorname{diag}(P_i(0), P_i(1), \ldots, P_i(d))$;

(iv) *the scalars $P_i(0), P_i(1), \ldots, P_i(d)$ are the roots of the characteristic polynomial of B_i.*

Proof. (i) By Definition 5.1 and

$$A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k \qquad (0 \leq j \leq d).$$

(ii) Since $A_i E_j = P_i(j) E_j$ for $0 \leq j \leq d$.
(iii) By (i), (ii) and the comments above the theorem statement.
(iv) B_i and B_i^t have the same characteristic polynomial. The result follows in view of (iii). □

Definition 5.6. For $0 \le i \le d$, define a matrix $B_i^* \in M_{d+1}(\mathbb{C})$ with (j,k)-entry $q_{i,j}^k$ for $0 \le j,k \le d$. We call B_i^* the i^{th} *dual intersection matrix* of $\mathcal{X}$.

Lemma 5.7. *The following* (i)–(vi) *hold.*

(i) $B_0^* = I$.

(ii) *For* $0 \le i \le d$, *the top row of* B_i^* *is* $(0,\dots,0,1,0,\dots,0)$, *with the 1 in column* i.

(iii) $\{B_i^*\}_{i=0}^d$ *are linearly independent.*

(iv) *For* $0 \le i,j \le d$,

$$B_i^* B_j^* = \sum_{k=0}^{d} q_{i,j}^k B_k^*.$$

(v) $B_i^* B_j^* = B_j^* B_i^*$ *for* $0 \le i,j \le d$.

(vi) *For* $0 \le i \le d$ *we have* $(B_i^*)^t = M^{-1} B_i^* M$ *where* $M = \mathrm{diag}(m_0, m_1, \dots, m_d)$.

Proof. Similar to the proof of Lemma 5.2. □

Definition 5.8. By Lemma 5.7 the matrices $\{B_i^*\}_{i=0}^d$ form a basis for a commutative subalgebra $\mathcal{B}^*$ of $M_{d+1}(\mathbb{C})$. We call $\mathcal{B}^*$ the *dual intersection algebra* of $\mathcal{X}$.

Definition 5.9. *Let* $\mathcal{M}^\circ$ *denote the algebra over* $\mathbb{C}$ *consisting of the vector space* $\mathcal{M}$ *together with the Hadamard multiplication* $\circ$. *The algebra* $\mathcal{M}^\circ$ *is commutative. Note that* J *is the multiplicative identity in* $\mathcal{M}^\circ$.

Theorem 5.10. *There exists an algebra isomorphism* $\mathcal{M}^\circ \to \mathcal{B}^*$ *that sends* $E_i \mapsto |X|^{-1} B_i^*$ *for* $0 \le i \le d$.

Proof. For $0 \le i,j \le d$ we have

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{i,j}^k E_k.$$

Compare this with Lemma 5.7(iv). □

For $A \in \mathcal{M}^\circ$, there exists a $\mathbb{C}$-linear map $L_A^\circ : \mathcal{M}^\circ \to \mathcal{M}^\circ$ that sends $B \mapsto A \circ B$ for all $B \in \mathcal{M}^\circ$.

Theorem 5.11. *With the above notation, the following* (i)–(iv) *hold for* $0 \le i \le d$:

(i) $|X|^{-1}(B_i^*)^t$ *represents* $L_{E_i}^\circ$ *with respect to the basis* $\{E_\ell\}_{\ell=0}^d$;

(ii) *the matrix* $|X|^{-1}\mathrm{diag}(Q_i(0), Q_i(1), \dots, Q_i(d))$ *represents* $L_{E_i}^\circ$ *with respect to* $\{A_\ell\}_{\ell=0}^d$;

(iii) $Q(B_i^*)^t Q^{-1} = \mathrm{diag}(Q_i(0), Q_i(1), \dots, Q_i(d))$;

(iv) *the scalars* $Q_i(0), Q_i(1), \ldots, Q_i(d)$ *are the roots of the characteristic polynomial of* B_i^*.

Proof. (i) By Definition 5.6 and

$$E_i \circ E_j = \sum_{k=0}^{d} q_{i,j}^k E_k \qquad (0 \leq j \leq d).$$

(ii) We have

$$E_i = |X|^{-1} \sum_{j=0}^{d} Q_i(j) A_j.$$

Therefore

$$E_i \circ A_j = |X|^{-1} Q_i(j) A_j \qquad (0 \leq j \leq d).$$

The result follows.
(iii) By (i), (ii) and since $|X|^{-1}Q$ is the transition matrix from the basis $\{A_\ell\}_{\ell=0}^d$ to the basis $\{E_\ell\}_{\ell=0}^d$.
(iv) B_i^* and $(B_i^*)^t$ have the same characteristic polynomial. The result follows in view of (iii). □

We have a comment.

Proposition 5.12. *We have* $KQ = \overline{P}^t M$, *where*

$$K = \text{diag}(k_0, k_1, \ldots, k_d), \qquad M = \text{diag}(m_0, m_1, \ldots, m_d).$$

Proof. We saw earlier that $k_i Q_j(i) = \overline{P_i(j)} m_j$ for $0 \leq i, j \leq d$. □

6 The dual Bose-Mesner algebra and the subconstituent algebra

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. Recall the standard module $V = \mathbb{C}^X$. For $y \in X$ define $\hat{y} \in V$ that has y-entry 1 and all other entries 0. The vectors $\{\hat{y}\}_{y \in X}$ is an orthonormal basis for V. We have

$$\mathbf{1} = \sum_{y \in X} \hat{y}.$$

For $z \in X$,

$$A_i \hat{z} = \sum_{(y,z) \in R_i} \hat{y} \qquad (0 \leq i \leq d).$$

Definition 6.1. Throughout this section, we fix a vertex $x \in X$. We call x the *base vertex.*

Definition 6.2. *For $0 \leq i \leq d$ we define a diagonal matrix $E_i^* = E_i^*(x)$ in $M_X(\mathbb{C})$ that has (y, y)-entry*

$$E_i^*(y,y) = \begin{cases} 1 \text{ if } (x,y) \in R_i; \\ 0 \text{ if } (x,y) \notin R_i \end{cases} \qquad y \in X.$$

Lemma 6.3. *With reference to Definition 6.2,*

(i) $E_i^* E_j^* = \delta_{i,j} E_i^* \qquad (0 \leq i, j \leq d)$;

(ii) $I = \sum_{i=0}^{d} E_i^*$;

(iii) *the matrices $\{E_i^*\}_{i=0}^d$ are linearly independent.*

Proof. By Definition 6.2. □

Definition 6.4. By Lemma 6.3, the matrices $\{E_i^*\}_{i=0}^d$ form a basis for a commutative subalgebra $\mathcal{M}^* = \mathcal{M}^*(x)$ of $M_X(\mathbb{C})$. We call $\mathcal{M}^*$ the *dual Bose-Mesner algebra of $\mathcal{X}$ with respect to x.* We call E_i^* the i^{th} *dual primitive idempotent of $\mathcal{X}$ with respect to x.*

Lemma 6.5. *We have*

$$V = \sum_{i=0}^{d} E_i^* V \qquad \text{(orthogonal direct sum).}$$

For $0 \leq i \leq d$ the subspace E_i^*V is a common eigenspace for $\mathcal{M}^*$, and E_i^* is the projection onto this eigenspace. The subspace E_i^*V has basis $\{\hat{y} | y \in \Gamma_i(x)\}$. Moreover $k_i = \dim E_i^*V$. The vector $\hat{x}$ is a basis for E_0^*V.

Proof. Routine consequence of Definition 6.2. □

Referring to Lemma 6.5, we call E_i^*V the i^{th} *subconstituent of $\mathcal{X}$ with respect to x.*

Next we describe how the algebras $\mathcal{M}^\circ$ and $\mathcal{M}^*$ are related.

Lemma 6.6. *There exists an algebra isomorphism $\natural : \mathcal{M}^\circ \to \mathcal{M}^*$ that sends $A_i \mapsto E_i^*$ for $0 \leq i \leq d$.*

Proof. For $0 \leq i, j \leq d$ we have $A_i \circ A_j = \delta_{i,j} A_i$ and $E_i^* E_j^* = \delta_{i,j} E_i^*$. □

We emphasize the nature of $\natural$. For $A, B \in \mathcal{M}$ we have

$$(A \circ B)^\natural = A^\natural B^\natural. \tag{23}$$

Lemma 6.7. *For $A \in \mathcal{M}$,*

$$(A^\natural)_{y,y} = A_{x,y} \qquad (y \in X). \tag{24}$$

Proof. Without loss, we may assume that A is an associate matrix A_i. In this case $A^\natural = E_i^*$. Now (24) holds by the definitions of A_i and E_i^*. □

Definition 6.8. For $0 \le i \le d$ let $A_i^* \in \mathcal{M}^*$ be the image of $|X|E_i$ under the map $\natural$ from Lemma 6.6. We call A_i^* the i^{th} *dual associate matrix of* $\mathcal{X}$ *with respect to* x.

Lemma 6.9. *For* $0 \le i \le d$,

$$(A_i^*)_{y,y} = |X|(E_i)_{x,y} \qquad (y \in X).$$

Proof. By Lemma 6.7 with $A = |X|E_i$. □

Lemma 6.10. *The matrices* $\{A_i^*\}_{i=0}^d$ *form a basis for* $\mathcal{M}^*$. *Moreover*

$$A_i^* = \sum_{j=0}^{d} Q_i(j) E_j^* \qquad (0 \le i \le d), \tag{25}$$

$$E_i^* = |X|^{-1} \sum_{j=0}^{d} P_i(j) A_j^* \qquad (0 \le i \le d). \tag{26}$$

Proof. The first assertion holds because $\natural : \mathcal{M}^\circ \to \mathcal{M}^*$ is a bijection and $\{E_i\}_{i=0}^d$ is a basis for $\mathcal{M}^\circ$. To get (25), (26) we apply $\natural$ to each side of (10), (11). □

Lemma 6.11. *For* $0 \le i,j \le d$ *the scalar* $Q_i(j)$ *is the eigenvalue of* A_i^* *associated to the common eigenspace* E_j^*V *of* $\mathcal{M}^*$.

Proof. By (25). □

Proposition 6.12. *The following* (i)–(iv) *hold:*

(i) $A_0^* = I$;

(ii) $|X|E_0^* = \sum_{i=0}^d A_i^*$;

(iii) $\overline{A_i^*} = A_i^* \qquad (0 \le i \le d)$;

(iv) *for* $0 \le i,j \le d$,

$$A_i^* A_j^* = \sum_{k=0}^{d} q_{i,j}^k A_k^*.$$

Proof. (i) We have

$$A_0^* = |X|(E_0)^\natural = J^\natural = I.$$

(ii) We have

$$\sum_{i=0}^{d} A_i^* = |X|(E_0 + E_1 + \cdots + E_d)^\natural = |X|I^\natural = |X|E_0^*.$$

(iii) We have

$$\overline{A_i^*} = |X|\overline{(E_i)^\natural} = |X|(\overline{E_i})^\natural = |X|(E_{\hat{i}})^\natural = A_{\hat{i}}^*.$$

(iv) Apply $\natural$ to each side of

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{i,j}^k E_k.$$

□

Next, we consider how $\mathcal{M}$ and $\mathcal{M}^*$ are related.

Definition 6.13. Let $T = T(x)$ denote the subalgebra of $M_X(\mathbb{C})$ generated by $\mathcal{M}$ and $\mathcal{M}^*$. We call T the *subconstituent algebra of $\mathfrak{X}$ with respect to x.*

We have some comments. By construction, the algebra T is finite-dimensional. Moreover T is noncommutative in general. The algebra T is closed under both the transpose map and complex-conjugation, because $\mathcal{M}$ and $\mathcal{M}^*$ are closed under both the transpose map and complex-conjugation.

We are going to show that for $0 \leq \alpha, \beta, \gamma \leq d$,

$$E_\alpha^* A_\beta E_\gamma^* = 0 \text{ iff } p_{\alpha,\beta}^\gamma = 0;$$
$$E_\alpha A_\beta^* E_\gamma = 0 \text{ iff } q_{\alpha,\beta}^\gamma = 0.$$

The above equations are called the *triple product relations.*

To obtain the triple product relations, we endow the vector space $M_X(\mathbb{C})$ with a bilinear form $(\ ,\ )$ such that $(A, B) = \operatorname{tr}(A^t \overline{B})$ for all $A, B \in M_X(\mathbb{C})$. Abbreviate $\|A\|^2 = (A, A)$. For $A, B, C \in M_X(\mathbb{C})$ and $\alpha \in \mathbb{C}$, we have

$$(B, A) = \overline{(A, B)}, \qquad (\alpha A, B) = \alpha(A, B),$$
$$(A + B, C) = (A, C) + (B, C), \qquad \|A\|^2 \in \mathbb{R},$$
$$\|A\|^2 \geq 0, \qquad \|A\|^2 = 0 \text{ iff } A = 0,$$
$$(AB, C) = (B, \overline{A}^t C) = (A, C\overline{B}^t).$$

Lemma 6.14. *For $0 \leq \alpha, \beta, \gamma, i, j, k \leq d$ we have*

(i) $(E_\alpha^* A_\beta E_\gamma^*, E_i^* A_j E_k^*) = \delta_{\alpha,i}\delta_{\beta,j}\delta_{\gamma,k} k_\gamma p_{\alpha,\beta}^\gamma;$

(ii) $(E_\alpha A_\beta^* E_\gamma, E_i A_j^* E_k) = \delta_{\alpha,i}\delta_{\beta,j}\delta_{\gamma,k} m_\gamma q_{\alpha,\beta}^\gamma.$

Proof. (i) Using $\operatorname{tr}(BC) = \operatorname{tr}(CB)$,

$$\begin{aligned}(E_\alpha^* A_\beta E_\gamma^*, E_i^* A_j E_k^*) &= \operatorname{tr}\big((E_\alpha^* A_\beta E_\gamma^*)^t \overline{E_i^* A_j E_k^*}\big) \\ &= \operatorname{tr}\big(E_\gamma^* A_{\beta'} E_\alpha^* E_i^* A_j E_k^*\big) \\ &= \delta_{\alpha,i}\delta_{\gamma,k} \operatorname{tr}\big(E_\gamma^* A_{\beta'} E_\alpha^* A_j\big)\end{aligned}$$