Proof. To get (17), eliminate the associate matrices in (15) using (10), and simplify the result. To get (18), evaluate (17) using Theorem 4.6(iii), and use the fact that $p^\ell_{i,j}$ is real. To get (19), eliminate the primitive idempotents E_i, E_j in (16) using (11), and simplify the result. To get (20), evaluate (19) using Theorem 4.6(iii), and use the fact that $q^\ell_{i,j}$ is real. $\square$

Lecture 6

We will use the following fact from linear algebra.

Lemma 4.11. *For $A, B, C \in M_X(\mathbb{C})$,*

$$\sum_{x\in X}\sum_{y\in X} A_{x,y}B_{x,y}C_{x,y} = \operatorname{tr}\big((A\circ B)C^t\big) = \operatorname{tr}\big((B\circ C)A^t\big) = \operatorname{tr}\big((C\circ A)B^t\big)$$
$$= \operatorname{tr}\big((A^t\circ B^t)C\big) = \operatorname{tr}\big((B^t\circ C^t)A\big) = \operatorname{tr}\big((C^t\circ A^t)B\big).$$

Proof. Use matrix multiplication. $\square$

Proposition 4.12. *We have*

(i) $q^k_{i,0} = \delta_{i,k} \qquad (0 \le i, k \le d)$;

(ii) $q^k_{0,j} = \delta_{j,k} \qquad (0 \le j, k \le d)$;

(iii) $q^0_{i,j} = \delta_{i,\hat{j}}m_i \qquad (0 \le i, j \le d)$;

(iv) $q^k_{i,j} = q^{\hat{k}}_{\hat{i},\hat{j}} \qquad (0 \le i, j, k \le d)$;

(v) $m_i = \sum_{j=0}^d q^k_{i,j} \qquad (0 \le i, k \le d)$;

(vi) $m_\ell q^\ell_{i,j} = m_i q^i_{\ell,\hat{j}} = m_j q^j_{\hat{i},\ell} \qquad (0 \le i, j, \ell \le d)$;

(vii) $\sum_{\alpha=0}^d q^\alpha_{i,j} q^\ell_{k,\alpha} = \sum_{\alpha=0}^d q^\alpha_{k,i} q^\ell_{\alpha,j} \qquad (0 \le i, j, k, \ell \le d)$.

Proof. (i)–(iii) Routine application of (16) and Lemma 4.11.
(iv) Using (16),

$$q^{\hat{k}}_{\hat{i},\hat{j}} = |X|m_k^{-1}\operatorname{tr}\big((E_{\hat{i}}\circ E_{\hat{j}})E_{\hat{k}}\big) = |X|m_k^{-1}\operatorname{tr}\big((\overline{E_i}\circ\overline{E_j})\overline{E_k}\big)$$
$$= |X|m_k^{-1}\overline{\operatorname{tr}\big((E_i\circ E_j)E_k\big)} = \overline{q^k_{i,j}} = q^k_{i,j}.$$

(v) Using (16),

$$\sum_{j=0}^d q^k_{i,j} = |X|m_k^{-1}\sum_{j=0}^d \operatorname{tr}\big((E_i\circ E_j)E_k\big) = |X|m_k^{-1}\operatorname{tr}\big((E_i\circ I)E_k\big) = m_k^{-1}\operatorname{tr}\big((m_iI)E_k\big) = m_i.$$

(vi) Use (16) and Lemma 4.11.
(vii) In the equation

$$(E_k\circ E_i)\circ E_j = E_k\circ(E_i\circ E_j),$$

write each side as a linear combination of $\{E_\ell\}_{\ell=0}^d$, and compare coefficients. $\square$

Our next goal is to show that the Krein parameters are nonnegative. As a warmup, let us review some facts about Hermitean matrices.

A matrix $A \in M_X(\mathbb{C})$ is *Hermitean* whenever $\overline{A}^t = A$. For example, the primitive idempotents $\{E_i\}_{i=0}^d$ are Hermitean. Assume that $A, B \in M_X(\mathbb{C})$ are Hermitean. Then $A \circ B$ is Hermitean. Assume that $A \in M_X(\mathbb{C})$ is Hermitean. Then A diagonalizable, and its eigenspaces are mutually orthogonal. Moreoever, the eigenvalues of A are real. We say that A is *positive semidefinite* (or *PSD*) whenever the eigenvalues of A are nonnegative. One checks that A is PSD if and only if $\overline{v}^t A v \geq 0$ for all $v \in V$, where V is the standard module. Let $\mathbb{A}$ denote a principle submatrix of A (rows/cols of $\mathbb{A}$ indexed by the same subset of X). Note that $\mathbb{A}$ is Hermitean. If A is PSD then $\mathbb{A}$ is PSD.

Lemma 4.13. *Given PSD Hermitean matrices $A, B \in M_X(\mathbb{C})$. Then $A \circ B$ is PSD.*

Proof. Define a matrix $A \otimes B \in M_{X \times X}(\mathbb{C})$ as follows. For vertices $r = (x, y) \in X \times X$ and $s = (u, v) \in X \times X$, the (r, s)-entry of $A \otimes B$ is $A_{x,u} B_{y,v}$. The matrix $A \otimes B$ is Hermitean. The characteristic polynomial of $A \otimes B$ has roots $\{\lambda_x \mu_y | (x, y) \in X \times X\}$, where $\{\lambda_x | x \in X\}$ (resp. $\{\mu_y | y \in X\}$) are the roots of the characteristic polynomial of A (resp. B). Therefore $A \otimes B$ is PSD. The matrix $A \circ B$ is the principle submatrix of $A \otimes B$ with rows/columns indexed by $\{(z, z) | z \in X\}$. By these comments $A \circ B$ is PSD. □

Theorem 4.14. (the Krein condition) *We have $q_{i,j}^k \geq 0$ for $0 \leq i, j, k \leq d$.*

Proof. Let i, j be given. We show that $q_{i,j}^k \geq 0$ for $0 \leq k \leq d$. The matrices E_i and E_j are Hermitean. They are both PSD, because their eigenvalues are zero or one. Therefore $E_i \circ E_j$ is PSD by Lemma 4.13. Recall that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{i,j}^k E_k.$$

So for $0 \leq k \leq d$, the scalar $|X|^{-1} q_{i,j}^k$ is the eigenvalue of $E_i \circ E_j$ for the common eigenspace $E_k V$ of $\mathcal{M}$. This eigenvalue is nonnegative, so $q_{i,j}^k \geq 0$. □

We mention some more inequalities.

Theorem 4.15. *For $0 \leq i, j \leq d$ we have*

$$|P_i(j)| \leq k_i, \qquad |Q_j(i)| \leq m_j. \tag{21}$$

Proof. We first prove the inequality on the left. Recall that $A_i E_j = P_i(j) E_j$. Pick a nonzero $v \in E_j V$, where V is the standard module. Then $A_i v = P_i(j) v$. In this equation, we compare the entries on either side. For $x \in X$, let v_x denote the x-entry of v. From the x-entry in $A_i v = P_i(j) v$, we obtain

$$\sum_{y \in \Gamma_i(x)} v_y = P_i(j) v_x. \tag{22}$$

Pick $x \in X$ such that $|v_x| \geq |v_y|$ for all $y \in X$. Observe that

$$|P_i(j)||v_x| = |P_i(j)v_x| = \left| \sum_{y \in \Gamma_i(x)} v_y \right| \leq \sum_{y \in \Gamma_i(x)} |v_y| \leq \sum_{y \in \Gamma_i(x)} |v_x| = k_i |v_x|.$$

We have $|v_x| > 0$ since $v \neq 0$, so $|P_i(j)| \leq k_i$. We also have

$$|Q_j(i)| = \left| \frac{\overline{P_i(j)} m_j}{k_i} \right| = \frac{|P_i(j)| m_j}{k_i} \leq m_j.$$

□

5 The intersection matrices

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

Earlier we computed the intersection numbers and Krein parameters in terms of the character tables. Our goal in this section is to compute the character tables in terms of the intersection numbers and also the Krein parameters.

Definition 5.1. For $0 \leq i \leq d$, define a matrix $B_i \in M_{d+1}(\mathbb{C})$ with (j,k)-entry $p_{i,j}^k$ for $0 \leq j, k \leq d$. We call B_i the i^{th} *intersection matrix* of $\mathcal{X}$.

Lemma 5.2. *The following* (i)–(vi) *hold.*

(i) $B_0 = I$.

(ii) *For* $0 \leq i \leq d$, *the top row of* B_i *is* $(0, \ldots, 0, 1, 0, \ldots, 0)$, *with the 1 in column* i.

(iii) $\{B_i\}_{i=0}^d$ *are linearly independent.*

(iv) *For* $0 \leq i, j \leq d$,

$$B_i B_j = \sum_{k=0}^{d} p_{i,j}^k B_k.$$

(v) $B_i B_j = B_j B_i$ *for* $0 \leq i, j \leq d$.

(vi) *For* $0 \leq i \leq d$ *we have* $B_i^t = K^{-1} B_{i'} K$ *where* $K = \operatorname{diag}(k_0, k_1, \ldots, k_d)$.

Proof. (i) For $0 \leq j, k \leq d$ the (j,k)-entry of B_0 is $p_{0,j}^k = \delta_{j,k}$.
(ii) For $0 \leq k \leq d$ the $(0,k)$-entry of B_i is $p_{i,0}^k = \delta_{i,k}$.
(iii) By (ii) above.
(iv) Compare the entries on either side using Proposition 3.2(vii).
(v) By (iv) and since $p_{i,j}^k = p_{j,i}^k$ for $0 \leq k \leq d$.
(vi) To get $KB_i^t = B_{i'}K$, compare the entries of each side using Proposition 3.2(vi). □

Definition 5.3. By Lemma 5.2 the matrices $\{B_i\}_{i=0}^d$ form a basis for a commutative subalgebra $\mathcal{B}$ of $M_{d+1}(\mathbb{C})$. We call $\mathcal{B}$ the *intersection algebra* of $\mathcal{X}$.

Theorem 5.4. *There exists an algebra isomorphism $\mathcal{M} \to \mathcal{B}$ that sends $A_i \mapsto B_i$ for $0 \leq i \leq d$.*

Proof. Clear from Lemma 5.2(iv). □

Let us recall some linear algebra. For the moment, let W denote any finite-dimensional vector space over $\mathbb{C}$, and let $\{w_i\}_{i=1}^n$ denote a basis for W. Let $A : W \to W$ denote a $\mathbb{C}$-linear map. There exists a unique $n \times n$ matrix B such that

$$Aw_j = \sum_{i=1}^n B_{i,j} w_i \qquad (1 \leq j \leq n).$$

We say that B *represents* A *with respect to* $\{w_i\}_{i=1}^n$. Let $\{w_i'\}_{i=1}^n$ denote a second basis for W. There exists a unique $n \times n$ matrix S such that

$$w_j' = \sum_{i=1}^n S_{i,j} w_i \qquad (1 \leq j \leq n).$$

The matrix S is invertible. We call S the *transition matrix from* $\{w_i\}_{i=1}^n$ *to* $\{w_i'\}_{i=1}^n$. By linear algebra, the matrix $S^{-1}BS$ represents A with respect to $\{w_i'\}_{i=1}^n$.

We return our attention to $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. The Bose-Mesner algebra $\mathcal{M}$ has bases $\{A_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$. Recall the first and second eigenmatrices P, Q. Then P is the transition matix from $\{E_i\}_{i=0}^d$ to $\{A_i\}_{i=0}^d$. Moreover, $|X|^{-1}Q$ is the transition matrix from $\{A_i\}_{i=0}^d$ to $\{E_i\}_{i=0}^d$.

For $A \in \mathcal{M}$, there exists a $\mathbb{C}$-linear map $L_A : \mathcal{M} \to \mathcal{M}$ that sends $N \mapsto AN$ for all $N \in \mathcal{M}$.

Theorem 5.5. *With the above notation, the following* (i)–(iv) *hold for $0 \leq i \leq d$:*

(i) *B_i^t represents L_{A_i} with respect to the basis $\{A_\ell\}_{\ell=0}^d$;*

(ii) *the matrix* $\operatorname{diag}(P_i(0), P_i(1), \ldots, P_i(d))$ *represents L_{A_i} with respect to $\{E_\ell\}_{\ell=0}^d$;*

(iii) $PB_i^tP^{-1} = \operatorname{diag}(P_i(0), P_i(1), \ldots, P_i(d))$;

(iv) *the scalars $P_i(0), P_i(1), \ldots, P_i(d)$ are the roots of the characteristic polynomial of B_i.*

Proof. (i) By Definition 5.1 and

$$A_iA_j = \sum_{k=0}^d p_{i,j}^k A_k \qquad (0 \leq j \leq d).$$

(ii) Since $A_iE_j = P_i(j)E_j$ for $0 \leq j \leq d$.
(iii) By (i), (ii) and the comments above the theorem statement.
(iv) B_i and B_i^t have the same characteristic polynomial. The result follows in view of (iii). □