4 Character tables for commutative association schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. Recall the standard module $V = \mathbb{C}^X$. Define the vector $\mathbf{1} \in V$ that has all entries 1.

Lemma 4.1. *We have*

$$V = \sum_{i=0}^{d} E_i V \qquad \text{(orthogonal direct sum).}$$

For $0 \leq i \leq d$ the subspace E_iV is a common eigenspace for $\mathcal{M}$, and E_i is the projection onto this eigenspace. Moreover $\mathbf{1}$ is a basis for E_0V.

Proof. Routine consequence of Proposition 3.7. □

The Bose-Mesner algebra $\mathcal{M}$ has a basis $\{A_i\}_{i=0}^d$ and a basis $\{E_i\}_{i=0}^d$. Let us consider how these bases are related. We have

$$A_i = \sum_{j=0}^{d} P_i(j) E_j \qquad (0 \leq i \leq d), \tag{10}$$

$$E_i = |X|^{-1} \sum_{j=0}^{d} Q_i(j) A_j \qquad (0 \leq i \leq d), \tag{11}$$

$$P_i(j) \in \mathbb{C}, \qquad Q_i(j) \in \mathbb{C} \qquad (0 \leq i, j \leq d). \tag{12}$$

For $0 \leq i, j \leq d$ the scalar $P_i(j)$ is the eigenvalue of A_i for the common eigenspace E_jV.

Let $M_{d+1}(\mathbb{C})$ denote the algebra over $\mathbb{C}$ consisting of the $d+1$ by $d+1$ matrices that have all entries in $\mathbb{C}$. We index the rows and columns by $0, 1, \ldots, d$. Define $P \in M_{d+1}(\mathbb{C})$ that has (i,j)-entry $P_j(i)$ for $0 \leq i, j \leq d$. Define $Q \in M_{d+1}(\mathbb{C})$ that has (i,j)-entry $Q_j(i)$ for $0 \leq i, j \leq d$. By construction,

$$PQ = |X| I = QP. \tag{13}$$

We call P the *first eigenmatrix* (or *first character table*) of $\mathcal{X}$. We call Q the *second eigenmatrix* (or *second character table*) of $\mathcal{X}$.

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Lemma 4.2. *The following hold for $0 \leq i \leq d$:*

(i) $\operatorname{tr}(A_i) = \delta_{i,0}|X|$;

(ii) $A_iJ = JA_i = k_iJ$.

Proof. (i) We have $A_0 = I$. For $1 \leq i \leq d$ the diagonal entries of A_i are zero.
(ii) The matrix A_i has constant row sum k_i by the definition of k_i. The matrix A_i has constant column sum k_i because $A_i^t = A_{i'}$ and $k_{i'} = k_i$. $\square$

Define

$$m_i = \dim E_i V \qquad (0 \leq i \leq d). \tag{14}$$

Lemma 4.3. *We have*

(i) $m_0 = 1$;

(ii) $m_i = m_{\hat{i}} \qquad (0 \leq i \leq d)$;

(iii) $|X| = \sum_{i=0}^{d} m_i$;

(iv) $m_i \neq 0 \qquad (0 \leq i \leq d)$.

Proof. (i) E_0 has basis $\mathbf{1}$.
(ii) Note that m_i is equal to the rank of E_i. The matrices E_i, E_i^t have the same rank, and $E_i^t = E_{\hat{i}}$.
(iii) By Lemma 4.1 and (14).
(iv) By construction. $\square$

Lemma 4.4. *The following hold for $0 \leq i \leq d$:*

(i) $\mathrm{tr}(E_i) = m_i$;

(ii) $E_i J = J E_i = \delta_{i,0} J$.

Proof. (i) The matrix E_i is similar to a diagonal matrix that has m_i diagonal entries 1 and all other diagonal entries 0.
(ii) Use $E_i E_0 = E_0 E_i = \delta_{i,0} E_0$ and $E_0 = |X|^{-1} J$. $\square$

Proposition 4.5. *The following hold for $0 \leq i \leq d$:*

(i) $P_0(i) = 1$;

(ii) $P_i(0) = k_i$;

(iii) $Q_0(i) = 1$;

(iv) $Q_i(0) = m_i$.

Proof. (i) Since $A_0 = I = \sum_{i=0}^{d} E_i$.
(ii) In the equation $A_i = \sum_{j=0}^{d} P_i(j) E_j$, multiply each side on the right by E_0 and simplify using $A_i E_0 = k_i E_0$.
(iii) Since $|X| E_0 = J = \sum_{i=0}^{d} A_i$.
(iv) In the equation $E_i = |X|^{-1} \sum_{j=0}^{d} Q_i(j) A_j$, take the trace of each side and evaluate the result using Lemmas 4.2, 4.4. $\square$

Theorem 4.6. *The following hold for* $0 \le i, j \le d$*:*

(i) $P_{i'}(j) = \overline{P_i(j)}$;

(ii) $Q_{\hat{i}}(j) = \overline{Q_i(j)}$;

(iii) $\frac{Q_j(i)}{m_j} = \frac{\overline{P_i(j)}}{k_i}$;

(iv) $\sum_{\ell=0}^{d} P_\ell(i)\overline{P_\ell(j)}k_\ell^{-1} = \delta_{i,j}|X|m_i^{-1}$ (first orthogonality relation);

(v) $\sum_{\ell=0}^{d} P_i(\ell)\overline{P_j(\ell)}m_\ell = \delta_{i,j}|X|k_i$ (second orthogonality relation).

Proof. (i) In the equation $A_i = \sum_{j=0}^{d} P_i(j)E_j$, take the conjugate-transpose of each side and simplify the result using $\overline{E}_j^t = E_j$ along with $A_i^t = A_{i'} = \sum_{j=0}^{d} P_{i'}(j)E_j$.
(ii) In the equation $E_i = |X|^{-1}\sum_{j=0}^{d} Q_i(j)A_j$, take the complex conjugate of each side and simplify the result using $\overline{E}_i = E_{\hat{i}} = |X|^{-1}\sum_{j=0}^{d} Q_{\hat{i}}(j)A_j$.
(iii) We compute the trace of $A_{i'}E_j$ in two ways. On one hand,

$$\mathrm{tr}(A_{i'}E_j) = P_{i'}(j)\mathrm{tr}(E_j) = \overline{P_i(j)}\mathrm{tr}(E_j) = \overline{P_i(j)}m_j.$$

On the other hand,

$$\begin{aligned}\mathrm{tr}(A_{i'}E_j) &= |X|^{-1}\mathrm{tr}\left(A_{i'}\sum_{\ell=0}^{d} Q_j(\ell)A_\ell\right) = |X|^{-1}\mathrm{tr}\left(\sum_{\ell=0}^{d} Q_j(\ell)\sum_{h=0}^{d} p_{i',\ell}^{h}A_h\right)\\ &= \sum_{\ell=0}^{d} Q_j(\ell)p_{i',\ell}^{0} = \sum_{\ell=0}^{d} Q_j(\ell)\delta_{i,\ell}k_i = Q_j(i)k_i.\end{aligned}$$

(iv) In the equation $PQ = |X|I$, compute the (i,j)-entry of each side, and evaluate the result using (iii) above.
(v) In the equation $QP = |X|I$, compute the (j,i)-entry of each side, and evaluate the result using (iii) above. □

Theorem 4.7. *The following hold for* $0 \le i, j, \ell \le d$*:*

(i) $P_i(\ell)P_j(\ell) = \sum_{k=0}^{d} p_{i,j}^{k}P_k(\ell)$;

(ii) $Q_i(\ell)Q_j(\ell) = \sum_{k=0}^{d} q_{i,j}^{k}Q_k(\ell)$;

(iii) $P_i(j)Q_j(\ell) = \sum_{k=0}^{d} p_{i,k}^{\ell}Q_j(k)$.

Proof. (i) In the equation $A_iA_j = \sum_{k=0}^{d} p_{i,j}^{k}A_k$, write each side as a linear combination of the primitive idempotents using (10).
(ii) In the equation $E_i \circ E_j = |X|^{-1}\sum_{k=0}^{d} q_{i,j}^{k}E_k$, write each side as a linear combination of the associate matrices using (11).
(iii) By (i) above,

$$P_{i'}(j)P_\ell(j) = \sum_{k=0}^{d} p_{i',\ell}^{k}P_k(j).$$

In this equation, eliminate $P_\ell(j)$ and $P_k(j)$ using Theorem 4.6(iii), and simplify the result using Proposition 3.2(vi). □

Our next goal is to compute the intersection numbers and Krein parameters from the character tables.

Lemma 4.8. *For $0 \leq i, j, \ell \leq d$ we have*

$$p^\ell_{i,j} = |X|^{-1} k_\ell^{-1} \operatorname{tr}\big(A_i A_j A_{\ell'}\big); \tag{15}$$

$$q^\ell_{i,j} = |X| m_\ell^{-1} \operatorname{tr}\big((E_i \circ E_j) E_\ell\big). \tag{16}$$

Proof. Concerning (15), we have

$$\begin{aligned}\operatorname{tr}\big(A_i A_j A_{\ell'}\big) &= \operatorname{tr}\left(\sum_{h=0}^{d} p^h_{i,j} A_h A_{\ell'}\right) = \operatorname{tr}\left(\sum_{h=0}^{d}\sum_{\nu=0}^{d} p^h_{i,j} p^\nu_{h,\ell'} A_\nu\right) \\ &= |X| \sum_{h=0}^{d} p^h_{i,j} p^0_{h,\ell'} = |X| \sum_{h=0}^{d} p^h_{i,j} \delta_{h,\ell} k_\ell = |X| k_\ell p^\ell_{i,j}.\end{aligned}$$

Concerning (16), we have

$$\operatorname{tr}\big((E_i \circ E_j) E_\ell\big) = \operatorname{tr}\left(|X|^{-1} \sum_{h=0}^{d} q^h_{i,j} E_h E_\ell\right) = |X|^{-1} q^\ell_{i,j} \operatorname{tr}(E_\ell) = |X|^{-1} m_\ell q^\ell_{i,j}.$$

□

Lemma 4.9. *We have $q^\ell_{i,j} \in \mathbb{R}$ for $0 \leq i, j, \ell \leq d$.*

Proof. By (16) we have

$$\begin{aligned}|X|^{-1} m_\ell \overline{q^\ell_{i,j}} &= \operatorname{tr}\big((\overline{E_i} \circ \overline{E_j})\overline{E_\ell}\big) = \operatorname{tr}\big((E_i^t \circ E_j^t) E_\ell^t\big) = \operatorname{tr}\big((E_i \circ E_j)^t E_\ell^t\big) \\ &= \operatorname{tr}\big(E_\ell (E_i \circ E_j)\big)^t = \operatorname{tr}\big(E_\ell (E_i \circ E_j)\big) = \operatorname{tr}\big((E_i \circ E_j) E_\ell\big) = |X|^{-1} m_\ell q^\ell_{i,j}.\end{aligned}$$

□

Theorem 4.10. *For $0 \leq i, j, \ell \leq d$ we have*

$$p^\ell_{i,j} = |X|^{-1} k_\ell^{-1} \sum_{\nu=0}^{d} P_i(\nu) P_j(\nu) \overline{P_\ell(\nu)} m_\nu \tag{17}$$

$$= |X|^{-1} k_i k_j \sum_{\nu=0}^{d} Q_\nu(i) Q_\nu(j) \overline{Q_\nu(\ell)} m_\nu^{-2}; \tag{18}$$

$$q^\ell_{i,j} = |X|^{-1} m_\ell^{-1} \sum_{\nu=0}^{d} Q_i(\nu) Q_j(\nu) \overline{Q_\ell(\nu)} k_\nu \tag{19}$$

$$= |X|^{-1} m_i m_j \sum_{\nu=0}^{d} P_\nu(i) P_\nu(j) \overline{P_\nu(\ell)} k_\nu^{-2}. \tag{20}$$

Proof. To get (17), eliminate the associate matrices in (15) using (10), and simplify the result. To get (18) , evaluate (17) using Theorem 4.6(iii), and use the fact that $p^\ell_{i,j}$ is real. To get (19), eliminate the primitive idempotents E_i, E_j in (16) using (11), and simplify the result. To get (20) , evaluate (19) using Theorem 4.6(iii), and use the fact that $q^\ell_{i,j}$ is real. □

We will use the following fact from linear algebra.

Lemma 4.11. *For $A, B, C \in M_X(\mathbb{C})$,*

$$\sum_{x\in X}\sum_{y\in X} A_{x,y}B_{x,y}C_{x,y} = \mathrm{tr}\big((A\circ B)C^t\big) = \mathrm{tr}\big((B\circ C)A^t\big) = \mathrm{tr}\big((C\circ A)B^t\big)$$
$$= \mathrm{tr}\big((A^t\circ B^t)C\big) = \mathrm{tr}\big((B^t\circ C^t)A\big) = \mathrm{tr}\big((C^t\circ A^t)B\big).$$

Proof. Use matrix multiplication. □

Proposition 4.12. *We have*

(i) $q^k_{i,0} = \delta_{i,k} \qquad (0 \le i, k \le d)$;

(ii) $q^k_{0,j} = \delta_{j,k} \qquad (0 \le j, k \le d)$;

(iii) $q^0_{i,j} = \delta_{i,\hat{j}}m_i \qquad (0 \le i, j \le d)$;

(iv) $q^k_{i,j} = q^{\hat{k}}_{\hat{i},\hat{j}} \qquad (0 \le i, j, k \le d)$;

(v) $m_i = \sum_{j=0}^d q^k_{i,j} \qquad (0 \le i, k \le d)$;

(vi) $m_\ell q^\ell_{i,j} = m_i q^i_{\ell,\hat{j}} = m_j q^j_{\hat{i},\ell} \qquad (0 \le i, j, \ell \le d)$;

(vii) $\sum_{\alpha=0}^d q^\alpha_{i,j} q^\ell_{k,\alpha} = \sum_{\alpha=0}^d q^\alpha_{k,i} q^\ell_{\alpha,j} \qquad (0 \le i, j, k, \ell \le d)$.

Proof. (i)–(iii) Routine application of (16) and Lemma 4.11.
(iv) Using (16),

$$q^{\hat{k}}_{\hat{i},\hat{j}} = |X|m_k^{-1}\mathrm{tr}\big((E_{\hat{i}}\circ E_{\hat{j}})E_{\hat{k}}\big) = |X|m_k^{-1}\mathrm{tr}\big((\overline{E_i}\circ\overline{E_j})\overline{E_k}\big)$$
$$= |X|m_k^{-1}\mathrm{tr}\overline{\big((E_i\circ E_j)E_k\big)} = \overline{q^k_{i,j}} = q^k_{i,j}.$$

(v) Using (16),

$$\sum_{j=0}^d q^k_{i,j} = |X|m_k^{-1}\sum_{j=0}^d \mathrm{tr}\big((E_i\circ E_j)E_k\big) = |X|m_k^{-1}\mathrm{tr}\big((E_i\circ I)E_k\big) = m_k^{-1}\mathrm{tr}\big((m_iI)E_k\big) = m_i.$$

(vi) Use (16) and Lemma 4.11.
(vii) In the equation

$$(E_k\circ E_i)\circ E_j = E_k\circ(E_i\circ E_j),$$

write each side as a linear combination of $\{E_\ell\}_{\ell=0}^d$, and compare coefficients. □