For $0 \le i, j, k \le d$ and $(x, y) \in R_k$,

$$p^k_{i,j} = |\Gamma_i(x) \cap \Gamma_{j'}(y)|.$$

Define

$$k_i = p^0_{i,i'} \qquad (0 \le i \le d). \tag{7}$$

For $x \in X$,

$$k_i = |\Gamma_i(x)| \qquad (0 \le i \le d).$$

Lemma 3.1. *We have*

(i) $k_0 = 1$;

(ii) $k_i = k_{i'} \qquad (0 \le i \le d)$;

(iii) $|X| = \sum_{i=0}^{d} k_i$;

(iv) $k_i \ne 0 \qquad (0 \le i \le d)$.

Proof. Routine. □

Lecture 4

Proposition 3.2. *We have*

(i) $p^k_{i,0} = \delta_{i,k} \qquad (0 \le i, k \le d)$;

(ii) $p^k_{0,j} = \delta_{j,k} \qquad (0 \le j, k \le d)$;

(iii) $p^0_{i,j} = \delta_{i,j'} k_i \qquad (0 \le i, j \le d)$;

(iv) $p^k_{i,j} = p^{k'}_{i',j'} \qquad (0 \le i, j, k \le d)$;

(v) $k_i = \sum_{j=0}^{d} p^k_{i,j} \qquad (0 \le i, k \le d)$;

(vi) $k_\ell p^\ell_{i,j} = k_i p^i_{\ell,j'} = k_j p^j_{i',\ell} \qquad (0 \le i, j, \ell \le d)$;

(vii) $\sum_{\alpha=0}^{d} p^\alpha_{i,j} p^\ell_{k,\alpha} = \sum_{\alpha=0}^{d} p^\alpha_{k,i} p^\ell_{\alpha,j} \qquad (0 \le i, j, k, \ell \le d)$.

Proof. (i)–(iv) Routine.
(v) Fix $(x, y) \in R_k$. Partition $\Gamma_i(x)$ according to how its elements are related to y. This gives

$$\Gamma_i(x) = \cup_{j=0}^{d} (\Gamma_i(x) \cap \Gamma_{j'}(y)) \qquad \text{(disjoint union)}.$$

In this equation, take the cardinality of each side.
(vi) The three common values are equal to $|X|^{-1}$ times the number of 3-tuples (x, y, z) such that $(x, y) \in R_\ell$ and $(x, z) \in R_i$ and $(z, y) \in R_j$.
(vii) In the equation $A_k(A_iA_j) = (A_kA_i)A_j$, write each side as a linear combination of $\{A_\ell\}_{\ell=0}^{d}$, and compare coefficients. □

As we study the Bose-Mesner algebra of $\mathcal{X}$, the following results will be useful.

Lemma 3.3. *Let $\mathcal{M}$ denote a nonzero subspace of the vector space $M_X(\mathbb{C})$. Assume that:*

(i) *$\mathcal{M}$ is closed under matrix multiplication;*

(ii) *$AB = BA$ for all $A, B \in \mathcal{M}$;*

(iii) *$\mathcal{M}$ is closed under the conjugate-transpose map.*

Then $\mathcal{M}$ has a basis $\{E_i\}_{i=0}^d$ such that $E_iE_j = \delta_{i,j}E_i$ for $0 \leq i, j \leq d$. This basis is unique up to permutation of $E_0, E_1, \ldots, E_d$.

Proof. Consider the action of $\mathcal{M}$ on the standard module V. We claim that V has a basis consisting of common eigenvectors for $\mathcal{M}$. Let U denote the sum of the common eigenspaces for $\mathcal{M}$. It suffices to show that $U = V$. Suppose $U \subsetneq V$. We have an orthogonal direct sum $V = U + U^\perp$. By construction U is $\mathcal{M}$-invariant. By this and (6) the subspace $U^\perp$ is $\mathcal{M}$-invariant. Since the elements of $\mathcal{M}$ mutually commute, there exists $0 \neq v \in U^\perp$ that is a common eigenvector for $\mathcal{M}$. The vector v is contained in a common eigenspace for $\mathcal{M}$, so $v \in U$. Now $v \in U \cap U^\perp = 0$, for a contradiction. By these comments $U = V$, and the claim is proved. By the claim, there exists an invertible $S \in M_X(\mathbb{C})$ such that SAS^{-1} is diagonal for all $A \in \mathcal{M}$. By construction, $S\mathcal{M}S^{-1}$ is a nonzero subspace of $M_X(\mathbb{C})$ that is closed under matrix multiplication and has all elements diagonal. For diagonal matrices $A, B \in M_X(\mathbb{C})$ we have $AB = A \circ B$. Therefore $S\mathcal{M}S^{-1}$ is closed under Hadamard multiplication. By Lemma 2.1 the subspace $S\mathcal{M}S^{-1}$ has a basis $\{A_i\}_{i=0}^d$ such that $A_i \circ A_j = \delta_{i,j}A_i$ for $0 \leq i, j \leq d$. Note that $A_iA_j = \delta_{i,j}A_i$ for $0 \leq i, j \leq d$. Define $E_i = S^{-1}A_iS$ for $0 \leq i \leq d$. Then $\{E_i\}_{i=0}^d$ is a basis for $\mathcal{M}$ such that $E_iE_j = \delta_{i,j}E_i$ for $0 \leq i, j \leq d$. The uniqueness assertion is routinely checked. □

Definition 3.4. Referring to Lemma 3.3, we call $\{E_i\}_{i=0}^d$ the *primitive idempotents* of $\mathcal{M}$.

Lemma 3.5. *For the subspace $\mathcal{M}$ in Lemma 3.3, its primitive idempotents satisfy $\overline{E}_i^t = E_i$ for $0 \leq i \leq d$.*

Proof. The subspace $\mathcal{M}$ contains $\overline{E}_i^t$ for $0 \leq i \leq d$. The matrices $\{\overline{E}_i^t\}_{i=0}^d$ form a basis for $\mathcal{M}$ such that $\overline{E}_i^t\overline{E}_j^t = \delta_{i,j}\overline{E}_i^t$ for $0 \leq i, j \leq d$. By the uniqueness statement in Lemma 3.3, the sequence $\{\overline{E}_i^t\}_{i=0}^d$ is a permutation of the sequence $\{E_i\}_{i=0}^d$. For $0 \leq i \leq d$ we have $\overline{E}_i^tE_i \neq 0$, so $\overline{E}_i^t = E_i$. □

Lemma 3.6. *We refer to the subspace $\mathcal{M}$ in Lemma 3.3.*

(i) *Assume that $I \in \mathcal{M}$. Then $I = \sum_{i=0}^d E_i$.*

(ii) *Assume that $J \in \mathcal{M}$. Then $|X|^{-1}J$ is a primitive idempotent of $\mathcal{M}$ (denoted E_0).*

(iii) *Assume that $\mathcal{M}$ is closed under both the transpose map and complex conjugation. Then for $0 \leq i \leq d$ there exists $\hat{i} \in \{0, 1, \ldots, d\}$ such that $E_i^t = E_{\hat{i}} = \overline{E}_i$.*

Proof. (i) There exists scalars $\{\alpha_i\}_{i=0}^d$ in $\mathbb{C}$ such that $I = \sum_{i=0}^d \alpha_i E_i$. For $0 \le i \le d$ we have

$$E_i = E_i I = E_i \sum_{j=0}^{d} \alpha_j E_j = \alpha_i E_i,$$

so $\alpha_i = 1$.
(ii) There exists scalars $\{\beta_i\}_{i=0}^d$ in $\mathbb{C}$ such that $J = \sum_{i=0}^d \beta_i E_i$. At least one of $\{\beta_i\}_{i=0}^d$ is nonzero. Without loss, we may assume $\beta_0 \neq 0$. We have $JE_0 = \beta_0 E_0$. Note that $J^2 = |X|J$, so

$$JE_0 = |X|^{-1} J^2 E_0 = |X|^{-1} J E_0 J = |X|^{-1} s J,$$

where s is the sum of all the entries of E_0. By these comments E_0 is a scalar multiple of J. Using $E_0^2 = E_0$ we obtain $E_0 = |X|^{-1} J$.
(iii) The subspace $\mathcal{M}$ contains E_i^t for $0 \le i \le d$. The matrices $\{E_i^t\}_{i=0}^d$ form a basis for $\mathcal{M}$ such that $E_i^t E_j^t = \delta_{i,j} E_i^t$ for $0 \le i, j \le d$. By the uniqueness statement in Lemma 3.3, the sequence $\{E_{\hat{i}}^t\}_{i=0}^d$ is a permutation of the sequence $\{E_i\}_{i=0}^d$. In other words, for $0 \le i \le d$ there exists $\hat{i} \in \{0, 1, \ldots, d\}$ such that $E_i^t = E_{\hat{i}}$. By Lemma 3.5 we have $\overline{E}_i = E_i^t = E_{\hat{i}}$. $\square$

We return our attention to the commutative association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$.

Proposition 3.7. *The Bose-Mesner algebra $\mathcal{M}$ of $\mathcal{X}$ has a basis $\{E_i\}_{i=0}^d$ that satisfies*

(i) $E_0 = |X|^{-1} J$;

(ii) $E_i E_j = \delta_{i,j} E_i$ *for* $0 \le i, j \le d$;

(iii) $I = \sum_{i=0}^d E_i$;

(iv) *for* $0 \le i \le d$ *there exists* $\hat{i} \in \{0, 1, \ldots, d\}$ *such that* $E_i^t = E_{\hat{i}} = \overline{E}_i$.

Proof. Note that $\mathcal{M}$ satisfies the conditions of Lemma 3.3 and Lemma 3.6. $\square$

The matrices $\{E_i\}_{i=0}^d$ form a basis for $\mathcal{M}$. Since $\mathcal{M}$ is closed under Hadamard multiplication, for $0 \le i, j \le d$ there exist $q_{i,j}^k \in \mathbb{C}$ $(0 \le k \le d)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{i,j}^k E_k. \tag{8}$$

By construction,

$$q_{i,j}^k = q_{j,i}^k \qquad (0 \le i, j, k \le d). \tag{9}$$

The scalars $q_{i,j}^k$ are called the *Krein parameters* of $\mathcal{X}$. Shortly we will show that $q_{i,j}^k$ is real and nonnegative for $0 \le i, j, k \le d$.

4 Character tables for commutative association schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. Recall the standard module $V = \mathbb{C}^X$. Define the vector $\mathbf{1} \in V$ that has all entries 1.

Lemma 4.1. *We have*

$$V = \sum_{i=0}^{d} E_i V \qquad \text{(orthogonal direct sum).}$$

For $0 \le i \le d$ the subspace E_iV is a common eigenspace for $\mathcal{M}$, and E_i is the projection onto this eigenspace. Moreover $\mathbf{1}$ is a basis for E_0V.

Proof. Routine consequence of Proposition 3.7. □

The Bose-Mesner algebra $\mathcal{M}$ has a basis $\{A_i\}_{i=0}^d$ and a basis $\{E_i\}_{i=0}^d$. Let us consider how these bases are related. We have

$$A_i = \sum_{j=0}^{d} P_i(j) E_j \qquad (0 \le i \le d), \tag{10}$$

$$E_i = |X|^{-1} \sum_{j=0}^{d} Q_i(j) A_j \qquad (0 \le i \le d), \tag{11}$$

$$P_i(j) \in \mathbb{C}, \qquad Q_i(j) \in \mathbb{C} \qquad (0 \le i, j \le d). \tag{12}$$

For $0 \le i, j \le d$ the scalar $P_i(j)$ is the eigenvalue of A_i for the common eigenspace E_jV.

Let $M_{d+1}(\mathbb{C})$ denote the algebra over $\mathbb{C}$ consisting of the $d+1$ by $d+1$ matrices that have all entries in $\mathbb{C}$. We index the rows and columns by $0, 1, \ldots, d$. Define $P \in M_{d+1}(\mathbb{C})$ that has (i,j)-entry $P_j(i)$ for $0 \le i, j \le d$. Define $Q \in M_{d+1}(\mathbb{C})$ that has (i,j)-entry $Q_j(i)$ for $0 \le i, j \le d$. By construction,

$$PQ = |X| I = QP. \tag{13}$$

We call P the *first eigenmatrix* (or *first character table*) of $\mathcal{X}$. We call Q the *second eigenmatrix* (or *second character table*) of $\mathcal{X}$.

Lemma 4.2. *The following hold for $0 \le i \le d$:*

(i) $\operatorname{tr}(A_i) = \delta_{i,0}|X|$;

(ii) $A_i J = J A_i = k_i J$.

Proof. (i) We have $A_0 = I$. For $1 \le i \le d$ the diagonal entries of A_i are zero.
(ii) The matrix A_i has constant row sum k_i by the definition of k_i. The matrix A_i has constant column sum k_i because $A_i^t = A_{i'}$ and $k_{i'} = k_i$. □