# Lecture 35

## 25 Some open problems

In this section we give some open problems related to association schemes and graph theory in general. These problems are at the research level; an elegant solution or substantial progress is surely publishable. The problems are in a raw form; feel free to adjust any given problem into a more elegant or suitable form.

All the graphs discussed in this section are assumed to be finite, undirected, and connected, without loops or multiple edges. Fix a finite set $X$ with $|X| \geq 2$.

We motivate the first problem with some comments about association schemes. Let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denote a symmetric association scheme with primitive idempotents $\{E_i\}_{i=0}^d$. Recall the standard module $V = \mathbb{R}^X$ and the function algebra product $\circ : V \times V \to V$. Recall that for $0 \leq i, j \leq d$,

$$\mathrm{Span}\big(E_i V \circ E_j V\big) = \sum_{\substack{0 \leq k \leq d \\ q_{i,j}^k \neq 0}} E_k V.$$

Assume that $\mathcal{X}$ is $Q$-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$. Then $E_1 V$ generates $V$ in the function algebra. Moreover

$$E_1 V \circ E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V \qquad (0 \leq i \leq d),$$

where $E_{-1} = 0$ and $E_{d+1} = 0$. For $0 \leq i \leq d$ define

$$(E_1 V)^{\circ i} = \mathrm{Span}\big(E_1 V \circ E_1 V \circ \cdots \circ E_1 V\big) \qquad (i \text{ copies}).$$

We interpret $(E_1 V)^{\circ 0} = E_0 V = \mathrm{Span}(\mathbf{1})$, where $\mathbf{1} = \sum_{y \in X} \hat{y}$. We have

$$\sum_{\ell=0}^{i} E_\ell V = \sum_{\ell=0}^{i} (E_1 V)^{\circ \ell} \qquad (0 \leq i \leq d).$$

We are done with the motivation. Now let $\Gamma = (X, \mathcal{R})$ denote any graph with vertex set $X$ and adjacency relation $\mathcal{R}$. Let $A \in M_X(\mathbb{R})$ denote the adjacency matrix of $\Gamma$. We assume that $\Gamma$ is regular with valency $k$; thus each vertex in $X$ is adjacent to exactly $k$ vertices in $X$. In this case $k$ is the maximal eigenvalue of $A$, and the corresponding eigenspace is spanned by $\mathbf{1}$. We denote this eigenspace by $V_0$ and call it trivial. Let $\{V_i\}_{i=1}^D$ denote an ordering of the nontrivial eigenspaces of $A$.

**Definition 25.1.** The above ordering $\{V_i\}_{i=0}^D$ is called *$Q$-polynomial* whenever

$$\sum_{\ell=0}^{i} V_\ell = \sum_{\ell=0}^{i} (V_1)^{\circ \ell} \qquad (0 \leq i \leq D).$$

**Definition 25.2.** A graph $\Gamma$ is said to be *$Q$-polynomial* whenever $\Gamma$ is regular, and there exists at least one $Q$-polynomial ordering of its eigenspaces.

**Example 25.3.** Assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a $Q$-polynomial association scheme. Let $S$ denote a subset of $\{1, 2, \ldots, d\}$ such that $A = \sum_{i \in S} A_i$ generates the Bose-Mesner algebra of $\mathcal{X}$. Then for the relation $\mathcal{R} = \cup_{i \in S} R_i$ the graph $(X, \mathcal{R})$ is $Q$-polynomial.

**Conjecture 25.4.** Let $\Gamma$ denote a $Q$-polynomial graph. Let $\mathcal{M}$ denote the subalgebra of $M_X(\mathbb{R})$ generated by the adjacency matrix $A$ of $\Gamma$. Then $\mathcal{M}$ is the Bose-Mesner algebra of a $Q$-polynomial association scheme.

We now describe the $Q$-polynomial property from another point of view.

**Definition 25.5.** Let $\Gamma = (X, \mathcal{R})$ denote a regular graph. Let $U$ denote a nontrivial eigenspace of $\Gamma$. We say that $\Gamma$ is *$Q$-polynomial with respect to $U$* whenever there exists a $Q$-polynomial ordering $\{V_i\}_{i=0}^D$ of the eigenspaces of $\Gamma$ such that $V_1 = U$.

Let $\Gamma = (X, \mathcal{R})$ denote a graph. Recall the bilinear form $\langle\, , \rangle : V \times V \to \mathbb{R}$ such that $\langle u, v \rangle = u^t v$ for all $u, v \in V$.

**Definition 25.6.** Assume that $\Gamma$ is regular, and let $U$ denote a nontrivial eigenspace of $\Gamma$ that generates $V$ in the function algebra. Define the integer

$$D = \min\Big\{ i \Big| i \geq 0,\ \sum_{\ell=0}^{i} U^{\circ \ell} = V \Big\}.$$

Next, we recursively define some subspaces $\{U^{(i)}\}_{i=0}^D$ of $V$. Define $U^{(0)} = V_0$ and $U^{(1)} = U$. For $2 \leq i \leq D$ define $U^{(i)}$ to be the orthogonal complement of $U^{(0)} + U^{(1)} + \cdots + U^{(i-1)}$ in $\sum_{\ell=0}^{i} U^{\circ \ell}$.

Referring to Definition 25.6, for $0 \leq i \leq D$ we have

$$\sum_{\ell=0}^{i} U^{(\ell)} = \sum_{\ell=0}^{i} U^{\circ \ell},$$

with the sum on the left being orthogonal and direct. In particular

$$V = U^{(0)} + U^{(1)} + \cdots + U^{(D)} \qquad \text{(orthogonal direct sum)}.$$

**Lemma 25.7.** *Let $U$ denote a nontrivial eigenspace of a regular graph $\Gamma = (X, \mathcal{R})$. Then the following are equivalent:*

(i) *$\Gamma$ is $Q$-polynomial with respect to $U$;*

(ii) *$U$ generates $V$ in the function algebra, and $U^{(i)}$ is an eigenspace of $\Gamma$ for $0 \leq i \leq D$.*

*Proof.* By the construction above the lemma, and since the eigenspaces of $\Gamma$ are mutually orthogonal. ∎

In order to clarify things, let us consider a special case. Referring to Lemma 25.7, assume that each eigenspace of $\Gamma$ has dimension one. Let the vector $u = \sum_{x \in X} u_x \hat{x}$ be a basis for $U$. Consider the vectors

$$\sum_{x \in X} (u_x)^i \hat{x} \qquad (0 \le i \le D). \tag{112}$$

Applying the Gram-Schmidt orthogonalization process to the vectors (112), we obtain a sequence of polynomials $f_i \in \mathbb{R}[\lambda]$ $(0 \le i \le D)$ such that $f_i$ has degree $i$ and the vector $\sum_{x \in X} f_i(u_x)\hat{x}$ is a basis for $U^{(i)}$ $(0 \le i \le D)$. The polynomials $\{f_i\}_{i=0}^D$ are orthogonal; the orthogonality is

$$\sum_{x \in X} f_i(u_x) f_j(u_x) = \begin{cases} 0 & \text{if } i \ne j; \\ \ne 0 & \text{if } i = j \end{cases} \qquad (0 \le i, j \le D).$$

By the theory of orthogonal polynomials,

$$\lambda f_i \in \operatorname{Span}\{f_{i-1}, f_i, f_{i+1}\} \qquad (0 \le i \le D),$$

where $f_{-1} = 0$ and $f_{D+1} = \prod_{x \in X}(\lambda - u_x)$. In summary we have the following result.

**Lemma 25.8.** *Assume that $\Gamma$ is regular, and every eigenspace of $\Gamma$ has dimension one. Let $U$ denote a nontrivial eigenspace of $\Gamma$, with basis $u = \sum_{x \in X} u_x \hat{x}$. Then the following are equivalent:*

(i) *$\Gamma$ is $Q$-polynomial with respect to $U$;*

(ii) *there exists a sequence of orthogonal polynomials $\{f_i\}_{i=0}^D$ such that*

$$\sum_{x \in X} f_i(u_x)\hat{x}$$

*is an eigenvector of $\Gamma$ for $0 \le i \le D$.*

**Problem 25.9.** Hunt for some $Q$-polynomial graphs that have all eigenspaces of dimension one.

We now consider a related problem. For this problem we view $V = \mathbb{C}^X$.

**Definition 25.10.** Let $\Gamma = (X, \mathcal{R})$ denote a graph. Let $\Phi \in \mathbb{C}[\lambda, \mu]$ denote a polynomial in two variables such that $\Phi(\lambda, \mu) = \pm\Phi(\mu, \lambda)$. By a *$\Phi$-hyper-eigenvector* for $\Gamma$ we mean a vector $u = \sum_{x \in X} u_x \hat{x} \in V$ such that for all $x \in X$ the multiset of scalars $\{u_y\}_{y \in \Gamma(x)}$ gives all the roots of the polynomial $\Phi(\lambda, u_x)$. We call $\Phi$ the *hyper-eigenvalue* of $u$.

Referring to Definition 25.10, if $u = \sum_{x \in X} u_x \hat{x}$ is a $\Phi$-hyper-eigenvector then $\Phi(u_x, u_y) = 0$ for all pairs of adjacent vertices $x, y \in X$.

**Example 25.11.** Assume that $\Gamma$ is the 3-cube $H(3,2)$. Let $\uparrow, \downarrow$ denote a basis for a 2-dimensional vector space $W$. We interpret $V = W \otimes W \otimes W$. We interpret $\{\hat{x}\}_{x \in X}$ to be the set of vectors

$$r \otimes s \otimes t \qquad r, s, t \in \{\uparrow, \downarrow\}.$$

We now give the hyper-eigenvalues and their hyper-eigenvectors.

(i) For $\Phi = (\lambda - \mu)^3$;

$$(\uparrow + \downarrow) \otimes (\uparrow + \downarrow) \otimes (\uparrow + \downarrow).$$

(ii) For $\Phi = (\lambda - \mu)^2(\lambda + \mu)$:

$$(\uparrow - \downarrow) \otimes (\uparrow + \downarrow) \otimes (\uparrow + \downarrow),$$
$$(\uparrow + \downarrow) \otimes (\uparrow - \downarrow) \otimes (\uparrow + \downarrow),$$
$$(\uparrow + \downarrow) \otimes (\uparrow + \downarrow) \otimes (\uparrow - \downarrow).$$

(iii) For $\Phi = (\lambda - \mu)(\lambda + \mu)^2$:

$$(\uparrow - \downarrow) \otimes (\uparrow - \downarrow) \otimes (\uparrow + \downarrow),$$
$$(\uparrow - \downarrow) \otimes (\uparrow + \downarrow) \otimes (\uparrow - \downarrow),$$
$$(\uparrow + \downarrow) \otimes (\uparrow - \downarrow) \otimes (\uparrow - \downarrow).$$

(iv) For $\Phi = (\lambda + \mu)^3$:

$$(\uparrow - \downarrow) \otimes (\uparrow - \downarrow) \otimes (\uparrow - \downarrow).$$

**Problem 25.12.** Find the hyper-eigenvalues and hyper-eigenvectors for your favorite graph.

**Definition 25.13.** Let $A$ and $B$ denote square matrices of the same size. We say that $A$ and $B$ are *linked* whenever there exists an invertible matrix $P$ such that $P^{-1}AP$ is diagonal and $PBP^{-1}$ is diagonal.

Every linked pair of matrices over $\mathbb{C}$ appears in the following way.

**Example 25.14.** Let $V$ denote a finite-dimensional vector space over $\mathbb{C}$. Let $S : V \to V$ and $T : V \to V$ denote diagonalizable linear transformations.

(i) Pick an eigenbasis for $S$. Let $A$ denote the matrix that represents $T$ in this basis.

(ii) Pick an eigenbasis for $T$. Let $B$ denote the matrix that represents $S$ in this basis.

(iii) Let $P$ denote the transition matrix from the basis in (i) to the basis in (ii).

The matrices $P^{-1}AP$ and $PBP^{-1}$ are diagonal, so $A$ and $B$ are linked.

**Problem 25.15.** What pairs of matrices are linked?

**Definition 25.16.** Let $\Gamma$ and $\Gamma'$ denote graphs with the same number of vertices. We say that $\Gamma$ and $\Gamma'$ are *linked* whenever their adjacency matrices are linked.

**Problem 25.17.** What pairs of graphs are linked?

**Problem 25.18.** What matrix is linked to itself?

**Problem 25.19.** What graph is linked to itself?

**Example 25.20.** It is routine to check that the complete graph $K_n$ is linked to itself, and any hypercube $H(d, 2)$ is linked to itself.