Proof. (i), (ii) By Corollary 23.9 we find that cE_i^Y $(0 \leq i \leq s-1)$ and $I - c\sum_{i=0}^{s-1} E_i^Y$ are mutually orthogonal idempotents. These are linearly independent and contained in $\mathcal{M}^Y$. They must form a basis for $\mathcal{M}^Y$, because $\mathcal{M}^Y$ has dimension $s+1$. By these comments the subspace $\mathcal{M}^Y$ is closed under matrix multiplication. Therefore $\mathcal{Y}$ is a symmetric association scheme.
(iii) By the construction and since $\mathcal{X}$ is Q-polynomial with respect to $\{E_i\}_{i=0}^d$.
(iv) We saw earlier that $c\sum_{i=0}^{s} E_i^Y = I^Y$. □

Lecture 34

24 Linear programming in the hypercube

We return our attention to the hypercube $H(d,2)$. When we first introduced linear programming, we considered an example involving the orthogonality graph Ω_d. For $d=4$ we worked out the solution by brute force. In this section we give the solution for all d.

Let X denote the vertex set of $H(d,2)$. Recall that $|X| = 2^d$. Recall the bipartition $X = X^+ \cup X^-$. Note that each of $X^\pm$ has size $|X|/2 = 2^{d-1}$.

We now recall the orthogonality graph.

Definition 24.1. For even $d = 2t \geq 2$, the *orthogonality graph* Ω_d has vertex set X; vertices y, z are adjacent in Ω_d whenever $(y,z) \in R_t$ in $H(d,2)$. A set of vertices $Y \subseteq X$ is called *independent* in Ω_d whenever no two vertices in Y are adjacent in Ω_d.

Problem 24.2. Find the maximal size of an independent set in Ω_d.

The above problem is easily solved for t odd, and much harder for t even. Let us first dispense with the case of t odd.

Lemma 24.3. *Assume that t is odd.*

(i) *2^{d-1} is the maximum size of an independent set in Ω_d;*

(ii) *each of $X^\pm$ is an independent set of size 2^{d-1};*

(iii) *there is no other independent set in Ω_d of size 2^{d-1}.*

Proof. The sets $X^\pm$ are independent in Ω_d, because t is odd and for $x, y \in X^\pm$ we have $(x,y) \in R_k$ with k even. Assume $Y \subseteq X$ is independent in Ω_d. We show $|Y| \leq 2^{d-1}$, with equality if and only if $Y = X^\pm$. Let $\overline{Y} = X \backslash Y$. Note that $Y = X^\pm$ if and only if $\overline{Y}$ is independent in Ω_d. The graph Ω_d is regular; let κ denote the valency. We count the edges in Ω_d between Y and $\overline{Y}$. Since Y is independent, every vertex in Y is adjacent to exactly κ vertices in $\overline{Y}$. Therefore the edge count is $|Y|\kappa$. Each vertex in $\overline{Y}$ is adjacent to at most κ vertices in Y. Therefore the edge count is at most $|\overline{Y}|\kappa$, with equality iff $\overline{Y}$ is independent in Ω_d. By these comments $|Y|\kappa \leq |\overline{Y}|\kappa$, with equality iff $\overline{Y}$ is independent in Ω_d. Therefore $|Y| \leq |\overline{Y}|$, with equality iff $\overline{Y}$ is independent in Ω_d. Therefore $|Y| \leq 2^{d-1}$, with equality iff $\overline{Y}$ is independent in Ω_d. The result follows. □

For the rest of this section, we assume that t is even. In this case, the above Problem 24.2 is open, so we consider the following related problem.

Problem 24.4. Use linear programming to find an upper bound on the size of an independent set in Ω_d.

We will prove the following result.

Theorem 24.5. *For t even, the linear programming upper bound is $2^d/d$ for the size of an independent set in Ω_d.*

We recall some facts about $H(d,2)$. The intersection numbers are

$$c_i = i, \qquad b_i = d - i \qquad (0 \le i \le d). \tag{101}$$

The valencies are

$$k_i = \binom{d}{i} \qquad (0 \le i \le d). \tag{102}$$

The eigenvalues and dual eigenvalues are

$$\theta_i = d - 2i, \qquad \theta_i^* = d - 2i \qquad (0 \le i \le d). \tag{103}$$

The eigenmatrices P and Q satisfy $P = Q$. Their entries are given by

$$P_i(j) = Q_i(j) = K_i(j) \qquad (0 \le i, j \le d), \tag{104}$$

where $\{K_i\}_{i=0}^d$ are the Krawtchouk polynomials. For $0 \le j \le d$ we have

$$K_0(j) = 1, \qquad K_1(j) = d - 2j, \qquad K_2(j) = \frac{(d-2j)^2 - d}{2}. \tag{105}$$

The Krawtchouk polynomial generating function is

$$\sum_{i=0}^{d} K_i(j) z^i = (1-z)^j (1+z)^{d-j} \qquad (0 \le j \le d). \tag{106}$$

The Krawtchouk polynomials satisfy

$$\frac{K_i(j)}{k_i} = \frac{K_j(i)}{k_j} \qquad (0 \le i, j \le d). \tag{107}$$

Lemma 24.6. *We have*

$$K_2(i) = \frac{\theta_i^2 - d}{2} \qquad K_i(2) = \frac{\binom{d}{i}}{\binom{d}{2}} K_2(i) \qquad (0 \le i \le d).$$

Proof. By (105) and (107). □

Recall $d = 2t$ with t even.

Lemma 24.7. *The following hold for $0 \le i \le d$.*

(i) *Assume i is odd. Then $K_i(t) = K_t(i) = 0$.*

(ii) *Assume $i = 2\ell$ is even. Then*

$$K_i(t) = (-1)^\ell \binom{t}{\ell}, \qquad K_t(i) = (-1)^\ell \binom{d}{t} \frac{(2\ell-1)(2\ell-3)\cdots 3\cdot 1}{(d-1)(d-3)\cdots(d-2\ell+1)}$$

Proof. To obtain $K_i(t)$ we use the generating function. We have

$$\sum_{i=0}^{d} K_i(t) z^i = (1-z)^t(1+z)^t = (1-z^2)^t = \sum_{\ell=0}^{t} (-1)^\ell \binom{t}{\ell} z^{2\ell}.$$

To obtain $K_t(i)$ from $K_i(t)$ we use (107). □

We are now ready to apply linear programming with

$$D = \{0, 1, \ldots, d\}, \qquad M = D \backslash \{t\}, \qquad C = Q.$$

Lemma 24.8. *The following is a program for Problem (Q, M):*

$$a_i = \frac{1}{d}\binom{d}{i} + \frac{d-1}{d} K_i(2) = \binom{d}{i} \frac{\theta_i^2}{d^2} \qquad (i \in M). \qquad (108)$$

For this program the objective function is $g = 2^d/d$. Moreover

$$a_2^* = \frac{(d-1)2^d}{d}, \qquad a_i^* = 0 \qquad (1 \le i \le d, i \ne 2),$$

where

$$a_j^* = \sum_{i \in M} a_i Q_j(i) \qquad (1 \le j \le d). \qquad (109)$$

Proof. By construction, $a_0 = 1$ and $a_i \ge 0$ for $i \in M^\times$. For notational convenience, define $a_t = 0$. Note that (108) holds at $i = t$ because $\theta_t = 0$. Define

$$(a_0^*, a_1^*, \ldots, a_d^*) = (a_0, a_1, \ldots, a_d)Q.$$

Note that

$$(a_0, a_1, \ldots, a_d) = \frac{1}{d}\Big(\text{row 0 of } P\Big) + \frac{d-1}{d}\Big(\text{row 2 of } P\Big).$$

Therefore

$$\begin{aligned}
(a_0^*, a_1^*, \ldots, a_d^*) &= (a_0, a_1, \ldots, a_d)Q \\
&= \frac{1}{d}\Big(\text{row 0 of } P\Big)Q + \frac{d-1}{d}\Big(\text{row 2 of } P\Big)Q \\
&= \frac{1}{d}\Big(\text{row 0 of } PQ\Big) + \frac{d-1}{d}\Big(\text{row 2 of } PQ\Big) \\
&= \frac{|X|}{d}\Big(\text{row 0 of } I\Big) + \frac{(d-1)|X|}{d}\Big(\text{row 2 of } I\Big) \\
&= \Big(\frac{2^d}{d}, 0, \frac{(d-1)2^d}{d}, 0, 0, \ldots, 0\Big).
\end{aligned}$$

Note that $a_j^* \geq 0$ for $1 \leq j \leq d$. By these comments $\{a_i\}_{i \in M}$ is a program for Problem (Q, M). Note that $g = a_0^* = 2^d/d$. The result follows. $\square$

Lemma 24.9. *The following is a program for Problem $(Q,M)'$: for $0 \leq i \leq d$,*

$$\alpha_i = \frac{1}{d} + \frac{d-1}{d}\frac{K_t(i)}{\binom{d}{t}} = \begin{cases} 1/d & \text{if } i \text{ is odd;} \\ 1/d + (-1)^\ell \frac{d-1}{d} \frac{(2\ell-1)(2\ell-3)\cdots 3\cdot 1}{(d-1)(d-3)\cdots(d-2\ell+1)} & \text{if } i = 2\ell \text{ is even} \end{cases} \tag{110}$$

For this program the objective function is $\gamma = 2^d/d$. Moreover

$$\alpha_j^* = 0 \qquad (j \in M^\times),$$

where

$$\alpha_j^* = \sum_{i \in D} \alpha_i Q_i(j) \qquad (j \in M^\times). \tag{111}$$

Proof. Using (110), we find $\alpha_0 = 1$ and $\alpha_i \geq 0$ for $1 \leq i \leq d$. Define

$$(\alpha_0^*, \alpha_1^*, \ldots, \alpha_d^*) = (\alpha_0, \alpha_1, \ldots, \alpha_d)Q^t.$$

By (110),

$$(\alpha_0, \alpha_1, \ldots, \alpha_d) = \frac{1}{d}\Big(\text{row } 0 \text{ of } P^t\Big) + \frac{d-1}{d}\frac{1}{\binom{d}{t}}\Big(\text{row } t \text{ of } P^t\Big).$$

Therefore

$$\begin{aligned}
(\alpha_0^*, \alpha_1^*, \ldots, \alpha_d^*) &= (\alpha_0, \alpha_1, \ldots, \alpha_d)Q^t \\
&= \frac{1}{d}\Big(\text{row } 0 \text{ of } P^t\Big)Q^t + \frac{d-1}{d}\frac{1}{\binom{d}{t}}\Big(\text{row } t \text{ of } P^t\Big)Q^t \\
&= \frac{1}{d}\Big(\text{row } 0 \text{ of } P^tQ^t\Big) + \frac{d-1}{d}\frac{1}{\binom{d}{t}}\Big(\text{row } t \text{ of } P^tQ^t\Big) \\
&= \frac{|X|}{d}\Big(\text{row } 0 \text{ of } I\Big) + \frac{(d-1)|X|}{d\binom{d}{t}}\Big(\text{row } t \text{ of } I\Big) \\
&= \Big(\frac{2^d}{d}, 0, \ldots, 0, \frac{(d-1)2^d}{d\binom{d}{t}}, 0, \ldots, 0\Big).
\end{aligned}$$

This shows that $\alpha_j^* = 0$ unless $j = t$ $(1 \leq j \leq d)$. Therefore $\alpha_j^* = 0$ for $j \in M^\times$. Consequently $\alpha_j^* \leq 0$ for $j \in M^\times$. By these comments $\{\alpha_i\}_{i=0}^d$ is a program for Problem $(Q, M)'$. Note that $\gamma = \alpha_0^* = 2^d/d$. The result follows. $\square$

We displayed a program for Problem (Q, M) and a program for Problem $(Q, M)'$ such that $g = 2^d/d = \gamma$. Therefore, every program for Problem (Q, M) has objective function at most $2^d/d$. Consequently, an independent set in Ω_d has cardinality at most $2^d/d$. Theorem 24.5 is proved.