Corollary 22.19. *The algebra* $\mathcal{M}^Y$ *is the Bose-Mesner algebra of the association scheme* $\mathcal{Y}$. *Moreover*

$$A_i^Y A_j^Y = \sum_{k=0}^{s} r_{i,j}^k A_k^Y \qquad (0 \le i, j \le s).$$

Proof. By construction $\{A_i^Y\}_{i=0}^s$ are the associate matrices of $\mathcal{Y}$, and the $r_{i,j}^k$ are the intersection numbers of $\mathcal{Y}$. □

Note 22.20. It is known that the association scheme $\mathcal{Y}$ is Q-polynomial. See the paper

Sho Suda. New parameters of subsets in polynomial schemes.
`arXiv:1008.0189`.

Lecture 33

23 Relative t-designs

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ that is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents.

Until further notice, Y denotes a nonempty subset of X.

Recall the inner distribution $\{a_i\}_{i=0}^d$ of Y and the dual distribution $\{a_i^*\}_{i=0}^d$ of Y. Recall that for $0 \le t \le d$, the set Y is a t-design whenever the strength of Y is at least t. This occurs if and only if $a_i^* = 0$ for $1 \le i \le t$, if and only if $E_i \psi_Y = 0$ for $1 \le i \le t$.

We now introduce the notion of a relative t-design.

Until further notice, fix $x \in X$ and write $T = T(x)$.

Definition 23.1. For $0 \le t \le d$, we call Y a *relative t-design* with respect to x, whenever the vectors $E_i \psi_Y$ and $E_i \hat{x}$ are linearly dependent for $1 \le i \le t$.

Note 23.2. The subset Y is always a relative 0-design with respect to x, because $E_0 \psi_Y$ and $E_0 \hat{x}$ are both scalar multiples of $\mathbf{1}$.

We now investigate the linear dependencies in Definition 23.1. Recall that

$$\|E_i \psi_Y\|^2 = \frac{|Y|}{|X|} a_i^*, \qquad \|E_i \hat{x}\|^2 = |X|^{-1} m_i \qquad (0 \le i \le d).$$

Lemma 23.3. *For* $0 \le i \le d$,

$$\langle E_i \psi_Y, E_i \hat{x} \rangle = |X|^{-1} \sum_{\ell=0}^{d} Q_i(\ell) |\Gamma_\ell(x) \cap Y|.$$

Proof. We have

$$\begin{aligned}\langle E_i\psi_Y, E_i\hat{x}\rangle &= \langle \psi_Y, E_i^2\hat{x}\rangle = \langle \psi_Y, E_i\hat{x}\rangle \\ &= |X|^{-1}\sum_{\ell=0}^{d} Q_i(\ell)\langle \psi_Y, A_\ell\hat{x}\rangle = |X|^{-1}\sum_{\ell=0}^{d} Q_i(\ell)|\Gamma_\ell(x)\cap Y|.\end{aligned}$$

□

Lemma 23.4. *For* $0 \le i \le d$ *we have*

$$a_i^* \ge \frac{\left(\sum_{\ell=0}^{d} Q_i(\ell)|\Gamma_\ell(x)\cap Y|\right)^2}{|Y|m_i}, \tag{98}$$

with equality if and only if $E_i\psi_Y$, $E_i\hat{x}$ *are linearly dependent. In this case*

$$E_i\psi_Y = \frac{\langle E_i\psi_Y, E_i\hat{x}\rangle}{\|E_i\hat{x}\|^2}E_i\hat{x}. \tag{99}$$

Proof. Compute the inner product matrix

$$\begin{pmatrix} \|E_i\psi_Y\|^2 & \langle E_i\psi_Y, E_i\hat{x}\rangle \\ \langle E_i\hat{x}, E_i\psi_Y\rangle & \|E_i\hat{x}\|^2 \end{pmatrix}$$

using Lemma 23.3 and the comments above it. The inner product matrix is positive semidefinite, so the determinate is nonnegative. The determinate is zero if and only if $E_i\psi_Y$, $E_i\hat{x}$ are linearly dependent. In this case, the dependency (99) is readily computed. □

Corollary 23.5. *For* $1 \le t \le d$ *the following are equivalent:*

(i) Y *is a relative* t*-design with respect to* x*;*

(ii) *equality holds in* (98) *for* $1 \le i \le t$.

Proof. By Definition 23.1 and Lemma 23.4. □

Theorem 23.6. *Define*

$$s = \big|\{k|1 \le k \le d,\ \Gamma_k(x)\cap Y \ne \emptyset\}\big|.$$

Assume that Y *is a* t*-design with* $t+1 \ge s$. *Then each nonempty* $\Gamma_k(x)\cap Y$ *is a relative* $(t+1-s)$*-design with respect to* x.

Proof. Define the set

$$S = \{k|1 \le k \le d,\ \Gamma_k(x)\cap Y \ne \emptyset\}.$$

Note that $s = |S|$. For $0 \le k \le d$,

$$E_k^*\psi_Y = \sum_{y\in\Gamma_k(x)\cap Y} \hat{y}.$$

For $1 \le k \le d$, $E_k^*\psi_Y \neq 0$ if and only if $k \in S$.

It suffices to show that for $k \in S$ the vectors

$$E_i E_k^* \psi_Y, \qquad E_i \hat{x}$$

are linearly dependent for $1 \le i \le t+1-s$.

We will be discussing the vector $E_0^*\psi_Y$. Note that $E_0^*\psi_Y = \hat{x}$ if $x \in Y$, and $E_0^*\psi_Y = 0$ if $x \notin Y$. Consider the following two sets of vectors:

(i) $\{E_k^*\psi_Y\}_{k\in S} \cup \{E_0^*\psi_Y\}$;

(ii) $\{E_i\hat{x} \circ \psi_Y\}_{i=0}^{s-1} \cup \{E_0^*\psi_Y\}$.

We claim that the above sets (i), (ii) have the same span. To prove the claim, note that for $0 \le i \le s-1$,

$$\begin{aligned} E_i\hat{x} \circ \psi_Y &= |X|^{-1} \sum_{k=0}^{d} Q_i(k) A_k \hat{x} \circ \psi_Y = |X|^{-1} \sum_{k=0}^{d} Q_i(k) E_k^* \psi_Y \\ &= |X|^{-1} \left(m_i E_0^* \psi_Y + \sum_{k \in S} Q_i(k) E_k^* \psi_Y \right). \end{aligned}$$

Consider the $s \times s$ submatrix of Q that has rows indexed by S and columns indexed by $\{0, 1, 2, \ldots, s-1\}$. This submatrix is essentially Vandermonde, and hence invertible. The claim follows.

For $k \in S$,

$$E_k^*\psi_Y \in \operatorname{Span}\{E_i\hat{x} \circ \psi_Y\}_{i=0}^{s-1} + \operatorname{Span}\{\hat{x}\}.$$

Also for $0 \le i \le s-1$,

$$\begin{aligned} E_i\hat{x} \circ \psi_Y &= E_i\hat{x} \circ \left(\psi_Y - \frac{|Y|}{|X|}\mathbf{1} \right) + \frac{|Y|}{|X|} E_i\hat{x} \circ \mathbf{1} \\ &= E_i\hat{x} \circ \left(\psi_Y - \frac{|Y|}{|X|}\mathbf{1} \right) + \frac{|Y|}{|X|} E_i\hat{x}. \end{aligned}$$

We saw earlier that

$$\psi_Y - \frac{|Y|}{|X|}\mathbf{1} \in E_{t+1}V + \cdots + E_dV.$$

So for $0 \le i \le s-1$,

$$\begin{aligned} E_i\hat{x} \circ \left(\psi_Y - \frac{|Y|}{|X|}\mathbf{1} \right) &\in E_{t+1-i}V + \cdots + E_dV \\ &\subseteq E_{t+2-s}V + \cdots + E_dV. \end{aligned}$$

The Bose-Mesner algebra $\mathcal{M}$ has basis $\{E_i\}_{i=0}^d$. The primary T-module $\mathcal{M}\hat{x}$ has basis $\{E_i\hat{x}\}_{i=0}^d$. Let $k \in S$. By the above comments,

$$E_k^*\psi_Y \in E_{t+2-s}V + \cdots + E_dV + \mathcal{M}\hat{x}.$$

So for $1 \leq i \leq t+1-s$,

$$E_iE_k^*\psi_Y \in E_i\mathcal{M}\hat{x} = \text{Span}\{E_i\hat{x}\}.$$

Therefore, the vectors

$$E_iE_k^*\psi_Y, \qquad E_i\hat{x}$$

are linearly dependent. The result follows. □

Lemma 23.7. *Let s denote the degree of Y. Assume that Y is a t-design with $t+1 \geq s$. Then for $x \in Y$ and $0 \leq k \leq d$,*

$$\left|\Gamma_k(x) \cap Y\right| = a_k.$$

Proof. In Lemma 22.13 we proved this under the assumption that $t = 2s$. However we did not use the full strength of that assumption. We only used $t+1 \geq s$. □

Recall the norm $\|A\|^2 = \text{tr}(A^tA)$ for $A \in M_X(\mathbb{R})$.

Lemma 23.8. *For $0 \leq k, \ell \leq d$ we have*

$$\Big\| |X|E_k\Delta_Y E_\ell - |Y|\delta_{k,\ell}E_k \Big\|^2 = |Y|\sum_{j=1}^{d} q_{k,\ell}^j a_j^*, \tag{100}$$

where Δ_Y is the diagonal matrix in $M_X(\mathbb{R})$ with (y,y)-entry 1 if $y \in Y$ and 0 if $y \notin Y$ $(y \in X)$.

Proof. Routine using

$$\psi_Y^t E_j \psi_Y = \frac{|Y|}{|X|}a_j^* \qquad (0 \leq j \leq d)$$

and

$$q_{k,\ell}^0 = \delta_{k,\ell}m_\ell$$

and

$$(E_k)_{y,y} = |X|^{-1}m_k \qquad (y \in X).$$

Write $\Delta = \Delta_Y$. First assume that $k \neq \ell$. We have

$$\begin{aligned}
\left\| |X| E_k \Delta E_\ell \right\|^2 &= |X|^2 \mathrm{tr}\Big((E_k \Delta E_\ell)^t E_k \Delta E_\ell \Big) \\
&= |X|^2 \mathrm{tr}\big(E_\ell \Delta E_k^2 \Delta E_\ell \big) \\
&= |X|^2 \mathrm{tr}\big(\Delta E_k \Delta E_\ell \big) \\
&= |X|^2 \sum_{y \in X} \sum_{z \in X} \Delta_{y,y} (E_k)_{y,z} \Delta_{z,z} (E_\ell)_{z,y} \\
&= |X|^2 \sum_{y \in Y} \sum_{z \in Y} (E_k)_{y,z} (E_\ell)_{z,y} \\
&= |X|^2 \sum_{y \in Y} \sum_{z \in Y} (E_k)_{y,z} (E_\ell)_{y,z} \\
&= |X|^2 \sum_{y \in Y} \sum_{z \in Y} (E_k \circ E_\ell)_{y,z} \\
&= |X|^2 \psi_Y^t (E_k \circ E_\ell) \psi_Y \\
&= |X| \sum_{j=1}^{d} q_{k,\ell}^j \psi_Y^t E_j \psi_Y \\
&= |Y| \sum_{j=1}^{d} q_{k,\ell}^j a_j^*.
\end{aligned}$$

The result is proved for $k \neq \ell$. The proof for $k = \ell$ is similar. □

Corollary 23.9. *For $0 \leq t \leq d$ the following are equivalent:*

(i) *Y is a t-design;*

(ii) *for $k, \ell \geq 0$ such that $k + \ell \leq t$,*

$$|X| E_k \Delta_Y E_\ell = |Y| \delta_{k,\ell} E_k.$$

Proof. Routine using Lemma 23.8. □

Let s denote the degree of Y. For notational convenience, assume that $a_i \neq 0$ for $1 \leq i \leq s$.

Theorem 23.10. *Assume that Y is a t-design with $t \geq 2s - 2$. Then:*

(i) *$\mathcal{Y} = (Y, \{R_i^Y\}_{i=0}^s)$ is a symmetric association scheme;*

(ii) *$\mathcal{Y}$ has primitive idempotents cE_i^Y $(0 \leq i \leq s-1)$ and $I - c\sum_{i=0}^{s-1} E_i^Y$, where $c = |X|/|Y|$;*

(iii) *$\mathcal{Y}$ is Q-polynomial with respect to the above ordering of the primitive idempotents;*

(iv) *assume that $t = 2s$. Then $I - c\sum_{i=0}^{s-1} E_i^Y = cE_s^Y$.*

Proof. (i), (ii) By Corollary 23.9 we find that cE_i^Y $(0 \leq i \leq s-1)$ and $I - c\sum_{i=0}^{s-1} E_i^Y$ are mutually orthogonal idempotents. These are linearly independent and contained in $\mathcal{M}^Y$. They must form a basis for $\mathcal{M}^Y$, because $\mathcal{M}^Y$ has dimension $s+1$. By these comments the subspace $\mathcal{M}^Y$ is closed under matrix multiplication. Therefore $\mathcal{Y}$ is a symmetric association scheme.
(iii) By the construction and since $\mathcal{X}$ is Q-polynomial with respect to $\{E_i\}_{i=0}^d$.
(iv) We saw earlier that $c\sum_{i=0}^{s} E_i^Y = I^Y$. □