Lecture 32

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ that is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents.

Until further notice, Y denotes a nonempty subset of X that has degree s and strength $t = 2s$.

Recall the inner distribution $\{a_i\}_{i=0}^d$ of Y. Recall the set $S = \{i | 1 \leq i \leq d, \ a_i \neq 0\}$. Recall that $|S| = s$. Reindexing the relations $\{R_i\}_{i=0}^d$ if necessary, we may assume without loss that

$$S = \{1, 2, \ldots, s\}.$$

Definition 22.7. For $0 \leq i \leq d$ let R_i^Y denote the restriction of R_i to $Y \times Y$. By construction, R_i^Y is nonempty if and only if $0 \leq i \leq s$.

Lemma 22.8. *The following hold:*

(i) $R_0^Y = \{(y, y) | y \in Y\}$;

(ii) *the relations* $\{R_i^Y\}_{i=0}^s$ *partition* $Y \times Y$;

(iii) R_i^Y *is symmetric for* $0 \leq i \leq s$.

Proof. By the construction. □

Our next general goal is to show that $\mathcal{Y} = (Y, \{R_i^Y\}_{i=0}^s)$ is a symmetric association scheme.

Definition 22.9. Let the matrix $Q^Y \in M_{s+1}(\mathbb{R})$ be the submatrix of Q associated with rows $0, 1, \ldots, s$ and columns $0, 1, \ldots, s$. So Q^Y has (i, j)-entry

$$Q_{i,j}^Y = Q_j(i) = v_j^*(\theta_i^*) \qquad (0 \leq i, j \leq s).$$

Lemma 22.10. *The matrix* Q^Y *is invertible.*

Proof. The polynomial $v_i^*(z)$ has degree i for $0 \leq i \leq s$. The scalars $\{\theta_j^*\}_{j=0}^s$ are mutually distinct. By these comments the matrix Q^Y is essentially Vandermonde, and hence invertible. □

The following example should clarify what is meant by essentially Vandermonde.

Example 22.11. Assume that $s = 2$. Then

$$Q^Y = \begin{pmatrix} 1 & \theta_0^* & \frac{(\theta_0^*)^2 - q_{1,1}^1 \theta_0^* - q_{1,1}^0}{q_{1,1}^2} \\ 1 & \theta_1^* & \frac{(\theta_1^*)^2 - q_{1,1}^1 \theta_1^* - q_{1,1}^0}{q_{1,1}^2} \\ 1 & \theta_2^* & \frac{(\theta_2^*)^2 - q_{1,1}^1 \theta_2^* - q_{1,1}^0}{q_{1,1}^2} \end{pmatrix}.$$

Via elementary column operations, we can reduce Q^Y to the Vandermonde matrix

$$\begin{pmatrix} 1 & \theta_0^* & (\theta_0^*)^2 \\ 1 & \theta_1^* & (\theta_1^*)^2 \\ 1 & \theta_2^* & (\theta_2^*)^2 \end{pmatrix}.$$

The above Vandermonde matrix is invertible. An elementary column operation changes the determinant by a nonzero scalar factor. Therefore Q^Y is invertible.

Definition 22.12. Define a matrix $P^Y \in M_{s+1}(\mathbb{R})$ such that

$$P^Y Q^Y = |Y| I.$$

For $0 \le i, j \le s$ the (i,j)-entry of P^Y is denoted $P_j^Y(i)$.

Lemma 22.13. *For $x \in Y$ and $0 \le i \le s$,*

$$|\Gamma_i(x) \cap Y| = a_i.$$

Proof. By definition, a_i is the average value of $|\Gamma_i(x) \cap Y|$, where the average is over all $x \in Y$. Therefore, It suffices to show that $|\Gamma_i(x) \cap Y|$ does not depend on the choice of x. Recall the vector ψ_Y. Recall that $E_j \psi_Y = 0$ for $1 \le j \le 2s$ and

$$E_0 \psi_Y = \frac{|Y|}{|X|} \mathbf{1}.$$

For $0 \le j \le 2s$ we have

$$\begin{aligned}
\delta_{0,j} \frac{|Y|}{|X|} &= \langle E_j \hat{x}, E_j \psi_Y \rangle = \sum_{y \in Y} \langle E_j \hat{x}, E_j \hat{y} \rangle = \sum_{k=0}^{s} \sum_{y \in \Gamma_k(x) \cap Y} \langle E_j \hat{x}, E_j \hat{y} \rangle \\
&= |X|^{-1} \sum_{k=0}^{s} \sum_{y \in \Gamma_k(x) \cap Y} Q_j(k) = |X|^{-1} \sum_{k=0}^{s} \sum_{y \in \Gamma_k(x) \cap Y} v_j^*(\theta_k^*) \\
&= |X|^{-1} \sum_{k=0}^{s} \big|\Gamma_k(x) \cap Y\big| v_j^*(\theta_k^*).
\end{aligned}$$

For notational convenience, define

$$z_k = \big|\Gamma_k(x) \cap Y\big| \qquad (0 \le k \le s).$$

By the above equations,

$$\big(z_0, z_1, \ldots, z_s\big) Q^Y = \big(|Y|, 0, \ldots, 0\big).$$

Therefore

$$\big(z_0, z_1, \ldots, z_s\big) = \big(1, 0, \ldots, 0\big) P^Y. \tag{97}$$

This shows that for $0 \le i \le s$ the number z_i does not depend on the choice of x. Therefore $z_i = a_i$ for $0 \le i \le s$. The result follows. □

We have some comments about P^Y.

Lemma 22.14. *The following hold for $0 \le i \le s$:*

(i) $P_0^Y(i) = 1$;

(ii) $P_i^Y(0) = a_i$.

Proof. (i) In the previous lecture we saw that

$$\frac{(z-\theta_1^*)(z-\theta_2^*)\cdots(z-\theta_s^*)}{(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)\cdots(\theta_0^*-\theta_s^*)} = \frac{v_0^*(z)+v_1^*(z)+\cdots+v_s^*(z)}{|Y|}.$$

Taking $z \in \{\theta_0^*, \theta_1^*, \dots, \theta_s^*\}$ we obtain

$$Q^Y \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} |Y| \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = P^Y \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(ii) In the proof of Lemma 22.13 we obtained

$$(a_0, a_1, \dots, a_s) = (1, 0, \dots, 0)P^Y.$$

□

The following result will help us understand the combinatorial regularity of $\mathcal{Y}$.

Lemma 22.15. *Let $0 \le i, j, k \le s$ and $x, y \in Y$ with $(x, y) \in R_k$. Then*

$$\langle E_i\hat{x} \circ E_j\hat{y}, \psi_Y \rangle = \delta_{i,j}\frac{|Y|}{|X|^2}v_i^*(\theta_k^*).$$

Proof. We have

$$\begin{aligned}\langle E_i\hat{x} \circ E_j\hat{y}, \psi_Y \rangle &= \sum_{\ell=0}^{d}\sum_{h=0}^{d}\langle E_\ell(E_i\hat{x} \circ E_j\hat{y}), E_h\psi_Y \rangle \\ &= \sum_{\ell=0}^{d}\langle E_\ell(E_i\hat{x} \circ E_j\hat{y}), E_\ell\psi_Y \rangle.\end{aligned}$$

In the above sum, the ℓ-summand is zero for $1 \le \ell \le d$, because

$$\begin{aligned}&E_\ell(E_i\hat{x} \circ E_j\hat{y}) = 0 \qquad (2s+1 \le \ell \le d), \\ &E_\ell\psi_Y = 0 \qquad (1 \le \ell \le 2s).\end{aligned}$$

By these comments

$$\begin{aligned}\langle E_i\hat{x} \circ E_j\hat{y}, \psi_Y \rangle &= \langle E_0(E_i\hat{x} \circ E_j\hat{y}), E_0\psi_Y \rangle = \frac{|Y|}{|X|}\langle E_0(E_i\hat{x} \circ E_j\hat{y}), \mathbf{1} \rangle \\ &= \frac{|Y|}{|X|}\langle E_i\hat{x} \circ E_j\hat{y}, E_0\mathbf{1} \rangle = \frac{|Y|}{|X|}\langle E_i\hat{x} \circ E_j\hat{y}, \mathbf{1} \rangle \\ &= \frac{|Y|}{|X|}\langle E_i\hat{x}, E_j\hat{y} \rangle = \delta_{i,j}\frac{|Y|}{|X|^2}Q_i(k) = \delta_{i,j}\frac{|Y|}{|X|^2}v_i^*(\theta_k^*).\end{aligned}$$

□

Recall the Bose-Mesner algebra $\mathcal{M}$.

Definition 22.16. For $M \in \mathcal{M}$ let M^Y denote the restriction of M to $Y \times Y$. Define

$$\mathcal{M}^Y = \text{Span}\{M^Y | M \in \mathcal{M}\}.$$

By construction $\mathcal{M}^Y$ is a subspace of $M_Y(\mathbb{R})$. It will turn out that $\mathcal{M}^Y$ is a subalgebra of $M_Y(\mathbb{R})$.

We make an observation. For $0 \leq i \leq d$, $A_i^Y \neq 0$ if and only if $0 \leq i \leq s$.

Lemma 22.17. *Each of the following is a basis for the vector space* $\mathcal{M}^Y$*:*

$$\{A_i^Y\}_{i=0}^s, \qquad \{E_i^Y\}_{i=0}^s.$$

Moreover the following hold for $0 \leq i \leq s$*:*

$$E_i^Y = |X|^{-1} \sum_{j=0}^s Q_i(j) A_j^Y, \qquad A_i^Y = \frac{|X|}{|Y|} \sum_{j=0}^s P_i^Y(j) E_j^Y.$$

Proof. The matrices $\{A_i\}_{i=0}^d$ form a basis for $\mathcal{M}$, so the matrices $\{A_i^Y\}_{i=0}^d$ span $\mathcal{M}^Y$. We have $A_i^Y = 0$ for $s+1 \leq i \leq d$, so $\{A_i^Y\}_{i=0}^s$ span $\mathcal{M}^Y$. The matrices $\{A_i^Y\}_{i=0}^s$ are linearly independent, since their nonzero entries are in disjoint locations. Therefore $\{A_i^Y\}_{i=0}^s$ is a basis for $\mathcal{M}^Y$. The remaining assertions follow from the construction and Definition 22.12. □

Theorem 22.18. $\mathcal{Y} = (Y, \{R_i^Y\}_{i=0}^s)$ *is a symmetric association scheme.*

Proof. Recall Lemma 22.8. It remains to show that for $0 \leq i, j, k \leq s$ and $x, y \in Y$ with $(x, y) \in R_k$, the number

$$r_{i,j}^k = \left|\Gamma_i(x) \cap \Gamma_j(y) \cap Y\right|$$

is independent of the choice of x, y. We have

$$|\Gamma_i(x) \cap \Gamma_j(y) \cap Y| = \langle A_i\hat{x} \circ A_j\hat{y}, \psi_Y\rangle = \langle A_i^Y\hat{x} \circ A_j^Y\hat{y}, \psi_Y\rangle$$

$$\begin{aligned}
&= \frac{|X|^2}{|Y|^2} \sum_{\ell=0}^s \sum_{h=0}^s P_i^Y(\ell) P_j^Y(h) \langle E_\ell^Y \hat{x} \circ E_h^Y \hat{y}, \psi_Y\rangle \\
&= \frac{|X|^2}{|Y|^2} \sum_{\ell=0}^s \sum_{h=0}^s P_i^Y(\ell) P_j^Y(h) \langle E_\ell \hat{x} \circ E_h \hat{y}, \psi_Y\rangle \\
&= |Y|^{-1} \sum_{\ell=0}^s P_i^Y(\ell) P_j^Y(\ell) v_\ell^*(\theta_k^*).
\end{aligned}$$

The result follows. □

Corollary 22.19. *The algebra* $\mathcal{M}^Y$ *is the Bose-Mesner algebra of the association scheme* $\mathcal{Y}$. *Moreover*

$$A_i^Y A_j^Y = \sum_{k=0}^{s} r_{i,j}^k A_k^Y \qquad (0 \le i, j \le s).$$

Proof. By construction $\{A_i^Y\}_{i=0}^s$ are the associate matrices of $\mathcal{Y}$, and the $r_{i,j}^k$ are the intersection numbers of $\mathcal{Y}$. □

Note 22.20. It is known that the association scheme $\mathcal{Y}$ is Q-polynomial. See the paper

Sho Suda. New parameters of subsets in polynomial schemes.
`arXiv:1008.0189`.