Lecture 31

(Adjusting the notes from Lecture 30 by inserting a few results) Our next general goal is to treat the case of equality in Theorem 22.1. We will make use of the following concepts.

Recall the standard module V.

Definition 22.2. We turn the vector space V into a commutative, associative, $\mathbb{R}$-algebra with multiplication $\circ$ defined as follows:

$$\hat{y} \circ \hat{z} = \delta_{y,z}\hat{y} \qquad y, z \in X. \tag{90}$$

The algebra V is isomorphic to the algebra of functions $X \to \mathbb{R}$. Motivated by this, we call the algebra V the *function algebra*.

In order to illustrate the multiplication $\circ$, let $v, w \in V$ and write

$$v = \sum_{y \in X} v_y \hat{y}, \qquad w = \sum_{y \in X} w_y \hat{y} \qquad v_y, w_y \in \mathbb{R}.$$

Then

$$v \circ w = \sum_{y \in X} v_y w_y \hat{y}.$$

Lemma 22.3. *For the function algebra V, the multiplicative identity is $\mathbf{1} = \sum_{y \in X} \hat{y}$.*

Proof. Routine. □

Lemma 22.4. *Fix $x \in X$ and write $T = T(x)$. For $v \in V$ and $0 \le i \le d$,*

$$A_i^* v = |X| E_i \hat{x} \circ v.$$

Proof. Write $v = \sum_{y \in X} v_y \hat{y}$. Pick $y \in X$. The y-coordinate of $A_i^* v$ is

$$(A_i^* v)_y = (A_i^*)_{y,y} v_y = |X| (E_i)_{x,y} v_y.$$

The y-coordinate of $E_i \hat{x} \circ v$ is

$$\big(E_i \hat{x} \circ v\big)_y = (E_i \hat{x})_y v_y = (E_i)_{y,x} v_y = (E_i)_{x,y} v_y.$$

The result follows. □

Proposition 22.5. *For $0 \le i, j \le d$,*

$$\operatorname{Span}\big(E_i V \circ E_j V\big) = \sum_{\substack{0 \le k \le d \\ q_{i,j}^k \ne 0}} E_k V. \tag{91}$$

Proof. We first establish the inclusion $\subseteq$. By construction $E_iV = \text{Span}\{E_i\hat{y}|y \in X\}$. We show that for $x \in X$,

$$E_i\hat{x} \circ E_jV \subseteq \sum_{\substack{0\le k\le d\\ q^k_{i,j}\neq 0}} E_kV.$$

Write $T = T(x)$. We have

$$E_i\hat{x} \circ E_jV = A^*_iE_jV \subseteq \sum_{\substack{0\le k\le d\\ q^k_{i,j}\neq 0}} E_kV.$$

Next we establish the inclusion $\supseteq$. For $0 \le k \le d$ such that $q^k_{i,j} \neq 0$, we show that $\text{Span}\big(E_iV \circ E_jV\big) \supseteq E_kV$. We have

$$\begin{aligned}
\text{Span}\big(E_iV \circ E_jV\big) &= \text{Span}\{E_i\hat{y} \circ E_j\hat{z}|y, z \in X\}\\
&\supseteq \text{Span}\{E_i\hat{y} \circ E_j\hat{y}|y \in X\}\\
&= \text{Span}\{(E_i \circ E_j)\hat{y}|y \in X\}\\
&= (E_i \circ E_j)V\\
&\supseteq (E_i \circ E_j)E_kV\\
&= \left(|X|^{-1}\sum_{\ell=0}^{d} q^{\ell}_{i,j}E_\ell\right)E_kV\\
&= |X|^{-1}q^k_{i,j}E_kV\\
&= E_kV.
\end{aligned}$$

□

We now consider the case of equality in Theorem 22.1.

Theorem 22.6. *Let Y denote a nonempty subset of X with degree s and stength t. Write $e = \lfloor t/2 \rfloor$. Then the following are equivalent:*

(i) $|Y| = \sum_{i=0}^{e} m_i$;

(ii) $|Y| = \sum_{i=0}^{s} m_i$;

(iii) $s = e$.

Assume (i)–(iii) *holds, and write $E = \sum_{i=0}^{s} E_i$. Then the vectors $\{E\hat{y}\}_{y\in Y}$ form a basis for $\sum_{i=0}^{s} E_iV$ such that*

$$\langle E\hat{y}, E\hat{z}\rangle = \delta_{y,z}\frac{|Y|}{|X|} \qquad (y, z \in Y).$$

Proof. (i) $\Rightarrow$ (iii) This is Theorem 20.2(iii).
(iii) $\Rightarrow$ (i), (ii) We have

$$m_0 + m_1 + \cdots + m_e \leq |Y| \leq m_0 + m_1 + \cdots + m_e$$

and hence equality thoughout.
(ii) $\Rightarrow$ (iii) It suffices to show that $t = 2s$. We adopt the notation from the proof of Theorem 22.1.

By the proof of Theorem 22.1, the vectors $\{E\hat{y}\}_{y\in Y}$ are linearly independent and contained in $\sum_{i=0}^{s} E_i V$. The subspace $\sum_{i=0}^{s} E_i V$ has dimension $\sum_{i=0}^{s} m_i = |Y|$. Therefore $\{E\hat{y}\}_{y\in Y}$ is a basis for $\sum_{i=0}^{s} E_i V$. Recall that

$$\langle E\hat{y}, F\hat{z}\rangle = \delta_{y,z} \qquad (y, z \in Y).$$

Therefore, the vectors $\{F\hat{y}\}_{y\in Y}$ form a basis for $\sum_{i=0}^{s} E_i V$ that is dual to the basis $\{E\hat{y}\}_{y\in Y}$. Let $H \in M_Y(\mathbb{R})$ denote the transition matrix from the basis $\{E\hat{y}\}_{y\in Y}$ to the basis $\{F\hat{y}\}_{y\in Y}$. For $z \in Y$,

$$F\hat{z} = \sum_{y\in Y} H_{y,z} E\hat{y}.$$

For $y, z \in Y$ we compute the (y,z)-entry of H. Write $(y,z) \in R_k$. We have

$$\langle F\hat{y}, F\hat{z}\rangle = \sum_{w\in Y} H_{w,z}\langle F\hat{y}, E\hat{w}\rangle = \sum_{w\in Y} H_{w,z}\delta_{y,w} = H_{y,z}.$$

Therefore

$$\begin{aligned} H_{y,z} = \langle F\hat{y}, F\hat{z}\rangle &= |X|^2 \sum_{i=0}^{s}\sum_{j=0}^{s} \gamma_i\gamma_j \langle E_i\hat{y}, E_j\hat{z}\rangle \\ &= |X|^2 \sum_{i=0}^{s} \gamma_i^2 \langle E_i\hat{y}, E_i\hat{z}\rangle = |X| \sum_{i=0}^{s} \gamma_i^2 Q_i(k) = |X| \sum_{i=0}^{s} \gamma_i^2 v_i^*(\theta_k^*). \end{aligned}$$

The matrix H^{-1} is the transition matrix from the basis $\{F\hat{y}\}_{y\in Y}$ to the basis $\{E\hat{y}\}_{y\in Y}$. For $z \in Y$,

$$E\hat{z} = \sum_{y\in Y} (H^{-1})_{y,z} F\hat{y}.$$

For $y, z \in Y$ we compute the (y,z)-entry of H^{-1}. Write $(y,z) \in R_k$. We have

$$\langle E\hat{y}, E\hat{z}\rangle = \sum_{w\in Y} (H^{-1})_{w,z}\langle E\hat{y}, F\hat{w}\rangle = \sum_{w\in Y} (H^{-1})_{w,z}\delta_{y,w} = (H^{-1})_{y,z}.$$

Therefore

$$\begin{aligned} (H^{-1})_{y,z} = \langle E\hat{y}, E\hat{z}\rangle &= \sum_{i=0}^{s}\sum_{j=0}^{s} \langle E_i\hat{y}, E_j\hat{z}\rangle \\ &= \sum_{i=0}^{s} \langle E_i\hat{y}, E_i\hat{z}\rangle = |X|^{-1} \sum_{i=0}^{s} Q_i(k) = |X|^{-1} \sum_{i=0}^{s} v_i^*(\theta_k^*). \end{aligned}$$

Recall the vector ψ_Y. Note that

$$\psi_Y = \sum_{i=0}^{d} E_i\psi_Y = E_0\psi_Y + \sum_{i=1}^{d} E_i\psi_Y = \frac{|Y|}{|X|}\mathbf{1} + \sum_{i=1}^{d} E_i\psi_Y.$$

Our goal is to show that $t = 2s$. It suffices to show that $a_i^* = 0$ for $1 \le i \le 2s$. It suffices to show that $E_i\psi_Y = 0$ for $1 \le i \le 2s$. It suffices to show that

$$\left\langle \psi_Y - \frac{|Y|}{|X|}\mathbf{1}, E_0V + E_1V + \cdots + E_{2s}V \right\rangle = 0. \tag{92}$$

By Proposition 22.5 and since the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial,

$$\begin{aligned} E_0V + E_1 + \cdots + E_{2s}V &= \operatorname{Span}\{u \circ v | u, v \in E_0V + E_1V + \cdots + E_sV\} \\ &= \operatorname{Span}\{E\hat{y} \circ F\hat{z} | y, z \in Y\}. \end{aligned}$$

By these comments, the requirement (92) becomes

$$\left\langle \psi_Y - \frac{|Y|}{|X|}\mathbf{1}, E\hat{y} \circ F\hat{z} \right\rangle = 0, \qquad y, z \in Y. \tag{93}$$

We will show (93). For $y, z \in Y$ we have

$$\langle \mathbf{1}, E\hat{y} \circ F\hat{z} \rangle = \langle E\hat{y}, F\hat{z} \rangle = \delta_{y,z}.$$

For $y, z \in Y$ we also have

$$\begin{aligned} \langle \psi_Y, E\hat{y} \circ F\hat{z} \rangle &= \sum_{w \in Y} \langle \hat{w}, E\hat{y} \circ F\hat{z} \rangle = \sum_{w \in Y} \langle E\hat{y}, \hat{w} \rangle \langle F\hat{z}, \hat{w} \rangle \\ &= \sum_{w \in Y} \langle E\hat{y}, E\hat{w} \rangle \langle F\hat{z}, E\hat{w} \rangle = \sum_{w \in Y} \langle E\hat{y}, E\hat{w} \rangle \delta_{z,w} = \langle E\hat{y}, E\hat{z} \rangle \\ &= |X|^{-1} \sum_{i=0}^{s} v_i^*(\theta_k^*), \qquad (y, z) \in R_k. \end{aligned}$$

With these comments in mind, we verify (93). Pick $y, z \in Y$. First assume that $y = z$. Then y, z satisfy (93) because

$$\langle \psi_Y, E\hat{y} \circ F\hat{z} \rangle = |X|^{-1} \sum_{i=0}^{s} v_i^*(\theta_0^*) = |X|^{-1} \sum_{i=0}^{s} m_i = \frac{|Y|}{|X|} = \frac{|Y|}{|X|}\langle \mathbf{1}, E\hat{y} \circ F\hat{z} \rangle.$$

Next assume that $y \neq z$. Write $(y, z) \in R_k$. In equation (93), the left-hand side is equal to

$$\langle \psi_Y, E\hat{y} \circ F\hat{z} \rangle = |X|^{-1} \sum_{i=0}^{s} v_i^*(\theta_k^*).$$

Therefore, the vertices y, z satisfy (93) provided that θ^*_k is a root of the polynomial $\sum_{i=0}^{s} v^*_i(z)$. By these comments, $t = 2s$ provided that θ^*_k is a root of $\sum_{i=0}^{s} v^*_i(z)$ for all $k \in S$. The polynomial $\sum_{i=0}^{s} v^*_i(z)$ has degree $s = |S|$. Recall that the polynomial $f(z) = \sum_{i=0}^{s} \gamma_i v^*_i(z)$ has roots $\{\theta^*_k\}_{k \in S}$. Therefore, $t = 2s$ provided that $f(z)$ and $\sum_{i=0}^{s} v^*_i(z)$ agree up to a scalar factor. In other words, $t = 2s$ provided that γ_i is independent of i for $0 \le i \le s$. We now compute $\{\gamma_i\}_{i=0}^{s}$. We have

$$1 = \sum_{i=0}^{s} \gamma_i m_i \tag{94}$$

because

$$1 = f(\theta^*_0) = \sum_{i=0}^{s} \gamma_i v^*_i(\theta^*_0) = \sum_{i=0}^{s} \gamma_i m_i.$$

If γ_i is independent of i for $0 \le i \le s$, then the common value must be $|Y|^{-1}$. Our next goal is to show

$$\gamma_i = |Y|^{-1} \qquad (0 \le i \le s). \tag{95}$$

For $0 \le i \le s$ we have $\gamma_i \ge 0$; otherwise the vectors $\{E'\hat{y}\}_{y \in Y}$ remain linearly independent, where $E' = E - E_i$. But these vectors are contained in the subspace $E_0V + \cdots + E_{i-1}V + E_{i+1}V + \cdots + E_sV$ of dimension $|Y| - m_i$, a contradiction. For notational convenience, define $\gamma_i = 0$ for $s+1 \le i \le d$. For $0 \le i \le s$ we have

$$m_i(1 - |Y|\gamma_i) = \sum_{k=1}^{d} \sum_{j=0}^{d} \gamma_j q^k_{i,j} a^*_k, \tag{96}$$

which is obtained using the definition of the a^*_k. By (96) we obtain $\gamma_i \le |Y|^{-1}$ for $0 \le i \le s$. Combining this with (94) we obtain (95). We have shown (95), so $t = 2s$. We have

$$F = |X| \sum_{i=0}^{s} \gamma_i E_i = \frac{|X|}{|Y|} \sum_{i=0}^{s} E_i = \frac{|X|}{|Y|} E.$$

Recall that $\langle E\hat{y}, F\hat{z} \rangle = \delta_{y,z}$ for $y, z \in Y$. By these comments

$$\langle E\hat{y}, E\hat{z} \rangle = \delta_{y,z} \frac{|Y|}{|X|} \qquad (y, z \in Y).$$

□