Lecture 30

22 On the strength and degree of a Q-polynomial association scheme

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Throughout this section we assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Abbreviate $\theta_i^* = Q_1(i)$ for $0 \leq i \leq d$. Recall that $\theta_0^* = m_1$ and

$$E_1 = |X|^{-1} \sum_{i=0}^{d} \theta_i^* A_i.$$

Let Y denote a nonempty subset of X. Recall the inner distribution $\{a_i\}_{i=0}^d$ and dual distribution $\{a_i^*\}_{i=0}^d$. Recall the degreee

$$s = \big|\{i | 1 \leq i \leq d,\ a_i \neq 0\}\big|$$

and the strength

$$t = \max\{i | 1 \leq i \leq d,\ a_1^* = a_2^* = \cdots = a_i^* = 0\}.$$

Abbreviate $e = \lfloor t/2 \rfloor$. We saw earlier that $s \geq e$ and

$$|Y| \geq m_0 + m_1 + \cdots + m_e. \tag{86}$$

Our next goal is to show that

$$|Y| \leq m_0 + m_1 + \cdots + m_s. \tag{87}$$

We will show that the following are equivalent: (i) equality holds in (86); (ii) equality holds in (87); (iii) $s = e$.

As a warmup, first assume that $s = 0$. Then $|Y| = 1 = m_0$. In this case $a_i^* = m_i \neq 0$ for $0 \leq i \leq d$. So $t = 0$ and $e = 0$.

To continue the warmup, assume that $s = 1$. Recall $t \leq 2s = 2$, so $t \in \{0, 1, 2\}$. We have $|Y| \geq 2$. There exists a unique integer i $(1 \leq i \leq d)$ such that $a_i \neq 0$. For distinct $y, z \in Y$ we have $(y, z) \in R_i$. We have $a_i = |Y| - 1$. Let M denote $|X|$ times the inner product matrix for $\{E_1 \hat{y}\}_{y \in Y}$. The entries of M are

$$M_{y,z} = \begin{cases} \theta_0^* & \text{if } y = z; \\ \theta_i^* & \text{if } y \neq z \end{cases} \qquad y, z \in Y.$$

The matrix M is positive semidefinite, so its eigenvalues are nonnegative. These eigenvalues are $\theta_0^* + (|Y| - 1)\theta_i^*$ (with multiplicity 1) and $\theta_0^* - \theta_i^*$ (with multiplicity $|Y| - 1$). We have $\theta_0^* > \theta_i^*$. We have $\theta_0^* + (|Y| - 1)\theta_i^* = a_1^* \geq 0$, with equality if and only if $t \geq 1$. Assume for the moment that $t = 0$. The matrix M has all eigenvalues positive, so M is invertible. The vectors $\{E_1\hat{y}\}_{y\in Y}$ are linearly independent. Therefore $|Y| \leq m_1$, so $|Y| < m_0 + m_1$.

Next assume that $t \geq 1$. We have $a_1^* = 0$. The matrix M has rank $|Y| - 1$. We have $|Y| - 1 \leq m_1$ so $|Y| \leq m_0 + m_1$. We have $\theta_0^* + (|Y| - 1)\theta_i^* = 0$ so

$$\theta_i^* = \frac{-\theta_0^*}{|Y| - 1}.$$

We now consider $a_2^* = Q_2(0) + (|Y| - 1)Q_2(i)$. Recall that $Q_2(j) = v_2^*(\theta_j^*)$ for $0 \leq j \leq d$, where

$$v_2^*(z) = \frac{z^2 - q_{1,1}^1 z - \theta_0^*}{q_{1,1}^2}.$$

We have

$$\begin{aligned} q_{1,1}^2 a_2^* &= (\theta_0^*)^2 - q_{1,1}^1\theta_0^* - \theta_0^* + \big(|Y| - 1\big)\big((\theta_i^*)^2 - q_{1,1}^1\theta_i^* - \theta_0^*\big) \\ &= (\theta_0^*)^2 - \theta_0^* + \big(|Y| - 1\big)\big((\theta_i^*)^2 - \theta_0^*\big) \\ &= (\theta_0^*)^2 + \big(|Y| - 1\big)(\theta_i^*)^2 - |Y|\theta_0^* \end{aligned}$$

so

$$\begin{aligned} \big(|Y| - 1\big)q_{1,1}^2 a_2^* &= \big(|Y| - 1\big)(\theta_0^*)^2 + \big(|Y| - 1\big)^2(\theta_i^*)^2 - \big(|Y| - 1\big)|Y|\theta_0^* \\ &= \big(|Y| - 1\big)(\theta_0^*)^2 + (\theta_0^*)^2 - \big(|Y| - 1\big)|Y|\theta_0^* \\ &= |Y|(\theta_0^*)^2 - \big(|Y| - 1\big)|Y|\theta_0^* \\ &= |Y|\theta_0^*\big(\theta_0^* + 1 - |Y|\big) \\ &= |Y|\theta_0^*\big(m_0 + m_1 - |Y|\big). \end{aligned}$$

Therefore $|Y| \leq m_0 + m_1$, with equality if and only if $a_2^* = 0$ if and only if $t = 2$.

In summary, for $s = 1$ we have $|Y| \leq m_0 + m_1$, with equality if and only if $t = 2$.

We now consider the case of general s.

Theorem 22.1. *Let Y denote a nonempty subset of X with degree s. Then*

$$|Y| \leq m_0 + m_1 + \cdots + m_s. \tag{88}$$

Proof. Recall the standard module $V = \mathbb{R}^X$. The subspace $\sum_{i=0}^s E_iV$ has dimension $\sum_{i=0}^s m_i$. Define

$$E = \sum_{i=0}^s E_i.$$

For $y \in Y$ we have

$$E\hat{y} \in \sum_{i=0}^s E_iV.$$

It suffices to show that the vectors $\{E\hat{y}\}_{y\in Y}$ are linearly independent.

Define the set

$$S = \{i|1 \leq i \leq d,\ a_i \neq 0\}.$$

We have $s = |S|$. Define the polynomial

$$f(z) = \prod_{i\in S} \frac{z-\theta_i^*}{\theta_0^*-\theta_i^*}.$$

We have $f(\theta_0^*) = 1$. Also for $1 \leq j \leq d$, $f(\theta_j^*) = 0$ if and only if $j \in S$. The polynomial $f(z)$ has degree s. Write

$$f(z) = \sum_{i=0}^{s} \gamma_i v_i^*(z) \qquad\qquad \gamma_i \in \mathbb{R},$$

where each $v_i^*(z)$ has degree i and $Q_i(j) = v_i^*(\theta_j^*)$ for $0 \leq j \leq d$. Define

$$F = |X| \sum_{i=0}^{s} \gamma_i E_i.$$

For $y \in Y$ we have

$$F\hat{y} \in \sum_{i=0}^{s} E_i V.$$

For $y, z \in Y$ we show that

$$\langle E\hat{y}, F\hat{z}\rangle = \delta_{y,z}. \tag{89}$$

Write $(y, z) \in R_k$. We have

$$\begin{aligned}\langle E\hat{y}, F\hat{z}\rangle &= |X| \sum_{i=0}^{s}\sum_{j=0}^{s} \gamma_j \langle E_i\hat{y}, E_j\hat{z}\rangle = |X| \sum_{i=0}^{s} \gamma_i \langle E_i\hat{y}, E_i\hat{z}\rangle \\ &= \sum_{i=0}^{s} \gamma_i Q_i(k) = \sum_{i=0}^{s} \gamma_i v_i^*(\theta_k^*) = f(\theta_k^*).\end{aligned}$$

If $y = z$, then $k = 0$ and $f(\theta_0^*) = 1$. If $y \neq z$, then $k \in S$ and $f(\theta_k^*) = 0$. By these comments we obtain (89).

We can now easily show that $\{E\hat{y}\}_{y\in Y}$ are linearly independent. Suppose we are given real numbers $\{\alpha_y\}_{y\in Y}$ such that

$$0 = \sum_{y\in Y} \alpha_y E\hat{y}.$$

We show that $\alpha_y = 0$ for $y \in Y$. For $y \in Y$,

$$0 = \sum_{z\in Y} \alpha_z \langle E\hat{z}, F\hat{y}\rangle = \sum_{z\in Y} \alpha_z \delta_{y,z} = \alpha_y.$$

We have shown the vectors $\{E\hat{y}\}_{y\in Y}$ are linearly independent. This implies the inequality (88). $\square$

Theorem 22.2. *Let Y denote a nonempty subset of X with degree s and stength t. Write $e = \lfloor t/2 \rfloor$. Then the following are equivalent:*

(i) $|Y| = \sum_{i=0}^{e} m_i$;

(ii) $|Y| = \sum_{i=0}^{s} m_i$;

(iii) $s = e$.

Proof. (i) $\Rightarrow$ (iii) This is Theorem 20.2(iii).
(iii) $\Rightarrow$ (i), (ii) We have

$$m_0 + m_1 + \cdots + m_e \leq |Y| \leq m_0 + m_1 + \cdots + m_e$$

and hence equality thoughout.
(ii) $\Rightarrow$ (iii). It suffices to show that $t = 2s$. We adopt the notation from the proof of Theorem 22.1.

The vectors $\{E\hat{y}\}_{y \in Y}$ form a basis for $\sum_{i=0}^{s} E_i V$. Recall that

$$\langle E\hat{y}, F\hat{z} \rangle = \delta_{y,z} \qquad\qquad (y, z \in Y).$$

Therefore, the vectors $\{F\hat{y}\}_{y \in Y}$ form a basis for $\sum_{i=0}^{s} E_i V$ that is dual to the basis $\{E\hat{y}\}_{y \in Y}$. Let $H \in M_Y(\mathbb{R})$ denote the transition matrix from the basis $\{E\hat{y}\}_{y \in Y}$ to the basis $\{F\hat{y}\}_{y \in Y}$. For $z \in Y$,

$$F\hat{z} = \sum_{y \in Y} H_{y,z} E\hat{y}.$$

For $y, z \in Y$ we compute the (y, z)-entry of H. Write $(y, z) \in R_k$. We have

$$\langle F\hat{y}, F\hat{z} \rangle = \sum_{w \in Y} H_{w,z} \langle F\hat{y}, E\hat{w} \rangle = \sum_{w \in Y} H_{w,z} \delta_{y,w} = H_{y,z}.$$

Therefore

$$\begin{aligned} H_{y,z} = \langle F\hat{y}, F\hat{z} \rangle &= |X|^2 \sum_{i=0}^{s} \sum_{j=0}^{s} \gamma_i \gamma_j \langle E_i\hat{y}, E_j\hat{z} \rangle \\ &= |X|^2 \sum_{i=0}^{s} \gamma_i^2 \langle E_i\hat{y}, E_i\hat{z} \rangle = |X| \sum_{i=0}^{s} \gamma_i^2 Q_i(k) = |X| \sum_{i=0}^{s} \gamma_i^2 v_i^*(\theta_k^*). \end{aligned}$$

The matrix H^{-1} is the transition matrix from the basis $\{F\hat{y}\}_{y \in Y}$ to the basis $\{E\hat{y}\}_{y \in Y}$. For $z \in Y$,

$$E\hat{z} = \sum_{y \in Y} (H^{-1})_{y,z} F\hat{y}.$$

For $y, z \in Y$ we compute the (y, z)-entry of H^{-1}. Write $(y, z) \in R_k$. We have

$$\langle E\hat{y}, E\hat{z} \rangle = \sum_{w \in Y} (H^{-1})_{w,z} \langle E\hat{y}, F\hat{w} \rangle = \sum_{w \in Y} (H^{-1})_{w,z} \delta_{y,w} = (H^{-1})_{y,z}.$$