The scheme $\mathcal{X}$ is commutative if and only if

$$A_iA_j = A_jA_i \qquad (0 \le i \le d).$$

The scheme $\mathcal{X}$ is symmetric if and only if

$$A_i^t = A_i \qquad (0 \le i \le d).$$

By the above conditions (i)–(iv), the matrices $\{A_i\}_{i=0}^d$ form a basis for a subalgebra $\mathcal{M}$ of $M_X(\mathbb{C})$ that contains J and is closed under transpose. Note that $\mathcal{M}$ is closed under Hadamard multiplication, because

$$A_i \circ A_j = \delta_{i,j}A_i \qquad (0 \le i, j \le d).$$

We call $\mathcal{M}$ the *adjacency algebra* of $\mathcal{X}$. If $\mathcal{X}$ is commutative, then we call $\mathcal{M}$ the *Bose-Mesner algebra* of $\mathcal{X}$.

Lecture 3

Our next goal is to define adjacency algebras in a more abstract way.

Lemma 2.1. *Let $\mathcal{M}$ denote a nonzero subspace of the vector space $M_X(\mathbb{C})$. Assume that $\mathcal{M}$ is closed under Hadamard multiplication. Then $\mathcal{M}$ has a basis $\{A_i\}_{i=0}^d$ such that $A_i \circ A_j = \delta_{i,j}A_i$ for $0 \le i, j \le d$. This basis is unique up to permutation of $A_0, A_1, \ldots, A_d$.*

Proof. For $A \in \mathcal{M}$ define the support set

$$\mathrm{Sup}(A) = \{(x, y) | x, y \in X, \quad A_{x,y} \neq 0\}.$$

For nonzero $\alpha \in \mathbb{C}$ we have

$$\mathrm{Sup}(\alpha A) = \mathrm{Sup}(A).$$

For $A, B \in \mathcal{M}$ we have

$$\mathrm{Sup}(A \circ B) = \mathrm{Sup}(A) \cap \mathrm{Sup}(B).$$

In particular,

$$\mathrm{Sup}(A \circ A) = \mathrm{Sup}(A).$$

For $A \in \mathcal{M}$, we say that A is *minimal* whenever (i) $A \neq 0$; and (ii) there does not exist a nonzero $B \in \mathcal{M}$ such that $\mathrm{Sup}(B) \subsetneq \mathrm{Sup}(A)$. Assume that $A \in \mathcal{M}$ is minimal. Then for all $B \in \mathcal{M}$, either $\mathrm{Sup}(A) \subseteq \mathrm{Sup}(B)$ or $\mathrm{Sup}(A) \cap \mathrm{Sup}(B) = \emptyset$. For minimal elements $A, B \in \mathcal{M}$, either $\mathrm{Sup}(A) = \mathrm{Sup}(B)$ or $\mathrm{Sup}(A) \cap \mathrm{Sup}(B) = \emptyset$. For minimal elements $A, B \in \mathcal{M}$ such that $\mathrm{Sup}(A) = \mathrm{Sup}(B)$, there exists a nonzero $\alpha \in \mathbb{C}$ such that $B = \alpha A$; otherwise there exists a linear combination of A, B that is nonzero and has its support properly contained in the common support of A and B. For a minimal element $A \in \mathcal{M}$ the nonzero entries of A are all the same; otherwise the previous assertion is contradicted with $B = A \circ A$. A minimal element $A \in \mathcal{M}$ is called *normalized* whenever its nonzero entries are equal to 1. Every minimal element of $\mathcal{M}$ is a scalar multiple of a normalized minimal element. Let $\{A_i\}_{i=0}^d$ denote an ordering of the normalized minimal elements of $\mathcal{M}$. By construction $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \leq i, j \leq d$. Consequently $\{A_i\}_{i=0}^d$ are linearly independent. For $A \in \mathcal{M}$ we have

$$A \in \mathrm{Span}\{A_i | 0 \leq i \leq d, \ \ \mathrm{Sup}(A_i) \subseteq \mathrm{Sup}(A)\}.$$

By these comments $\{A\}_{i=0}^d$ is a basis for the vector space $\mathcal{M}$. The uniqueness assertion is clear. □

Lemma 2.2. *For $A \in M_X(\mathbb{C})$ the following are equivalent:*

(i) *the diagonal entries of A are all the same;*

(ii) *$I \circ A$ is a scalar multiple of I.*

Proof. Routine. □

Definition 2.3. A subspace $\mathcal{M}$ of $M_X(\mathbb{C})$ is *homogeneous* whenever each $A \in \mathcal{M}$ satisfies the equivalent conditions (i), (ii) in Lemma 2.2.

Proposition 2.4. *Let $\mathcal{M}$ denote a subspace of the vector space $M_X(\mathbb{C})$ that satisfies* (i)–(v) *below:*

(i) *$I, J \in \mathcal{M}$;*

(ii) *$\mathcal{M}$ is closed under matrix multiplication;*

(iii) *$\mathcal{M}$ is closed under Hadamard multiplication;*

(iv) *$\mathcal{M}$ is closed under the transpose map;*

(v) *$\mathcal{M}$ is homogeneous.*

Then there exists an association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ that has adjacency algebra $\mathcal{M}$. Also, $\mathcal{X}$ is commutative if and only if $AB = BA$ for all $A, B \in \mathcal{M}$. Moreover, $\mathcal{X}$ is symmetric if and only if $A^t = A$ for all $A \in \mathcal{M}$.

Proof. Since $\mathcal{M}$ is closed under Hadamard multiplication, by Lemma 2.1 there exists a basis $\{A_i\}_{i=0}^d$ for $\mathcal{M}$ such that $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \leq i, j \leq d$. Since $\mathcal{M}$ contains J, we have $J = \sum_{i=0}^d A_i$. Since $\mathcal{M}$ is homogeneous and contains I, we see that one of the matrices $\{A_i\}_{i=0}^d$ must equal I; without loss we many assume that $A_0 = I$. Since $\mathcal{M}$ is closed under the transpose map, $\mathcal{M}$ contains the matrices $\{A_i^t\}_{i=0}^d$. Observe that the matrices $\{A_i^t\}_{i=0}^d$ form a basis for $\mathcal{M}$, and satisfy $A_i^t \circ A_j^t = \delta_{i,j} A_i^t$ for $0 \leq i, j \leq d$. By the uniqueness assertion in Lemma 2.1, the sequence $\{A_i^t\}_{i=0}^d$ is a permutation of the sequence $\{A_i\}_{i=0}^d$. In other words, for $0 \leq i \leq d$ there exists $i' \in \{0, 1, \ldots, d\}$ such that $A_i^t = A_{i'}$. Since $\mathcal{M}$ is closed under matrix multiplication, for $0 \leq i, j \leq d$ there exist scalars $p_{i,j}^k \in \mathbb{C}$ $(0 \leq k \leq d)$ such that

$$A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k.$$

For $0 \leq k \leq d$ we have $p^k_{i,j} \in \mathbb{N}$ because the nonzero entries of A_i, A_j, A_k are equal to 1. For $0 \leq i \leq d$ define

$$R_i = \{(x,y) | A_i(x,y) = 1\}.$$

By the above comments, the sequence $(X, \{R_i\}_{i=0}^d)$ is an association scheme, with associate matrices $\{A_i\}_{i=0}^d$ and adjacency algebra $\mathcal{M}$. The assertions about commutativity and symmetry are clear. □

We mention some concepts for later use. Let $\mathbb{R}$ denote the field of real numbers.

Let X denote a nonempty finite set. Let $V = \mathbb{C}^X$ denote the $\mathbb{C}$-vector space consisting of the column vectors that have coordinates indexed by X and entries in $\mathbb{C}$. Note that $M_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with a bilinear form $\langle \, , \, \rangle$ such that $\langle u, v \rangle = u^t \overline{v}$ for all $u, v \in V$. Abbreviate $\|u\|^2 = \langle u, u \rangle$. For $u, v, w \in V$ and $\alpha \in \mathbb{C}$, we have

$$\langle v, u \rangle = \overline{\langle u, v \rangle}, \qquad \langle \alpha u, v \rangle = \alpha \langle u, v \rangle,$$
$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, \qquad \|u\|^2 \in \mathbb{R},$$
$$\|u\|^2 \geq 0, \qquad \|u\|^2 = 0 \text{ iff } u = 0.$$

For $u, v \in V$ and $A \in M_X(\mathbb{C})$ we have

$$\langle Au, v \rangle = \langle u, \overline{A}^t v \rangle. \tag{6}$$

For a subspace $U \subseteq V$ define

$$U^\perp = \{v \in V | \langle u, v \rangle = 0 \, \forall \, u \in U\}.$$

Note that

$$V = U + U^\perp \qquad \text{(orthogonal direct sum)}.$$

We call $U^\perp$ the *orthgonal complement of* U.

3 Commutative association schemes

Throughout this section, we assume that $\mathcal{X} = (X\{R_i\}_{i=0}^d)$ is a commutative association scheme. By assumption,

$$p^k_{i,j} = p^k_{j,i} \qquad (0 \leq i, j, k \leq d).$$

Recall that for $x, y \in X$ and $0 \leq i \leq d$,

$$(x,y) \in R_i \text{ iff } (y,x) \in R_{i'}.$$

For $x \in X$ and $0 \leq i \leq d$ define

$$\Gamma_i(x) = \{y \in X | (x,y) \in R_i\}.$$

For $0 \leq i, j, k \leq d$ and $(x, y) \in R_k$,

$$p^k_{i,j} = |\Gamma_i(x) \cap \Gamma_{j'}(y)|.$$

Define

$$k_i = p^0_{i,i'} \qquad (0 \leq i \leq d). \tag{7}$$

For $x \in X$,

$$k_i = |\Gamma_i(x)| \qquad (0 \leq i \leq d).$$

Lemma 3.1. *We have*

(i) $k_0 = 1$;

(ii) $k_i = k_{i'} \qquad (0 \leq i \leq d)$;

(iii) $|X| = \sum_{i=0}^d k_i$;

(iv) $k_i \neq 0 \qquad (0 \leq i \leq d)$.

Proof. Routine. □

Proposition 3.2. *We have*

(i) $p^k_{i,0} = \delta_{i,k} \qquad (0 \leq i, k \leq d)$;

(ii) $p^k_{0,j} = \delta_{j,k} \qquad (0 \leq j, k \leq d)$;

(iii) $p^0_{i,j} = \delta_{i,j'} k_i \qquad (0 \leq i, j \leq d)$;

(iv) $p^k_{i,j} = p^{k'}_{i',j'} \qquad (0 \leq i, j, k \leq d)$;

(v) $k_i = \sum_{j=0}^d p^k_{i,j} \qquad (0 \leq i, k \leq d)$;

(vi) $k_\ell p^\ell_{i,j} = k_i p^i_{\ell,j'} = k_j p^j_{i',\ell} \qquad (0 \leq i, j, \ell \leq d)$;

(vii) $\sum_{\alpha=0}^d p^\alpha_{i,j} p^\ell_{k,\alpha} = \sum_{\alpha=0}^d p^\alpha_{k,i} p^\ell_{\alpha,j} \qquad (0 \leq i, j, k, \ell \leq d)$.

Proof. (i)–(iv) Routine.
(v) Fix $(x, y) \in R_k$. Partition $\Gamma_i(x)$ according to how its elements are related to y. This gives

$$\Gamma_i(x) = \cup_{j=0}^d (\Gamma_i(x) \cap \Gamma_{j'}(y)) \qquad \text{(disjoint union)}.$$

In this equation, take the cardinality of each side.
(vi) The three common values are equal to $|X|^{-1}$ times the number of 3-tuples (x, y, z) such that $(x, y) \in R_\ell$ and $(x, z) \in R_i$ and $(z, y) \in R_j$.
(vii) In the equation $A_k(A_i A_j) = (A_k A_i) A_j$, write each side as a linear combination of $\{A_\ell\}_{\ell=0}^d$, and compare coefficients. □