The vectors $\{E_j\hat{x}\}_{j=0}^d$ form a basis for the primary T-module, so $E_{e+1}\hat{x} \neq 0$. Also $K_e^* \neq 0$. Therefore

$$E_{e+1}\Psi_e^*(A^*)\psi_Y \neq 0.$$

On the other hand,

$$\begin{aligned}
E_{e+1}\Psi_e^*(A^*)\psi_Y &= E_{e+1}\Psi_e^*(A^*)I\psi_Y \\
&= E_{e+1}\Psi_e^*(A^*)\left(\sum_{j=0}^d E_j\right)\psi_Y \\
&= E_{e+1}\Psi_e^*(A^*)E_0\psi_Y + \sum_{j=t+1}^d E_{e+1}\Psi_e^*(A^*)E_j\psi_Y.
\end{aligned}$$

The polynomial $\Psi_e^*(z)$ has degree e, so by the triple product relations,

$$E_i\Psi_e^*(A^*)E_j = 0 \quad \text{if} \quad |i-j| > e \qquad\qquad (0 \leq i, j \leq d).$$

Consequently

$$E_{e+1}\Psi_e^*(A^*)E_0 = 0.$$

Also for $t+1 \leq j \leq d$,

$$E_{e+1}\Psi_e^*(A^*)E_j = 0$$

because

$$j - e - 1 \geq t + 1 - e - 1 = t - e = e + 1.$$

By these comments

$$E_{e+1}\Psi_e^*(A^*)\psi_Y = 0.$$

This is a contradiction, so $t \neq 2e+1$. We have shown that $t = 2e$ is even. □

Lecture 29

Until further notice, we assume that our association scheme $\mathcal{X}$ is the Hamming scheme $H(d,q)$. Let Y denote a nonempty subset of X. Recall the inner distribution $\{a_i\}_{i=0}^d$ and the dual distribution $\{a_j^*\}_{j=0}^d$. Recall that

$$a_j^* = \sum_{i=0}^d a_i Q_j(i) \qquad\qquad (0 \leq j \leq d). \tag{83}$$

We bring in some generating functions. Let x, y denote commuting indeterminates. Define

$$W(x,y) = \sum_{i=0}^{d} a_i x^{d-i} y^i, \qquad W^*(x,y) = \sum_{i=0}^{d} a_i^* x^{d-i} y^i.$$

We are going to show that

$$W^*(x,y) = W\big(x + (q-1)y, x-y\big).$$

As we will see, this identity depends only on (83). So let us state the result a bit abstractly. Consider any vectors $\{\alpha_i\}_{i=0}^d$ and $\{\alpha_j^*\}_{j=0}^d$ such that

$$\alpha_j^* = \sum_{i=0}^{d} \alpha_i Q_j(i) \qquad (0 \le j \le d). \tag{84}$$

Define

$$W(x,y) = \sum_{i=0}^{d} \alpha_i x^{d-i} y^i, \qquad W^*(x,y) = \sum_{i=0}^{d} \alpha_i^* x^{d-i} y^i. \tag{85}$$

Proposition 20.3. *Referring to* (84), (85) *we have*

$$W^*(x,y) = W\big(x + (q-1)y, x-y\big).$$

Proof. Recall the Krawtchouk polynomials $\{K_i(z)\}_{i=0}^d$. Recall that $Q_i(j) = K_i(j)$ for $0 \le i, j \le d$. Recall that for $0 \le j \le d$,

$$\sum_{i=0}^{d} K_i(j) z^i = (1-z)^j \big(1 + (q-1)z\big)^{d-j}.$$

Observe that

$$\begin{aligned}
W^*(x,y) &= \sum_{j=0}^{d} \alpha_j^* x^{d-j} y^j \\
&= \sum_{j=0}^{d} x^{d-j} y^j \sum_{i=0}^{d} \alpha_i K_j(i) \\
&= \sum_{i=0}^{d} \alpha_i \sum_{j=0}^{d} K_j(i) x^{d-j} y^j \\
&= x^d \sum_{i=0}^{d} \alpha_i \sum_{j=0}^{d} K_j(i) z^j \qquad z = y/x \\
&= x^d \sum_{i=0}^{d} \alpha_i (1-z)^i \big(1 + (q-1)z\big)^{d-i} \\
&= \sum_{i=0}^{d} \alpha_i (x-y)^i \big(x + (q-1)y\big)^{d-i} \\
&= W\big(x + (q-1)y, x-y\big).
\end{aligned}$$

□

21 Some constraints on the inner distribution and dual distribution

In this section, we discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$.

Let Y denote a nonempty subset of X. Recall the inner distribution vector $\{a_i\}_{i=0}^d$ of Y and the dual distribution vector $\{a_i^*\}_{i=0}^d$ of Y.

Theorem 21.1. *Assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Then in the vector $\{a_i^*\}_{i=0}^d$, the number of consecutive zero entries is at most $2s$, where s is the degree of Y. In particular, the strength t of Y is at most $2s$.*

Proof. We assume that there exists an integer ξ $(0 \le \xi \le d - 2s)$ such that $a_j^* = 0$ for $\xi \le j \le \xi + 2s$. We will get a contradiction. For Y we have the inner distribution vector $\{a_i\}_{i=0}^d$. Define the set

$$S = \{i | 1 \le i \le d,\ a_i \neq 0\}.$$

Note that $s = |S|$. Abbreviate $\theta_i^* = Q_1(i)$ for $0 \le i \le d$. Define a polynomial

$$f(z) = \prod_{i \in S} (z - \theta_i^*).$$

The polynomial $f(z)$ has degree s. Note that

$$f(\theta_0^*) = \prod_{i \in S} (\theta_0^* - \theta_i^*) \neq 0.$$

Fix a vertex $x \in Y$ and consider the subconstituent algebra $T = T(x)$. Recall the dual adjacency matrix $A^* = A_1^*(x)$. Recall the characteristic vector ψ_Y. We have

$$\psi_Y = I\psi_Y = \sum_{j=0}^d E_j^* \psi_Y = \hat{x} + \sum_{j=1}^d E_j^* \psi_Y.$$

Therefore

$$f(A^*)\psi_Y = f(\theta_0^*)\hat{x} + \sum_{j=1}^d f(\theta_j^*) E_j^* \psi_Y.$$

For $1 \le j \le d$ we have $f(\theta_j^*) E_j^* \psi_Y = 0$, because $E_j^* \psi_Y = 0$ (if $a_j = 0$) and $f(\theta_j^*) = 0$ (if $a_j \neq 0$). By these comments,

$$f(A^*)\psi_Y = f(\theta_0^*)\hat{x}.$$

We consider the vector

$$E_{\xi+s} f(A^*)\psi_Y$$

from two points of view. On one hand,

$$E_{\xi+s} f(A^*)\psi_Y = f(\theta_0^*) E_{\xi+s}\hat{x}.$$

The vectors $\{E_j\hat{x}\}_{j=0}^d$ form a basis for the primary T-module, so $E_{\xi+s}\hat{x} \neq 0$. Also $f(\theta_0^*) \neq 0$. Therefore

$$E_{\xi+s} f(A^*)\psi_Y \neq 0.$$

On the other hand, for $\xi \leq j \leq \xi + 2s$ we have $E_j\psi_Y = 0$ because $a_j^* = 0$. Therefore,

$$\begin{aligned} E_{\xi+s} f(A^*)\psi_Y &= E_{\xi+s} f(A^*) I \psi_Y \\ &= E_{\xi+s} f(A^*) \left(\sum_{j=0}^{d} E_j \right) \psi_Y \\ &= \sum_{j=0}^{\xi-1} E_{\xi+s} f(A^*) E_j \psi_Y + \sum_{j=\xi+2s+1}^{d} E_{\xi+s} f(A^*) E_j \psi_Y. \end{aligned}$$

The polynomial $f(z)$ has degree s, so by the triple product relations,

$$E_i f(A^*) E_j = 0 \quad \text{if} \quad |i-j| > s \qquad\qquad (0 \leq i, j \leq d).$$

For $0 \leq j \leq \xi - 1$ we have

$$E_{\xi+s} f(A^*) E_j = 0$$

because

$$\xi + s - j \geq \xi + s - (\xi - 1) = s + 1.$$

For $\xi + 2s + 1 \leq j \leq d$ we have

$$E_{\xi+s} f(A^*) E_j = 0$$

because

$$j - \xi - s \geq \xi + 2s + 1 - \xi - s = s + 1.$$

By these comments

$$E_{\xi+s} f(A^*)\psi_Y = 0.$$

This is a contradiction, and the result follows. □

Theorem 21.2. *Assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Then in the vector $\{a_i\}_{i=0}^d$, the number of consecutive zero entries is at most $2s^*$, where s^* is the dual degree of Y. In particular, the minimum distance of Y is at most $2s^* + 1$.*

Proof. We assume that there exists an integer ξ $(0 \le \xi \le d - 2s^*)$ such that $a_i = 0$ for $\xi \le i \le \xi + 2s^*$. We will get a contradiction. For Y we have the dual distribution vector $\{a_j^*\}_{j=0}^d$. Define the set

$$S^* = \{j | 1 \le j \le d,\ a_j^* \ne 0\}.$$

Note that $s^* = |S^*|$. Abbreviate $\theta_i = P_1(i)$ for $0 \le i \le d$. Define a polynomial

$$f^*(z) = \prod_{j \in S^*} (z - \theta_j).$$

The polynomial $f^*(z)$ has degree s^*. Note that

$$f^*(\theta_0) = \prod_{j \in S^*} (\theta_0 - \theta_j) \ne 0.$$

Recall the adjacency matrix $A = A_1$. Recall the characteristic vector ψ_Y and the vector $\mathbf{1} = \sum_{y \in X} \hat{y}$. Note that

$$E_0 \psi_Y = |X|^{-1} J \psi_Y = \frac{|Y|}{|X|} \mathbf{1}.$$

We have

$$\psi_Y = I \psi_Y = \sum_{j=0}^{d} E_j \psi_Y = \frac{|Y|}{|X|} \mathbf{1} + \sum_{j=1}^{d} E_j \psi_Y.$$

Therefore

$$f^*(A) \psi_Y = \frac{|Y|}{|X|} f^*(\theta_0) \mathbf{1} + \sum_{j=1}^{d} f^*(\theta_j) E_j \psi_Y.$$

For $1 \le j \le d$ we have $f^*(\theta_j) E_j \psi_Y = 0$, because $E_j \psi_Y = 0$ (if $a_j^* = 0$) and $f^*(\theta_j) = 0$ (if $a_j^* \ne 0$). By these comments,

$$f^*(A) \psi_Y = \frac{|Y|}{|X|} f^*(\theta_0) \mathbf{1}.$$

Fix a vertex $x \in Y$ and consider the subconstituent algebra $T = T(x)$. We consider the vector

$$E^*_{\xi+s^*} f^*(A) \psi_Y$$

from two points of view. On one hand,

$$E^*_{\xi+s^*} f^*(A) \psi_Y = \frac{|Y|}{|X|} f^*(\theta_0) E^*_{\xi+s^*} \mathbf{1}.$$

The vectors $\{E_i^*\mathbf{1}\}_{i=0}^d$ form a basis for the primary T-module, so $E_{\xi+s^*}^*\mathbf{1} \neq 0$. Also $f^*(\theta_0) \neq 0$. Therefore

$$E_{\xi+s^*}^* f^*(A)\psi_Y \neq 0.$$

On the other hand, for $\xi \leq i \leq \xi + 2s^*$ we have $E_i^*\psi_Y = 0$ because $a_i = 0$. Therefore,

$$\begin{aligned} E_{\xi+s^*}^* f^*(A)\psi_Y &= E_{\xi+s^*}^* f^*(A) I \psi_Y \\ &= E_{\xi+s^*}^* f^*(A) \left(\sum_{i=0}^{d} E_i^* \right) \psi_Y \\ &= \sum_{i=0}^{\xi-1} E_{\xi+s^*}^* f^*(A) E_i^* \psi_Y + \sum_{i=\xi+2s^*+1}^{d} E_{\xi+s^*}^* f^*(A) E_i^* \psi_Y. \end{aligned}$$

The polynomial $f^*(z)$ has degree s^*, so by the triple product relations,

$$E_i^* f^*(A) E_j^* = 0 \quad \text{if} \quad |i-j| > s^* \qquad\qquad (0 \leq i, j \leq d).$$

For $0 \leq i \leq \xi - 1$ we have

$$E_{\xi+s^*}^* f^*(A) E_i^* = 0$$

because

$$\xi + s^* - i \geq \xi + s^* - (\xi - 1) = s^* + 1.$$

For $\xi + 2s^* + 1 \leq i \leq d$ we have

$$E_{\xi+s^*}^* f^*(A) E_i^* = 0$$

because

$$i - \xi - s^* \geq \xi + 2s^* + 1 - \xi - s^* = s^* + 1.$$

By these comments

$$E_{\xi+s^*}^* f^*(A)\psi_Y = 0.$$

This is a contradiction, and the result follows. □