Equality is attained in (80) if and only if $g = \gamma$ if and only if $\{a_i\}_{i\in M}$ is a maximal program for (Q, M) and $\{\alpha_j\}_{j=0}^d$ is a minimal program for $(Q, M)'$. Assume this is the case. The code Y is perfect, so $\delta = 2e + 1$ is odd. We have

$$\alpha_j \left(\sum_{i\in M} a_i Q_j(i) \right) = 0 \qquad\qquad (1 \le j \le d).$$

Thus for $1 \le j \le d$ such that $\sum_{i\in M} a_i Q_j(i) \ne 0$, we have $\alpha_j = 0$ and hence $\Psi_e(\theta_j) = 0$. The number of such j is equal to the dual degree s^* of Y, and the degree of $\Psi_e(z)$ is equal to e, so $s^* \le e$. We have $s^* \ge e$ by the MacWilliams inequality, so $s^* = e$. For the polynomial $\Psi_e(z)$ the set of roots is $\{\theta_j | 1 \le j \le d,\ \sum_{i\in M} a_i Q_j(i) \ne 0\}$. Recall that for $1 \le j \le d$, $E_j \psi_Y \ne 0$ if and only if $\sum_{i\in M} a_i Q_j(i) \ne 0$. So for $\Psi_e(z)$ the set of roots is $\{\theta_j | 1 \le j \le d,\ E_j \psi_Y \ne 0\}$.

Lecture 28

20 Designs in a Q-polynomial association scheme

We continue to discuss a symmetric association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$. Throughout this section we assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Abbreviate $\theta_i^* = Q_1(i)$ for $0 \le i \le d$.

Let Y denote a nonempty subset of X. Recall the inner distribution $\{a_i\}_{i=0}^d$ and dual distribution $\{a_i^*\}_{i=0}^d$ of Y.

Definition 20.1. For an integer t $(0 \le t \le d)$, we call Y a *t-design* whenever $a_i^* = 0$ for $1 \le i \le t$.

Assume that Y is a t-design, and write $e = \lfloor t/2 \rfloor$. Recall the degree

$$s = \left| \{i | 1 \le i \le d,\ a_i \ne 0\} \right|.$$

By Corollary 18.14,

$$s \ge e.$$

Recall the multiplicities $\{m_i\}_{i=0}^d$ of $\mathfrak{X}$. The Lloyd theorem has the following dual.

Theorem 20.2. *Assume that Y is a t-design, and write $e = \lfloor t/2 \rfloor$.*

(i) *We have*

$$|Y| \ge m_0 + m_1 + \cdots + m_e. \qquad (81)$$

(ii) *Assume that equality holds in* (81). *Then $t = 2e$ is even.*

(iii) *Assume that equality holds in* (81). *Then $s = e$.*

(iv) *Assume that equality holds in* (81). *Define a polynomial*

$$\Psi_e^*(z) = \sum_{i=0}^{e} v_i^*(z),$$

where the polynomial $v_i^*(z)$ *has degree* i *and* $Q_i(j) = v_i^*(\theta_j^*)$ *for* $0 \leq j \leq d$. *Then the roots of* $\Psi_e^*(z)$ *are*

$$\{\theta_j^* | 1 \leq j \leq d,\ a_j \neq 0\}.$$

Proof. We use the linear programming method. Let us take $M = \{0, 2e+1, 2e+2, \ldots, d\}$ and $C = P$. Recall the dual distribution vector $\{a_i^*\}_{i=0}^d$ of Y. We have $a_0^* = |Y|$ and $a_i^* = 0$ for $1 \leq i \leq 2e$. For $0 \leq j \leq d$ we have

$$a_j = |X|^{-1} \sum_{i=0}^{d} a_i^* P_j(i).$$

Define

$$b_i = \frac{a_i^*}{|Y|} \qquad (0 \leq i \leq d).$$

We have $b_0 = 1$ and $b_i = 0$ for $1 \leq i \leq 2e$. Moreover for $0 \leq j \leq d$,

$$\sum_{i=0}^{d} b_i P_j(i) = \frac{|X|}{|Y|} a_j \geq 0.$$

By these comments, the vector $\{b_i\}_{i \in M}$ is a program for (P, M). Next we display a program $\{\beta_j\}_{j=0}^d$ for $(P, M)'$. Define

$$K_e^* = \sum_{i=0}^{e} m_i.$$

Note that

$$K_e^* = \sum_{i=0}^{e} Q_i(0) = \sum_{i=0}^{e} v_i^*(\theta_0^*) = \Psi_e^*(\theta_0^*).$$

Define

$$\beta_j = \left(\frac{\Psi_e^*(\theta_j^*)}{K_e^*}\right)^2 \qquad (0 \leq j \leq d).$$

By construction, $\beta_0 = 1$ and $\beta_j \geq 0$ for $1 \leq j \leq d$. Recall that

$$P_j(i) = \frac{Q_i(j) k_j}{m_i} = \frac{v_i^*(\theta_j^*) k_j}{m_i} \qquad (0 \leq i, j \leq d).$$

So for $i \in M^\times$,

$$\sum_{j=0}^{d} \beta_j P_j(i) = \sum_{j=0}^{d} \left(\frac{\Psi_e^*(\theta_j^*)}{K_e^*} \right)^2 \frac{v_i^*(\theta_j^*)k_j}{m_i}$$
$$= \frac{1}{(K_e^*)^2 m_i} \sum_{j=0}^{d} \left(\Psi_e^*(\theta_j^*)\right)^2 v_i^*(\theta_j^*)k_j.$$

By construction, the polynomial $\Psi_e^*(z)$ has degree e. Write

$$\left(\Psi_e^*(z)\right)^2 = \sum_{\ell=0}^{2e} c_\ell^* v_\ell^*(z). \qquad c_\ell^* \in \mathbb{R}.$$

We state the orthogonality relation for Q. For $0 \le i, j \le d$ we have

$$\delta_{i,j}|X|m_i = \sum_{\ell=0}^{d} Q_i(\ell)Q_j(\ell)k_\ell = \sum_{\ell=0}^{d} v_i^*(\theta_\ell^*)v_j^*(\theta_\ell^*)k_\ell.$$

For $i \in M^\times$,

$$\sum_{j=0}^{d} \beta_j P_j(i) = \frac{1}{(K_e^*)^2 m_i} \sum_{j=0}^{d} \left(\Psi_e^*(\theta_j^*)\right)^2 v_i^*(\theta_j^*)k_j$$
$$= \frac{1}{(K_e^*)^2 m_i} \sum_{j=0}^{d} \sum_{\ell=0}^{2e} c_\ell^* v_\ell^*(\theta_j^*) v_i^*(\theta_j^*)k_j$$
$$= \frac{1}{(K_e^*)^2 m_i} \sum_{\ell=0}^{2e} c_\ell^* \sum_{j=0}^{d} v_\ell^*(\theta_j^*) v_i^*(\theta_j^*)k_j$$
$$= \frac{1}{(K_e^*)^2 m_i} \sum_{\ell=0}^{2e} c_\ell^* |X| \delta_{\ell,i} m_i$$
$$= 0.$$

We have shown that the vector $\{\beta_j\}_{j=0}^d$ is a program for $(P, M)'$. Next, we compute the objective function γ for the program $\{\beta_j\}_{j=0}^d$. We have

$$\gamma = \sum_{\ell=0}^{d} \beta_\ell P_\ell(0) = \frac{1}{(K_e^*)^2} \sum_{\ell=0}^{d} \left(\Psi_e^*(\theta_\ell^*)\right)^2 k_\ell$$
$$= \frac{1}{(K_e^*)^2} \sum_{\ell=0}^{d} \sum_{i=0}^{e} \sum_{j=0}^{e} v_i^*(\theta_\ell^*)v_j^*(\theta_\ell^*)k_\ell = \frac{1}{(K_e^*)^2} \sum_{i=0}^{e} \sum_{j=0}^{e} \sum_{\ell=0}^{d} v_i^*(\theta_\ell^*)v_j^*(\theta_\ell^*)k_\ell$$
$$= \frac{1}{(K_e^*)^2} \sum_{i=0}^{e} \sum_{j=0}^{e} |X|\delta_{i,j} m_i = \frac{|X|}{(K_e^*)^2} \sum_{i=0}^{e} m_i = \frac{|X|}{K_e^*}.$$

Using the linear programming bound,

$$\frac{|X|}{|Y|} = \sum_{i \in M} b_i = g \le \gamma = \frac{|X|}{K_e^*}.$$

In other words,

$$|Y| \ge m_0 + m_1 + \cdots + m_e. \tag{82}$$

Equality is attained in (82) if and only if $g = \gamma$ if and only if $\{b_i\}_{i \in M}$ is a maximal program for (P, M) and $\{\beta_j\}_{j=0}^d$ is a minimal program for $(P, M)'$. Assume this is the case. We have

$$\beta_j \left(\sum_{i \in M} b_i P_j(i) \right) = 0 \qquad\qquad (1 \le j \le d).$$

Thus for $1 \le j \le d$ such that $\sum_{i \in M} b_i P_j(i) \ne 0$, we have $\beta_j = 0$ and hence $\Psi_e^*(\theta_j^*) = 0$. The number of such j is equal to the degree s of Y, and the degree of $\Psi_e^*(z)$ is equal to e, so $s \le e$. We mentioned earlier that $s \ge e$, so $s = e$. For the polynomial $\Psi_e^*(z)$ the set of roots is equal to $\{\theta_j^* | 1 \le j \le d, \sum_{i \in M} b_i P_j(i) \ne 0\}$, which is equal to $\{\theta_j^* | 1 \le j \le d, \ a_j \ne 0\}$.

It remains to show that $t = 2e$ is even. We suppose not, and get a contradiction. We have $t = 2e + 1$. For the rest of this proof, fix a vertex $x \in Y$ and consider the subconstituent algebra $T = T(x)$. For $0 \le i \le d$, T contains the dual primitive idempotent $E_i^* = E_i^*(x)$ and the dual associate matrix $A_i^* = A_i^*(x)$. Recall that $A_i^* = v_i^*(A^*)$, where $A^* = A_1^*$ is the dual adjacency matrix. Recall the characteristic vector ψ_Y. We have

$$\psi_Y = I\psi_Y = \sum_{j=0}^{d} E_j^* \psi_Y = \hat{x} + \sum_{j=1}^{d} E_j^* \psi_Y.$$

Therefore

$$\Psi_e^*(A^*)\psi_Y = \Psi_e^*(\theta_0^*)\hat{x} + \sum_{j=1}^{d} \Psi_e^*(\theta_j^*) E_j^* \psi_Y.$$

Recall that $\Psi_e^*(\theta_0^*) = K_e^*$. Also for $1 \le j \le d$ we have $\Psi_e^*(\theta_j^*) E_j^* \psi_Y = 0$, because $E_j^* \psi_Y = 0$ (if $a_j = 0$) and $\Psi_e^*(\theta_j^*) = 0$ (if $a_j \ne 0$). By these comments,

$$\Psi_e^*(A^*)\psi_Y = K_e^* \hat{x}.$$

We consider the vector

$$E_{e+1} \Psi_e^*(A^*) \psi_Y$$

from two points of view. On one hand,

$$E_{e+1} \Psi_e^*(A^*) \psi_Y = K_e^* E_{e+1} \hat{x}.$$

The vectors $\{E_j\hat{x}\}_{j=0}^d$ form a basis for the primary T-module, so $E_{e+1}\hat{x} \neq 0$. Also $K_e^* \neq 0$. Therefore

$$E_{e+1}\Psi_e^*(A^*)\psi_Y \neq 0.$$

On the other hand,

$$\begin{aligned}
E_{e+1}\Psi_e^*(A^*)\psi_Y &= E_{e+1}\Psi_e^*(A^*)I\psi_Y \\
&= E_{e+1}\Psi_e^*(A^*)\left(\sum_{j=0}^{d} E_j\right)\psi_Y \\
&= E_{e+1}\Psi_e^*(A^*)E_0\psi_Y + \sum_{j=t+1}^{d} E_{e+1}\Psi_e^*(A^*)E_j\psi_Y.
\end{aligned}$$

The polynomial $\Psi_e^*(z)$ has degree e, so by the triple product relations,

$$E_i\Psi_e^*(A^*)E_j = 0 \quad \text{if} \quad |i-j| > e \qquad\qquad (0 \leq i, j \leq d).$$

Consequently

$$E_{e+1}\Psi_e^*(A^*)E_0 = 0.$$

Also for $t+1 \leq j \leq d$,

$$E_{e+1}\Psi_e^*(A^*)E_j = 0$$

because

$$j - e - 1 \geq t + 1 - e - 1 = t - e = e + 1.$$

By these comments

$$E_{e+1}\Psi_e^*(A^*)\psi_Y = 0.$$

This is a contradiction, so $t \neq 2e+1$. We have shown that $t = 2e$ is even. $\square$