Proof. (i)–(iii) By our above comments.
(iv) Suppose not. Then $\delta = 2e+2$ is even. Pick $y \in Y$. Pick $z \in X$ such that $\partial(y,z) = e+1$. The vertex z is not contained in any of the subsets (78). Therefore the subsets (78) do not partition X, a contradiction. $\square$

Definition 19.5. The code Y is called *perfect* whenever equality holds in (79).

Recall the characteristic vector ψ_Y. Recall the Bose-Mesner algebra $\mathcal{M}$ and the vector $\mathbf{1} = \sum_{x \in X} \hat{x}$.

Theorem 19.6. (Lloyd type theorem, I). *Assume that Y is perfect, and write $\delta = 2e+1$.*

(i) *The vectors $\{A_i \psi_Y\}_{i=0}^{e}$ are linearly independent;*

(ii) $(A_0 + A_1 + \cdots + A_e)\psi_Y = \mathbf{1}$;

(iii) $(A - kI)(A_0 + A_1 + \cdots + A_e)\psi_Y = 0$;

(iv) *the vectors $\{A_i \psi_Y\}_{i=0}^{e}$ span $\mathcal{M}\psi_Y$;*

(v) $\dim \mathcal{M}\psi_Y = 1 + e$;

(vi) *the dual degree $s^* = e$.*

Proof. (i) For these vectors the nonzero coordinates are in disjoint locations.
(ii) Because X is partitioned by the subsets

$$\{z \in X | \partial(y,z) \le e\} \qquad\qquad y \in Y.$$

(iii) The graph Γ is regular with valency k, so $A\mathbf{1} = k\mathbf{1}$. The result follows by this and (ii).
(iv) Let W denote the span of $\{A_i\psi_Y\}_{i=0}^{e}$. Note that $\psi_Y \in W$. We show that $W = \mathcal{M}\psi_Y$. To do this, it suffices to show that $AW \subseteq W$. For $0 \le i \le e-1$ we have $AA_i\psi_Y \in W$, because AA_i is a linear combination of A_{i-1}, A_i, A_{i+1}. Also, $AA_e\psi_Y \in W$ in view of (iii).
(v) By (i), (iv) above.
(vi) We show that the dimension of $\mathcal{M}\psi_Y$ is equal to $1 + s^*$. The vector space $\mathcal{M}$ has a basis $\{E_i\}_{i=0}^{d}$. The vector space $\mathcal{M}\psi_Y$ has a basis consisting of the nonzero vectors among $\{E_i\psi_Y\}_{i=0}^{d}$. The cardinality of this basis is $1+s^*$, in view of Lemma 18.11. So the dimension of $\mathcal{M}\psi_Y$ is equal to $1 + s^*$. The result follows by this and (v). $\square$

Lecture 27

Theorem 19.7. (Lloyd type theorem, II). *Assume that Y is perfect, and write $\delta = 2e+1$. Define a polynomial*

$$\Psi_e(z) = \sum_{i=0}^{e} v_i(z),$$

where the polynomial $v_i(z)$ has degree i and $A_i = v_i(A)$. Then the roots of $\Psi_e(z)$ are

$$\{\theta_i | 1 \le i \le d,\ E_i\psi_Y \ne 0\}.$$

Proof. The polynomial $\Psi_e(z)$ has degree e. There are e many integers i ($1 \le i \le d$) such that $E_i\psi_Y \neq 0$. For these i we show that $\Psi_e(\theta_i) = 0$. We have

$$\begin{aligned} 0 &= (A - kI)(A_0 + A_1 + \cdots + A_e)\psi_Y \\ &= (A - kI)\Psi_e(A)\psi_Y. \end{aligned}$$

For $1 \le i \le d$,

$$\begin{aligned} 0 &= E_i(A - kI)\Psi_e(A)\psi_Y \\ &= (\theta_i - k)\Psi_e(\theta_i)E_i\psi_Y. \end{aligned}$$

If $E_i\psi_Y \neq 0$ then $\Psi_e(\theta_i) = 0$, by the above equation and $\theta_i \neq k$. The result follows. $\square$

The polynomial $\Psi_e(z)$ in Theorem 19.7 is called the *Lloyd polynomial.* This polynomial is determined by e and the intersection numbers of the scheme.

Next we consider Lloyd I, II using linear programming. Let Y denote a code with minimum distance δ, and write $e = \lfloor(\delta - 1)/2\rfloor$. Note that $\delta = 2e + 1$ or $\delta = 2e + 2$. Let us take $M = \{0, 2e+1, 2e+2, \ldots, d\}$ and $C = Q$. Recall the inner distribution vector $\{a_i\}_{i=0}^{d}$ of Y. We have $a_0 = 1$ and $a_i = 0$ for $1 \le i \le 2e$. The vector $\{a_i\}_{i\in M}$ is a program for (Q, M). Next we display a program $\{\alpha_j\}_{j=0}^{d}$ for $(Q, M)'$. Define

$$K_e = \sum_{i=0}^{e} k_i.$$

Note that

$$K_e = \sum_{i=0}^{e} P_i(0) = \sum_{i=0}^{e} v_i(\theta_0) = \Psi_e(\theta_0).$$

Define

$$\alpha_j = \left(\frac{\Psi_e(\theta_j)}{K_e}\right)^2 \qquad (0 \le j \le d).$$

By construction, $\alpha_0 = 1$ and $\alpha_j \ge 0$ for $1 \le j \le d$. Recall that

$$Q_j(i) = \frac{P_i(j)m_j}{k_i} = \frac{v_i(\theta_j)m_j}{k_i} \qquad (0 \le i, j \le d).$$

So for $i \in M^\times$,

$$\begin{aligned} \sum_{j=0}^{d} \alpha_j Q_j(i) &= \sum_{j=0}^{d} \left(\frac{\Psi_e(\theta_j)}{K_e}\right)^2 \frac{v_i(\theta_j)m_j}{k_i} \\ &= \frac{1}{K_e^2 k_i} \sum_{j=0}^{d} \left(\Psi_e(\theta_j)\right)^2 v_i(\theta_j)m_j. \end{aligned}$$

By construction, the polynomial $\Psi_e(z)$ has degree e. Write

$$\left(\Psi_e(z)\right)^2 = \sum_{\ell=0}^{2e} c_\ell v_\ell(z). \qquad c_\ell \in \mathbb{R}.$$

We recall the orthogonality relation for P. For $0 \leq i, j \leq d$ we have

$$\delta_{i,j}|X|k_i = \sum_{\ell=0}^{d} P_i(\ell)P_j(\ell)m_\ell = \sum_{\ell=0}^{d} v_i(\theta_\ell)v_j(\theta_\ell)m_\ell.$$

For $i \in M^\times$,

$$\begin{aligned}
\sum_{j=0}^{d} \alpha_j Q_j(i) &= \frac{1}{K_e^2 k_i} \sum_{j=0}^{d} \left(\Psi_e(\theta_j)\right)^2 v_i(\theta_j)m_j \\
&= \frac{1}{K_e^2 k_i} \sum_{j=0}^{d} \sum_{\ell=0}^{2e} c_\ell v_\ell(\theta_j)v_i(\theta_j)m_j \\
&= \frac{1}{K_e^2 k_i} \sum_{\ell=0}^{2e} c_\ell \sum_{j=0}^{d} v_\ell(\theta_j)v_i(\theta_j)m_j \\
&= \frac{1}{K_e^2 k_i} \sum_{\ell=0}^{2e} c_\ell |X| \delta_{\ell,i} k_i \\
&= 0.
\end{aligned}$$

We have shown that the vector $\{\alpha_j\}_{j=0}^d$ is a program for $(Q, M)'$. Next, we compute the objective function γ for the program $\{\alpha_j\}_{j=0}^d$. We have

$$\begin{aligned}
\gamma &= \sum_{\ell=0}^{d} \alpha_\ell Q_\ell(0) = \frac{1}{K_e^2} \sum_{\ell=0}^{d} \left(\Psi_e(\theta_\ell)\right)^2 m_\ell \\
&= \frac{1}{K_e^2} \sum_{\ell=0}^{d} \sum_{i=0}^{e} \sum_{j=0}^{e} v_i(\theta_\ell)v_j(\theta_\ell)m_\ell = \frac{1}{K_e^2} \sum_{i=0}^{e} \sum_{j=0}^{e} \sum_{\ell=0}^{d} v_i(\theta_\ell)v_j(\theta_\ell)m_\ell \\
&= \frac{1}{K_e^2} \sum_{i=0}^{e} \sum_{j=0}^{e} |X|\delta_{i,j}k_i = \frac{|X|}{K_e^2} \sum_{i=0}^{e} k_i = \frac{|X|}{K_e}.
\end{aligned}$$

Using the linear programming bound,

$$|Y| = \sum_{i \in M} a_i = g \leq \gamma = \frac{|X|}{K_e}.$$

In other words,

$$|Y|(k_0 + k_1 + \cdots + k_e) \leq |X|. \tag{80}$$

Equality is attained in (80) if and only if $g = \gamma$ if and only if $\{a_i\}_{i \in M}$ is a maximal program for (Q, M) and $\{\alpha_j\}_{j=0}^d$ is a minimal program for $(Q, M)'$. Assume this is the case. The code Y is perfect, so $\delta = 2e + 1$ is odd. We have

$$\alpha_j \left(\sum_{i \in M} a_i Q_j(i) \right) = 0 \qquad\qquad (1 \le j \le d).$$

Thus for $1 \le j \le d$ such that $\sum_{i \in M} a_i Q_j(i) \ne 0$, we have $\alpha_j = 0$ and hence $\Psi_e(\theta_j) = 0$. The number of such j is equal to the dual degree s^* of Y, and the degree of $\Psi_e(z)$ is equal to e, so $s^* \le e$. We have $s^* \ge e$ by the MacWilliams inequality, so $s^* = e$. For the polynomial $\Psi_e(z)$ the set of roots is $\{\theta_j | 1 \le j \le d,\ \sum_{i \in M} a_i Q_j(i) \ne 0\}$. Recall that for $1 \le j \le d$, $E_j \psi_Y \ne 0$ if and only if $\sum_{i \in M} a_i Q_j(i) \ne 0$. So for $\Psi_e(z)$ the set of roots is $\{\theta_j | 1 \le j \le d,\ E_j \psi_Y \ne 0\}$.

20 Designs in a Q-polynomial association scheme

We continue to discuss a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Throughout this section we assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Abbreviate $\theta_i^* = Q_1(i)$ for $0 \le i \le d$.

Let Y denote a nonempty subset of X. Recall the inner distribution $\{a_i\}_{i=0}^d$ and dual distribution $\{a_i^*\}_{i=0}^d$ of Y.

Definition 20.1. For an integer t $(0 \le t \le d)$, we call Y a *t-design* whenever $a_i^* = 0$ for $1 \le i \le t$.

Assume that Y is a t-design, and write $e = \lfloor t/2 \rfloor$. Recall the degree

$$s = \left| \{ i | 1 \le i \le d,\ a_i \ne 0 \} \right|.$$

By Corollary 18.14,

$$s \ge e.$$

Recall the multiplicities $\{m_i\}_{i=0}^d$ of $\mathcal{X}$. The Lloyd theorem has the following dual.

Theorem 20.2. *Assume that Y is a t-design, and write $e = \lfloor t/2 \rfloor$.*

(i) *We have*

$$|Y| \ge m_0 + m_1 + \cdots + m_e. \qquad (81)$$

(ii) *Assume that equality holds in* (81). *Then $t = 2e$ is even.*

(iii) *Assume that equality holds in* (81). *Then $s = e$.*

(iv) *Assume that equality holds in* (81). *Define a polynomial*

$$\Psi_e^*(z) = \sum_{i=0}^{e} v_i^*(z),$$

where the polynomial $v_i^*(z)$ *has degree* i *and* $Q_i(j) = v_i^*(\theta_j^*)$ *for* $0 \le j \le d$. *Then the roots of* $\Psi_e^*(z)$ *are*

$$\{\theta_j^* | 1 \le j \le d,\ a_j \ne 0\}.$$

Proof. We use the linear programming method. Let us take $M = \{0, 2e+1, 2e+2, \ldots, d\}$ and $C = P$. Recall the dual distribution vector $\{a_i^*\}_{i=0}^d$ of Y. We have $a_0^* = |Y|$ and $a_i^* = 0$ for $1 \le i \le 2e$. For $0 \le j \le d$ we have

$$a_j = |X|^{-1} \sum_{i=0}^{d} a_i^* P_j(i).$$

Define

$$b_i = \frac{a_i^*}{|Y|} \qquad (0 \le i \le d).$$

We have $b_0 = 1$ and $b_i = 0$ for $1 \le i \le 2e$. Moreover for $0 \le j \le d$,

$$\sum_{i=0}^{d} b_i P_j(i) = \frac{|X|}{|Y|} a_j \ge 0.$$

By these comments, the vector $\{b_i\}_{i \in M}$ is a program for (P, M). Next we display a program $\{\beta_j\}_{j=0}^d$ for $(P, M)'$. Define

$$K_e^* = \sum_{i=0}^{e} m_i.$$

Note that

$$K_e^* = \sum_{i=0}^{e} Q_i(0) = \sum_{i=0}^{e} v_i^*(\theta_0^*) = \Psi_e^*(\theta_0^*).$$

Define

$$\beta_j = \left(\frac{\Psi_e^*(\theta_j^*)}{K_e^*} \right)^2 \qquad (0 \le j \le d).$$

By construction, $\beta_0 = 1$ and $\beta_j \ge 0$ for $1 \le j \le d$. Recall that

$$P_j(i) = \frac{Q_i(j) k_j}{m_i} = \frac{v_i^*(\theta_j^*) k_j}{m_i} \qquad (0 \le i, j \le d).$$