We also have

$$\begin{aligned}
\gamma &= a_0\gamma \\
&= a_0 \sum_{j\in D} \alpha_j C_j(0) \\
&\geq a_0 \sum_{j\in D} \alpha_j C_j(0) + \sum_{i\in M^\times} a_i \left(\sum_{j\in D} \alpha_j C_j(i) \right) \\
&= \sum_{i\in M} \sum_{j\in D} a_i \alpha_j C_j(i).
\end{aligned}$$

□

We now state the duality theorem for linear programming.

Theorem 17.7. *Assume that Problems (C, M) and $(C, M)'$ are feasible. Then there exists a program $\{a_i\}_{i\in M}$ for (C, M) and a program $\{\alpha_j\}_{j\in D}$ for $(C, M)'$ such that $g = \gamma$. Moreoever $g_0 = g = \gamma = \gamma_0$.*

The proof of Theorem 17.7 can be found in the textbook, pages 110–112.

Next, we consider how to find the programs $\{a_i\}_{i\in M}$ and $\{\alpha_j\}_{j\in D}$ in Theorem 17.7.

Lemma 17.8. *Given a program $\{a_i\}_{i\in M}$ for (C, M) and a program $\{\alpha_j\}_{j\in D}$ for $(C, M)'$. Then $g = \gamma$ if and only if the following* (i), (ii) *hold:*

(i) *for $i \in M^\times$,*

$$a_i \sum_{j\in D} \alpha_j C_j(i) = 0.$$

(ii) *for $j \in D^\times$,*

$$\alpha_j \sum_{i\in M} a_i C_j(i) = 0.$$

Proof. Immediate from the proof of Lemma 17.6. □

Lecture 25

Example 17.9. For even $d = 2t$, the *orthogonality graph* Ω_d has the same vertex set as the hypercube $H(d, 2)$; vertices y, z are adjacent in Ω_d whenever $(y, z) \in R_t$ in $H(d, 2)$. A set of vertices Y for Ω_d is called independent whenever no two vertices in Y are adjacent in Ω_d. Our problem is to find the maximal size of an independent set in Ω_d. First assume that t is odd. Recall that $H(d, 2)$ is bipartite, and note that either half of the bipartition is an independent set in Ω_d. This independent set has cardinality 2^{d-1}, which is maximal. Next assume that t is even. In this case, the problem is open. The above linear programming technique gives an upper bound of $g_0 = 2^n/n$ for the size of an independent subset. Thus for $H(4,2)$ we have $g_0 = 16/4 = 4$. For $H(4,2)$ the linear programming details are shown below. We have $d = 4$. We have $D = \{0,1,2,3,4\}$ and $M = \{0,1,3,4\}$. We take $C = Q$ where

$$Q = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Problem (C, M): Maximize

$$g = a_0 + a_1 + a_3 + a_4$$

subject to

$$a_1 \geq 0, \quad a_3 \geq 0, \quad a_4 \geq 0, \qquad 4a_0 + 2a_1 - 2a_3 - 4a_4 \geq 0,$$
$$6a_0 + 6a_4 \geq 0, \qquad 4a_0 - 2a_1 + 2a_3 - 4a_4 \geq 0, \qquad a_0 - a_1 - a_3 + a_4 \geq 0.$$

Problem $(C, M)'$: Minimize

$$\gamma = \alpha_0 + 4\alpha_1 + 6\alpha_2 + 4\alpha_3 + \alpha_4$$

subject to

$$\alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad \alpha_3 \geq 0, \quad \alpha_4 \geq 0, \qquad \alpha_0 + 2\alpha_1 - 2\alpha_3 - \alpha_4 \leq 0,$$
$$\alpha_0 - 2\alpha_1 + 2\alpha_3 - \alpha_4 \leq 0, \qquad \alpha_0 - 4\alpha_1 + 6\alpha_2 - 4\alpha_3 + \alpha_4 \leq 0.$$

Suppose we are given a program $\{a_i\}_{i \in M}$ for (C, M) and a program $\{\alpha_j\}_{j \in D}$ for $(C, M)'$ such that $g = \gamma$. Then $a_0 = 1$ and

$$\alpha_1(4a_0 + 2a_1 - 2a_3 - 4a_4) = 0, \qquad \alpha_2(6a_0 + 6a_4) = 0,$$
$$\alpha_3(4a_0 - 2a_1 + 2a_3 - 4a_4) = 0, \qquad \alpha_4(a_0 - a_1 - a_3 + a_4) = 0.$$

Moreover, $\alpha_0 = 1$ and

$$a_1(\alpha_0 + 2\alpha_1 - 2\alpha_3 - \alpha_4) = 0, \qquad a_3(\alpha_0 - 2\alpha_1 + 2\alpha_3 - \alpha_4) = 0,$$
$$a_4(\alpha_0 - 4\alpha_1 + 6\alpha_2 - 4\alpha_3 + \alpha_4) = 0.$$

For the above 9 equations, there are 13 solutions (found using Maple). Among these solutions, only one satisfies the inequalities in Problem (C, M) and Problem $(C, M)'$. This unique solution is

$$a_0 = 1, \quad a_1 = 1, \quad a_3 = 1, \quad a_4 = 1,$$
$$\alpha_0 = 1, \quad \alpha_1 = 1/4, \quad \alpha_2 = 0, \quad \alpha_3 = 1/4, \quad \alpha_4 = 1.$$

For this solution $g = 4 = \gamma$. Therefore $g_0 = 4 = \gamma_0$.

For more information see

E. de Klerk, D. V. Pasechnik.
A note on the stability number of an orthogonality graph. `arXiv:math/0505038`.

Ferdinand Ihringer, Hajime Tanaka.
The Independence Number of the Orthogonality Graph in Dimension 2^k.
`arXiv:1901.04860`.

18 Subsets of an association scheme

In this section, we investigate the linear programming approach in more detail. Recall the symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ with eigenmatrices P and Q.

Let Y denote a nonempty subset of X.

Definition 18.1. The *inner distribution of* Y is the row vector $\{a_i\}_{i=0}^d$ where

$$a_i = \frac{|(Y \times Y) \cap R_i|}{|Y|} \qquad (0 \le i \le d).$$

We sometimes let a_Y denote the inner distribution of Y.

The *dual distribution of* Y is the row vector $\{a_j^*\}_{j=0}^d$ where

$$a_j^* = \sum_{i=0}^d a_i Q_j(i) \qquad (0 \le j \le d).$$

We sometimes let a_Y^* denote the dual distribution of Y.

Observe that

$$a_Y^* = a_Y Q, \qquad a_Y = |X|^{-1} a_Y^* P.$$

Moreover

$$a_j = |X|^{-1} \sum_{i=0}^d a_i^* P_j(i) \qquad (0 \le j \le d).$$

Lemma 18.2. *With reference to Definition 18.1,*

(i) $a_0 = 1$;

(ii) $a_0^* = \sum_{i=0}^d a_i = |Y|$.

Proof. Recall that $Q_0(i) = 1$ for $0 \le i \le d$. □

Definition 18.3. By the *characteristic vector of* Y we mean the vector

$$\psi_Y = \sum_{y \in X} \hat{y}.$$

Lemma 18.4. *For $0 \le i \le d$ we have*

$$a_i = \frac{\psi_Y^t A_i \psi_Y}{|Y|}, \qquad\qquad a_i^* = \frac{|X|}{|Y|}\,\psi_Y^t E_i \psi_Y.$$

Proof. We have

$$\psi_Y^t A_i \psi_Y = \sum_{y,z \in Y} (A_i)_{y,z} = |(Y \times Y) \cap R_i| = |Y| a_i.$$

We also have

$$a_i^* = \sum_{\ell=0}^{d} a_\ell Q_i(\ell) = |Y|^{-1} \sum_{\ell=0}^{d} \psi_Y^t A_\ell \psi_Y Q_i(\ell) = |Y|^{-1} \psi_Y^t \left(\sum_{\ell=0}^{d} Q_i(\ell) A_\ell \right) \psi_Y = \frac{|X|}{|Y|}\,\psi_Y^t E_i \psi_Y.$$

□

Corollary 18.5. *For $0 \le i \le d$,*

$$\|E_i \psi_Y\|^2 = \frac{|Y|}{|X|}\,a_i^*.$$

Moreover $a_i^* \ge 0$.

Proof. Use Lemma 18.4 and

$$\|E_i \psi_Y\|^2 = (E_i \psi_Y)^t E_i \psi_Y = \psi_Y^t E_i^2 \psi_Y = \psi_Y^t E_i \psi_Y.$$

□

Recall the set $D = \{0, 1, \ldots, d\}$.

Definition 18.6. We define a matrix B_Y with entries indexed by $X \times D$. For $x \in X$ and $i \in D$ the (x, i)-entry of B_Y is

$$B_Y(x, i) = |Y \cap \Gamma_i(x)|.$$

We call B_Y the *outer distribution of* Y.

Lemma 18.7. *We have*

$$a_Y = \frac{\psi_Y^t B_Y}{|Y|}.$$

Proof. For $0 \le i \le d$ the i^{th} entry of either side is equal to

$$|Y|^{-1} \sum_{x \in Y} |Y \cap \Gamma_i(x)|.$$

□

Lemma 18.8. *The following hold for $0 \le i \le d$:*

(i) *the vector* $A_i\psi_Y$ *is equal to column* i *of* B_Y*;*

(ii) *the vector* $|X|E_i\psi_Y$ *is equal to column* i *of* $B_Y Q$.

Proof. (i) For $x \in X$ the x-coordinate of $A_i\psi_Y$ is equal to

$$\sum_{y\in X}(A_i)_{x,y}(\psi_Y)_y = \sum_{y\in Y}(A_i)_{x,y} = \sum_{y\in Y\cap\Gamma_i(x)} 1 = |Y\cap\Gamma_i(x)| = B_Y(x,i).$$

(ii) Use (i) and $E_i = |X|^{-1}\sum_{j=0}^{d} Q_i(j)A_j$. □

Our next general goal is to compute the rank of the matrix B_Y.

We bring in some notation. For any vector $u = (u_1, u_2, \ldots, u_r)$ let Δ_u denote the diagonal matrix with diagonal entries $u_1, u_2, \ldots, u_r$.

Theorem 18.9. *With the above notation,*

$$B_Y^t B_Y = \frac{|Y|}{|X|}\, P^t \Delta_{a_Y^*} P.$$

Proof. For $0 \le i, j \le d$ we show that the (i,j)-entry of each side is equal to

$$|Y|\sum_{k=0}^{d} p_{i,j}^k a_k.$$

Using Lemma 18.8 the (i,j)-entry of $B_Y^t B_Y$ is equal to

$$(A_i\psi_Y)^t A_j\psi_Y = \psi_Y^t A_i A_j \psi_Y = \sum_{k=0}^{d} p_{i,j}^k \psi_Y^t A_k \psi_Y = |Y|\sum_{k=0}^{d} p_{i,j}^k a_k.$$

The (i,j)-entry of $P^t\Delta_{a_Y^*}P$ is equal to

$$\begin{aligned}
\sum_{\ell=0}^{d} P_{i,\ell}^t a_\ell^* P_{\ell,j} &= \sum_{\ell=0}^{d} P_i(\ell) a_\ell^* P_j(\ell) \\
&= \sum_{\ell=0}^{d} a_\ell^* \sum_{k=0}^{d} p_{i,j}^k P_k(\ell) \\
&= \sum_{k=0}^{d} p_{i,j}^k \sum_{\ell=0}^{d} a_\ell^* P_k(\ell) \\
&= |X|\sum_{k=0}^{d} p_{i,j}^k a_k.
\end{aligned}$$

□

Corollary 18.10. *The rank of the matrix* B_Y *is equal to the number of nonzero scalars among* $\{a_i^*\}_{i=0}^d$.