Therefore

$$\alpha = p^k_{1,2}\frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_k},$$

and Claim 1 is proved.

Lecture 23

Claim 2. We have

$$AA^*A_2 - A_2A^*A = \sum_{k=1}^{d} r^k_{1,2}(A^*A_k - A_kA^*). \tag{68}$$

Proof of Claim 2. For $y, z \in X$ we compute the (y,z)-entry of the left-hand side of (68) minus the right-hand side of (68). For $y = z$ the (y,z)-entry is zero. For $y \neq z$ the (y,z)-entry is equal to

$$\left\langle \rho(x), \sum_{w\in\Gamma(y)\cap\Gamma_2(z)} \rho(w) - \sum_{w\in\Gamma_2(y)\cap\Gamma(z)} \rho(w) - r^k_{1,2}\big(\rho(y) - \rho(z)\big) \right\rangle,$$

where $(y,z) \in R_k$. The above scalar is zero by Claim 1. Claim 2 is proved.
Conceivably $\theta^*_1 = \theta^*_2$. In this case $r^k_{1,2} = 0$ for $1 \le k \le d$. So by Claim 2, $AA^*A_2 = A_2A^*A$. In this equation we eliminate A_2 using $A_2 = (A^2 - a_1A - \kappa I)/c_2$ ($\kappa = b_0$) and get

$$A^2A^*A - AA^*A^2 = \kappa(A^*A - AA^*). \tag{69}$$

We will return to this equation shortly.
Claim 3. Assume that $\theta^*_1 \neq \theta^*_2$. Then there exist scalars $\beta, \gamma, \varrho \in \mathbb{R}$ such that

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*], \tag{70}$$

where $[r,s] = rs - sr$.
Proof of Claim 3. Referring to (67), the scalar $p^k_{1,2}$ is zero if $k > 3$ and nonzero if $k = 3$. Therefore $r^k_{1,2}$ is zero if $k > 3$ and nonzero if $k = 3$. The matrices A_2 and A_3 appear in (68). Recall that A_2 and A_3 are polynomials in A that have degrees 2 and 3, respectively. Evaluating (68) using this fact, we obtain

$$A^3A^* - A^*A^3 \in \operatorname{Span}\Big(A^2A^*A - AA^*A^2, A^2A^* - A^*A^2, AA^* - A^*A\Big).$$

Therefore there exist $\beta, \gamma, \varrho \in \mathbb{R}$ such that

$$A^3A^* - A^*A^3 = (\beta+1)(A^2A^*A - AA^*A^2) + \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A).$$

In this equation we rearrange the terms to obtain (70). Claim 3 is proved.
For $0 \le i \le d$ let θ_i denote the eigenvalue of A for E_i. Recall the reduced representation diagram Δ^R_E. The graph Δ^R_E is connected since ρ is weakly nondegenerate. Recall that in Δ^R_E, vertex 0 is adjacent to vertex 1 and no other vertex. We will show that Δ^R_E is a path. To do this, it suffices to show that each vertex i in Δ^R_E is adjacent to at most 2 vertices in Δ^R_E.
Claim 4. For distinct vertices i, j in Δ^R_E that are adjacent,

(i) if $\theta^*_1 = \theta^*_2$ then $\theta_i\theta_j = -\kappa$;

(ii) if $\theta^*_1 \neq \theta^*_2$ then $\mathcal{P}(\theta_i, \theta_j) = 0$, where

$$\mathcal{P}(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \varrho.$$

Proof of Claim 4. First assume that $\theta^*_1 = \theta^*_2$. Then (69) holds. In (69), multiply each term on the left by E_i and on the right by E_j. Simplify the result to get

$$0 = E_iA^*E_j(\theta_i - \theta_j)(\theta_i\theta_j + \kappa).$$

We have $E_iA^*E_j \neq 0$ since i, j are adjacent in Δ^R_E. The scalar $\theta_i - \theta_j$ is nonzero since $i \neq j$. Therefore $\theta_i\theta_j + \kappa = 0$ so $\theta_i\theta_j = -\kappa$. Next assume that $\theta^*_1 \neq \theta^*_2$. Then (70) holds. In (70), multiply each term on the left by E_i and the right by E_j. Simplify the result to get

$$0 = E_iA^*E_j(\theta_i - \theta_j)\mathcal{P}(\theta_i, \theta_j).$$

We have $E_iA^*E_j \neq 0$ since i, j are adjacent in Δ^R_E. The scalar $\theta_i - \theta_j$ is nonzero since $i \neq j$. Therefore $\mathcal{P}(\theta_i, \theta_j) = 0$.
Claim 5. We have $\theta^*_1 \neq \theta^*_2$.
Proof of Claim 5. Suppose that $\theta^*_1 = \theta^*_2$. By Claim 4 and since vertex 0 is adjacent to vertex 1, we have $\theta_0\theta_1 = -\kappa$. We have $\theta_0 = \kappa$ so $\theta_1 = -1$. The graph Δ^R_E is connected, so vertex 1 is adjacent to some nonzero vertex j. By Claim 4 we have $\theta_1\theta_j = -\kappa$. By this and $\theta_1 = -1$, we obtain $\theta_j = \kappa$. This implies $j = 0$, for a contradiction. Claim 5 is proved.
Claim 6. Each vertex i in Δ^R_E is adjacent at most two vertices in Δ^R_E.
Proof of Claim 6. By Claims 4, 5 we see that for each vertex j in Δ^R_E that is adjacent vertex i, the eigenvalue θ_j is a root of the polynomial

$$\mathcal{P}(\theta_i, \mu) = \theta_i^2 - \beta\theta_i\mu + \mu^2 - \gamma(\theta_i + \mu) - \varrho.$$

This polynomial is quadratic in μ, so it has at most two distinct roots. Claim 6 is proved. We have shown that the graph Δ^R_E is a path. Consequently $\mathcal{X}$ is Q-polynomial with respect to E. □

The balanced set condition is very useful. We illustrate with some applications.

Theorem 16.20. *Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then*

$$\frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \qquad (71)$$

is independent of i for $2 \leq i \leq d-1$.

Proof. Fix an integer i $(2 \leq i \leq d-1)$. Pick $x \in X$ and $z \in \Gamma_{i+1}(x)$ and $y \in \Gamma_{i-2}(x) \cap \Gamma_3(z)$. By the balanced set condition,

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} \rho(w) - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} \rho(w) = r^3_{1,2}\big(\rho(y) - \rho(z)\big), \qquad (72)$$

where

$$r^3_{1,2} = p^3_{1,2}\frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_3}. \tag{73}$$

Take the inner product of $\rho(x)$ with each side of (72); this yields

$$p^3_{1,2}(\theta^*_{i-1} - \theta^*_i) = r^3_{1,2}(\theta^*_{i-2} - \theta^*_{i+1}).$$

Evaluating this using (73) and rearranging terms, we obtain

$$\frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} = \frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}.$$

The result follows. □

Definition 16.21. Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Define $\beta \in \mathbb{R}$ such that $\beta + 1$ is the common value of (71). We call β the *fundamental parameter* of E.

Corollary 16.22. *Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then the dual eigenvalues have the following closed forms.*

(i) *Assume $\beta \neq \pm 2$. Then*

$$\theta^*_i = a + bq^i + cq^{-i} \qquad (0 \leq i \leq d),$$

where $\beta = q + q^{-1}$.

(ii) *Assume $\beta = 2$. Then*

$$\theta^*_i = a + bi + ci^2 \qquad (0 \leq i \leq d).$$

(iii) *Assume $\beta = -2$. Then*

$$\theta^*_i = a + b(-1)^i + ci(-1)^i \qquad (0 \leq i \leq d).$$

Proof. The dual eigenvalues satisfy the three-term recurrence

$$\theta^*_{i+1} - (\beta + 1)\theta^*_i + (\beta + 1)\theta^*_{i-1} - \theta^*_{i-2} = 0 \qquad (2 \leq i \leq d - 1).$$

For this recurrence the characteristic polynomial is

$$\lambda^3 - (\beta + 1)\lambda^2 + (\beta + 1)\lambda - 1.$$

This polynomial has roots $1, q, q^{-1}$ where $\beta = q + q^{-1}$. The result follows by linear algebra. □

Note 16.23. Under the assumptions of Corollary 16.22, the eigenvalues $\{\theta_i\}_{i=0}^d$ have similar closed forms.

Theorem 16.24. *Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then for $x \in X$ the subgraph of (X, R_1) induced on $\Gamma_{d-1}(x) \cup \Gamma_d(x)$ is connected.*

Proof. The graph $\Gamma = (X, R_1)$ is distance-regular; let ∂ denote its path-length distance function. A path $\{y_i\}_{i=0}^r$ in Γ will be called *geodesic* whenever $\partial(y_0, y_r) = r$. We will use a proof by contradiction, and assume the subgraph induced on $\Gamma_{d-1}(x) \cup \Gamma_d(x)$ is disconnected. Let C be the vertex set of a connected component of the subgraph induced on $\Gamma_{d-1}(x) \cup \Gamma_d(x)$. Let the set Δ consist of the vertices in X that lie on a geodesic from x to C. Note that $\Delta \neq X$, since $C \neq \Gamma_{d-1}(x) \cup \Gamma_d(x)$. We partition $\Delta = \cup_{j=0}^d \Delta_j$ where $\Delta_j = \Delta \cap \Gamma_j(x)$ for $0 \leq j \leq d$. Note that $C = \Delta_{d-1} \cup \Delta_d$. Each vertex in Δ_d is adjacent to c_d vertices in Δ_{d-1}. Each vertex in Δ_{d-1} is adjacent to b_{d-1} vertices in Δ_d. For $0 \leq j \leq d-1$, each vertex in Δ_j is adjacent to at least one vertex in Δ_{j+1}.

A vertex in Δ will be called a *border* whenever it is adjacent to a vertex in $X \backslash \Delta$. Since $\Delta \neq X$ and Γ is connected, Δ contains at least one border vertex. Let t denote the maximal integer j $(0 \leq j \leq d)$ such that Δ_j contains a border vertex. By the construction $1 \leq t \leq d-2$. Pick a border vertex $w \in \Delta_t$. There exists $y \in \Delta_{t+2}$ such that $\partial(y, w) = 2$. Let $z \in X \backslash \Delta$ be adjacent to w. Define $\xi = \partial(x, z)$. By the triangle inequality $\xi \in \{t-1, t, t+1\}$. Note that $\xi \neq t-1$; otherwise z is on a geodesic from x to C passing through w, forcing $z \in \Delta$ for a contradiction. Therefore $\xi = t$ or $\xi = t+1$.

We next show that $\partial(y, z) = 3$. Because $\partial(y, w) = 2$ and $\partial(w, z) = 1$, the triangle inequality implies that $\partial(y, z) \leq 3$. By the maximality of t and since $y \in \Delta_{t+2}$, we see that y is not a border and not adjacent to a border. Therefore Δ contains all the vertices in X that are at distance at most 2 from y. The vertex z is not in Δ, so $\partial(y, z) \geq 3$. We have shown that $\partial(y, z) = 3$.

Note that $\Gamma(y) \cap \Gamma_2(z) \subseteq \Gamma_{t+1}(x)$ and $\Gamma_2(y) \cap \Gamma(z) \subseteq \Gamma_t(x)$. We apply the balanced set condition to y and z using $i = 1, j = 2, k = 3$ and then take the inner product of each side with $\rho(x)$; this gives

$$p^3_{1,2}(\theta^*_{t+1} - \theta^*_t) = p^3_{1,2}\frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_3}(\theta^*_{t+2} - \theta^*_\xi).$$

Rearranging terms, we obtain

$$\frac{\theta^*_\xi - \theta^*_{t+2}}{\theta^*_t - \theta^*_{t+1}} = \frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}. \tag{74}$$

By Theorem 16.20,

$$\frac{\theta^*_{t-1} - \theta^*_{t+2}}{\theta^*_t - \theta^*_{t+1}} = \frac{\theta^*_0 - \theta^*_3}{\theta^*_1 - \theta^*_2}. \tag{75}$$

Comparing (74), (75) we obtain $\theta^*_\xi = \theta^*_{t-1}$. We have $\xi = t-1$ since $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are mutually distinct. We mentioned earlier that $\xi \neq t-1$, for a contradiction. We conclude that the subgraph induced on $\Gamma_{d-1}(x) \cup \Gamma_d(x)$ is connected. □