Similarly

$$\left\langle \rho(y), \sum_{w\in\Gamma_j(y)\cap\Gamma_i(z)} \rho(w) \right\rangle = p^k_{i,j}\theta^*_j.$$

We also have

$$\langle\rho(y), \rho(y)\rangle = \theta^*_0, \qquad \langle\rho(y), \rho(z)\rangle = \theta^*_k.$$

By these comments,

$$p^k_{i,j}(\theta^*_i - \theta^*_j) = \alpha(\theta^*_0 - \theta^*_k).$$

The result follows. □

We have some comments about the representation diagram Δ_E. This diagram has vertex set $0, 1, 2, \ldots, d$. Since $\mathcal{X}$ is symmetric, the edges in Δ_E are undirected. Some of the vertices might have a loop. Let Δ^R_E denote the diagram obtained from Δ_E by removing the loops. We call Δ^R_E the *reduced representation diagram* for E.

We now state the next main result.

Theorem 16.12. *The following are equivalent:*

(i) *ρ is balanced;*

(ii) *Δ^R_E is a tree.*

We will prove Theorem 16.12 shortly. First we mention a corollary. Note that Δ^R_E is a path if and only if $\mathcal{X}$ is Q-polynomial with respect to E.

Corollary 16.13. *Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then ρ is balanced.*

Proof. The diagram Δ^R_E is a path and hence a tree. □

Lecture 22

To prove Theorem 16.12, we will use the subconstituent algebra. For the rest of this section, fix a vertex $x \in X$. Recall that $T = T(x)$ is generated by $\mathcal{M}$ and $\mathcal{M}^* = \mathcal{M}^*(x)$. Abbreviate $A^* = A^*_1 \in \mathcal{M}^*$. By construction

$$A^* = \sum_{i=0}^{d} \theta^*_i E^*_i.$$

We define a subspace $\mathcal{L}$ of the vector space T:

$$\mathcal{L} = \text{Span}\{MA^*N - NA^*M | M, N \in \mathcal{M}\}.$$

Each of $\{A_i\}^d_{i=0}$ and $\{E_i\}^d_{i=0}$ is a basis for $\mathcal{M}$. Therefore

$$\mathcal{L} = \text{Span}\{A_iA^*A_j - A_jA^*A_i | 0 \le i, j \le d\} \tag{59}$$

and

$$\mathcal{L} = \text{Span}\{E_iA^*E_j - E_jA^*E_i | 0 \le i, j \le d\}. \tag{60}$$

We now give a refined version of (60).

Lemma 16.14. *The set*

$$\{E_iA^*E_j - E_jA^*E_i | 0 \leq i < j \leq d,\ q^1_{i,j} \neq 0\} \tag{61}$$

is a basis for $\mathcal{L}$.

Proof. We saw earlier that for $0 \leq i, j \leq d$ the matrix $E_iA^*E_j = 0$ if and only if $q^1_{i,j} = 0$. Also, the nonzero matrices among $\{E_iA^*E_j | 0 \leq i, j \leq d\}$ are mutually orthogonal, and therefore linearly independent. □

Corollary 16.15. *The dimension of* $\mathcal{L}$ *is equal to the number of edges in* Δ^R_E.

Proof. For $0 \leq i < j \leq d$ the vertices i, j of Δ^R_E are adjacent if and only if $q^1_{i,j} \neq 0$. The result follows from this and Lemma 16.14. □

Corollary 16.16. *Assume that* ρ *is weakly nondegenerate. Then* $d \leq \dim \mathcal{L}$, *with equality if and only if* Δ^R_E *is a tree.*

Proof. The graph Δ^R_E is connected. An undirected connected graph with $d + 1$ vertices has at least d edges, with equality if and only if the graph is a tree. The result follows. □

Lemma 16.17. *Assume that* ρ *is weakly nondegenerate. Then the set* $\{A^*A_k - A_kA^* | 1 \leq k \leq d\}$ *is a linearly independent subset of* $\mathcal{L}$.

Proof. By construction the given set is contained in $\mathcal{L}$. For $1 \leq k \leq d$ and $y \in X$ we compute the (x, y)-entry of $A^*A_k - A_kA^*$. This entry is equal to

$$A^*_{x,x}(A_k)_{x,y} - (A_k)_{x,y}A^*_{y,y} = \begin{cases} \theta^*_0 - \theta^*_k & \text{if } (x, y) \in R_k; \\ 0 & \text{if } (x, y) \notin R_k. \end{cases}$$

The linear independence is a routine consequence of this. □

Corollary 16.18. *Assume that* ρ *is weakly nondegenerate. Then the following are equivalent:*

(i) $\dim \mathcal{L} = d$;

(ii) Δ^R_E *is a tree;*

(iii) *the matrices* $\{A^*A_k - A_kA^* | 1 \leq k \leq d\}$ *span* $\mathcal{L}$;

(iv) *the matrices* $\{A^*A_k - A_kA^* | 1 \leq k \leq d\}$ *form a basis for* $\mathcal{L}$.

Proof. By Corollary 16.15 and Lemma 16.17. □

Proof of Theorem 16.12. (i) ⇒ (ii) We assume ρ is balanced, so ρ is weakly nondegenerate. We show that for $0 \leq i, j \leq d$,

$$A_iA^*A_j - A_jA^*A_i = \sum_{k=1}^{d} r^k_{i,j}(A^*A_k - A_kA^*), \tag{62}$$

where

$$r_{i,j}^k = p_{i,j}^k \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_k^*} \qquad\qquad (1 \le k \le d).$$

To establish (62), we will show that the following matrix is equal to 0:

$$A_i A^* A_j - A_j A^* A_i - \sum_{k=1}^{d} r_{i,j}^k (A^* A_k - A_k A^*). \tag{63}$$

For $y, z \in X$ we compute the (y, z)-entry of (63). The (y, z)-entry of $A_i A^* A_j$ is equal to

$$\begin{aligned}\sum_{w \in X} (A_i)_{y,w} A^*_{w,w} (A_j)_{w,z} &= \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} A^*_{w,w} \\ &= \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} \langle \rho(x), \rho(w) \rangle \\ &= \Big\langle \rho(x), \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} \rho(w) \Big\rangle.\end{aligned}$$

Similarly, the (y, z)-entry of $A_j A^* A_i$ is equal to

$$\Big\langle \rho(x), \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} \rho(w) \Big\rangle.$$

For $1 \le k \le d$ the (y, z)-entry of $A^* A_k - A_k A^*$ is equal to

$$A^*_{y,y}(A_k)_{y,z} - (A_k)_{y,z} A^*_{z,z} = \begin{cases} \langle \rho(x), \rho(y) - \rho(z) \rangle & \text{if } (y, z) \in R_k; \\ 0 & \text{if } (y, z) \notin R_k. \end{cases}$$

By these comments, the (y, z)-entry of (63) is equal to

$$\Big\langle \rho(x), \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} \rho(w) - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} \rho(w) - r_{i,j}^k \big(\rho(y) - \rho(z)\big) \Big\rangle, \tag{64}$$

provided that $y \neq z$ and $(y, z) \in R_k$. In this case the scalar (64) is equal to zero, because in the inner product the vector on the right is zero. Note that for $y = z$ the (y, z)-entry of (63) is equal to zero. We have shown that the matrix (63) is equal to zero, so (62) holds. By (59) and (62) we get Corollary 16.18(iii), which implies Corollary 16.18(ii). We have shown that Δ_E^R is a tree.

(ii) $\Rightarrow$ (i) The graph Δ_E^R is connected since it is a tree. Therefore ρ is weakly nondegenerate. We show that ρ satisfies the balanced set condition. By Corollary 16.18 the matrices $\{A^* A_k - A_k A^* | 1 \le k \le d\}$ form a basis for $\mathcal{L}$. Consequently, for $0 \le i, j \le d$ there exist $r_{i,j}^k \in \mathbb{R}$ $(1 \le k \le d)$ such that

$$A_i A^* A_j - A_j A^* A_i = \sum_{k=1}^{d} r_{i,j}^k (A^* A_k - A_k A^*). \tag{65}$$

For distinct $y, z \in X$ we examine the (y,z)-entry in (65). The result shows that $\rho(x)$ is orthogonal to

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} \rho(w) - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} \rho(w) - r^k_{i,j}\big(\rho(y) - \rho(z)\big), \tag{66}$$

where $(y,z) \in R_k$. Since the choice of x is arbitrary, the vector (66) must be orthogonal to EV. The vector (66) is contained in EV, so the vector (66) is equal to zero. Therefore

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} \rho(w) - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} \rho(w) = r^k_{i,j}\big(\rho(y) - \rho(z)\big).$$

Consequently ρ satisfies the balanced set condition. We conclude that ρ is balanced. □

As we saw earlier, if $\mathcal{X}$ is Q-polynomial with respect to E, then ρ is balanced. We are going to show that the converse is true, provided that $\mathcal{X}$ is P-polynomial. This converse is implied by the following theorem. To avoid trivialities, we will assume $d \geq 3$.

Theorem 16.19. *Assume that $d \geq 3$, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Assume that $\rho = \rho_E$ satisfies:*

(i) *ρ is weakly nondegenerate;*

(ii) *for all $y, z \in X$,*

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} \rho(w) - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} \rho(w) \in \operatorname{Span}\big(\rho(y) - \rho(z)\big).$$

Then $\mathcal{X}$ is Q-polynomial with respect to E.

Proof. We abbreviate $A = A_1$. Fix $x \in X$ and write $T = T(x)$. We assume that ρ is weakly nondegenerate, so $\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$.
Claim 1. Pick an integer k $(1 \leq k \leq d)$ and $y, z \in X$ such that $(y,z) \in R_k$. Then

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} \rho(w) - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} \rho(w) = r^k_{1,2}\big(\rho(y) - \rho(z)\big),$$

where

$$r^k_{1,2} = p^k_{1,2}\frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_k}. \tag{67}$$

Proof of Claim 1. By assumption there exists $\alpha \in \mathbb{R}$ such that

$$\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} \rho(w) - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} \rho(w) = \alpha\big(\rho(y) - \rho(z)\big).$$

For each term in the above equation, take the inner product with $\rho(y)$. A brief calculation yields

$$p^k_{1,2}(\theta^*_1 - \theta^*_2) = \alpha(\theta^*_0 - \theta^*_k).$$

Therefore

$$\alpha = p^k_{1,2}\frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_k},$$

and Claim 1 is proved.
Claim 2. We have

$$AA^*A_2 - A_2A^*A = \sum_{k=1}^{d} r^k_{1,2}(A^*A_k - A_kA^*). \tag{68}$$

Proof of Claim 2. For $y, z \in X$ we compute the (y,z)-entry of the left-hand side of (68) minus the right-hand side of (68). For $y = z$ the (y,z)-entry is zero. For $y \neq z$ the (y,z)-entry is equal to

$$\left\langle \rho(x), \sum_{w\in\Gamma(y)\cap\Gamma_2(z)} \rho(w) - \sum_{w\in\Gamma_2(y)\cap\Gamma(z)} \rho(w) - r^k_{1,2}\big(\rho(y) - \rho(z)\big) \right\rangle,$$

where $(y,z) \in R_k$. The above scalar is zero by Claim 1. Claim 2 is proved.
Conceivably $\theta^*_1 = \theta^*_2$. In this case $r^k_{1,2} = 0$ for $1 \leq k \leq d$. So by Claim 2, $AA^*A_2 = A_2A^*A$. In this equation we eliminate A_2 using $A_2 = (A^2 - a_1A - \kappa I)/c_2$ $(\kappa = b_0)$ and get

$$A^2A^*A - AA^*A^2 = \kappa(A^*A - AA^*). \tag{69}$$

We will return to this equation shortly.
Claim 3. Assume that $\theta^*_1 \neq \theta^*_2$. Then there exist scalars $\beta, \gamma, \varrho \in \mathbb{R}$ such that

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*], \tag{70}$$

where $[r,s] = rs - sr$.
Proof of Claim 3. Referring to (67), the scalar $p^k_{1,2}$ is zero if $k > 3$ and nonzero if $k = 3$. Therefore $r^k_{1,2}$ is zero if $k > 3$ and nonzero if $k = 3$. The matrices A_2 and A_3 appear in (68). Recall that A_2 and A_3 are polynomials in A that have degrees 2 and 3, respectively. Evaluating (68) using this fact, we obtain

$$A^3A^* - A^*A^3 \in \mathrm{Span}\Big(A^2A^*A - AA^*A^2, A^2A^* - A^*A^2, AA^* - A^*A\Big).$$

Therefore there exist $\beta, \gamma, \varrho \in \mathbb{R}$ such that

$$A^3A^* - A^*A^3 = (\beta+1)(A^2A^*A - AA^*A^2) + \gamma(A^2A^* - A^*A^2) + \varrho(AA^* - A^*A).$$

In this equation we rearrange the terms to obtain (70). Claim 3 is proved.
For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A for E_i. Recall the reduced representation diagram Δ^R_E. The graph Δ^R_E is connected since ρ is weakly nondegenerate. Recall that in Δ^R_E, vertex 0 is adjacent to vertex 1 and no other vertex. We will show that Δ^R_E is a path. To do this, it suffices to show that each vertex i in Δ^R_E is adjacent to at most 2 vertices in Δ^R_E.
Claim 4. For distinct vertices i, j in Δ^R_E that are adjacent,

(i) if $\theta^*_1 = \theta^*_2$ then $\theta_i\theta_j = -\kappa$;

(ii) if $\theta^*_1 \neq \theta^*_2$ then $\mathcal{P}(\theta_i, \theta_j) = 0$, where

$$\mathcal{P}(\lambda, \mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda + \mu) - \varrho.$$

Proof of Claim 4. First assume that $\theta^*_1 = \theta^*_2$. Then (69) holds. In (69), multiply each term on the left by E_i and on the right by E_j. Simplify the result to get

$$0 = E_iA^*E_j(\theta_i - \theta_j)(\theta_i\theta_j + \kappa).$$

We have $E_iA^*E_j \neq 0$ since i, j are adjacent in Δ^R_E. The scalar $\theta_i - \theta_j$ is nonzero since $i \neq j$. Therefore $\theta_i\theta_j + \kappa = 0$ so $\theta_i\theta_j = -\kappa$. Next assume that $\theta^*_1 \neq \theta^*_2$. Then (70) holds. In (70), multiply each term on the left by E_i and the right by E_j. Simplify the result to get

$$0 = E_iA^*E_j(\theta_i - \theta_j)\mathcal{P}(\theta_i, \theta_j).$$

We have $E_iA^*E_j \neq 0$ since i, j are adjacent in Δ^R_E. The scalar $\theta_i - \theta_j$ is nonzero since $i \neq j$. Therefore $\mathcal{P}(\theta_i, \theta_j) = 0$.
Claim 5. We have $\theta^*_1 \neq \theta^*_2$.
Proof of Claim 5. Suppose that $\theta^*_1 = \theta^*_2$. By Claim 4 and since vertex 0 is adjacent to vertex 1, we have $\theta_0\theta_1 = -\kappa$. We have $\theta_0 = \kappa$ so $\theta_1 = -1$. The graph Δ^R_E is connected, so vertex 1 is adjacent to some nonzero vertex j. By Claim 4 we have $\theta_1\theta_j = -\kappa$. By this and $\theta_1 = -1$, we obtain $\theta_j = \kappa$. This implies $j = 0$, for a contradiction. Claim 5 is proved.
Claim 6. Each vertex i in Δ^R_E is adjacent at most two vertices in Δ^R_E.
Proof of Claim 6. By Claims 4, 5 we see that for each vertex j in Δ^R_E that is adjacent vertex i, the eigenvalue θ_j is a root of the polynomial

$$\mathcal{P}(\theta_i, \mu) = \theta_i^2 - \beta\theta_i\mu + \mu^2 - \gamma(\theta_i + \mu) - \varrho.$$

This polynomial is quadratic in μ, so it has at most two distinct roots. Claim 6 is proved. We have shown that the graph Δ^R_E is a path. Consequently $\mathcal{X}$ is Q-polynomial with respect to E. □