Lecture 21

16 Embeddings into spheres

Throughout this section we consider a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Recall that

$$i' = i, \qquad \hat{i} = i \qquad (0 \leq i \leq d),$$

and that

$$P_i(j) \in \mathbb{R}, \qquad Q_i(j) \in \mathbb{R} \qquad (0 \leq i, j \leq d).$$

To avoid trivialities, we assume that $d \geq 1$.

It will be convenient to work over the field $\mathbb{R}$ instead of $\mathbb{C}$. We take the standard module to be $V = \mathbb{R}^X$. We endow V with a bilinear form $\langle\, ,\, \rangle$ such that $\langle u, v \rangle = u^t v$ for all $u, v \in V$. Abbreviate $\|u\|^2 = \langle u, u \rangle$.

Throughout this section we fix a nontrivial primitive idempotent E. Without loss of generality, we may assume that $E = E_1$. We abbreviate

$$\theta_i^* = Q_1(i) \qquad (0 \leq i \leq d).$$

Note that

$$E = |X|^{-1} \sum_{i=0}^{d} \theta_i^* A_i.$$

Recall that EV is a common eigenspace for the Bose-Mesner algebra $\mathcal{M}$.

Definition 16.1. We define the map

$$\rho : \begin{array}{c} X \to EV \\ y \mapsto |X|^{1/2} E\hat{y} \end{array}$$

We call ρ the *spherical representation of* $\mathcal{X}$ associated with E.

By construction,

$$EV = \mathrm{Span}\{\rho(y) | y \in X\}.$$

Lemma 16.2. *The following hold for* $0 \leq i \leq d$.

(i) *For* $y, z \in X$ *such that* $(y, z) \in R_i$,

$$\langle \rho(y), \rho(z) \rangle = \theta_i^*.$$

(ii) *For $y \in X$,*

$$\sum_{z\in\Gamma_i(y)} \rho(z) = P_i(1)\rho(y),$$

where we recall

$$\Gamma_i(y) = \{z \in X \,|\, (y,z) \in R_i\}.$$

Proof. (i) We have

$$\begin{aligned}
\langle \rho(y), \rho(z)\rangle &= |X|\langle E\hat{y}, E\hat{z}\rangle \\
&= |X|\langle \hat{y}, E^t E\hat{z}\rangle \\
&= |X|\langle \hat{y}, E^2\hat{z}\rangle \\
&= |X|\langle \hat{y}, E\hat{z}\rangle \\
&= |X|\Big((y,z)\text{-entry of } E\Big) \\
&= \theta_i^*.
\end{aligned}$$

(ii) We have

$$\begin{aligned}
\sum_{z\in\Gamma_i(y)} \rho(z) &= |X|^{1/2} \sum_{z\in\Gamma_i(y)} E\hat{z} \\
&= |X|^{1/2} E \sum_{z\in\Gamma_i(y)} \hat{z} \\
&= |X|^{1/2} E A_i \hat{y} \\
&= |X|^{1/2} A_i E \hat{y} \\
&= |X|^{1/2} P_i(1) E\hat{y} \\
&= P_i(1)\rho(y).
\end{aligned}$$

□

Note that $\theta_0^* = Q_1(0) = m_1$. By Lemma 16.2(i),

$$\|\rho(y)\|^2 = \theta_0^* \qquad (y \in X).$$

Lemma 16.3. *For $y, z \in X$ the angle between $\rho(y), \rho(z)$ has cosine θ_i^*/θ_0^*, where $(y,z) \in R_i$.*

Proof. By Lemma 16.2(i) and the comment above the lemma statement. □

Definition 16.4. The spherical representation ρ is said to be *nondegenerate* whenever $\{\theta_i^*\}_{i=0}^d$ are mutually distinct.

Recall the Q-polynomial property from Definition 12.1.

Definition 16.5. We say that $\mathcal{X}$ is *Q-polynomial with respect to E* whenever the there exists a Q-polynomial ordering $\{E_i\}_{i=0}^d$ of the primitive idempotents such that $E = E_1$.

Lemma 16.6. *Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then ρ is nondegenerate.*

Proof. We saw earlier that $\{\theta_i^*\}_{i=0}^d$ are mutually distinct. □

Definition 16.7. The spherical representation ρ is said to be *weakly nondegenerate* whenever $\theta_i^* \neq \theta_0^*$ for $1 \leq i \leq d$.

Let us clarify the meaning of weakly nondegenerate.

Lemma 16.8. *The spherical representation ρ is weakly nondegenerate if and only if the vectors $\{\rho(y)|y \in X\}$ are mutually distinct.*

Proof. By Lemma 16.3. □

Recall the representation diagram Δ_E from Definition 9.15.

Lemma 16.9. *The spherical representation ρ is weakly nondegenerate if and only if Δ_E is connected.*

Proof. By Proposition 9.17. □

Definition 16.10. The spherical representation ρ is said to be *balanced* whenever:

(i) ρ is weakly nondegenerate;

(ii) for distinct $y, z \in X$ and $0 \leq i, j \leq d$ we have

$$\sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \rho(w) - \sum_{w\in\Gamma_j(y)\cap\Gamma_i(z)} \rho(w) \in \operatorname{Span}\big(\rho(y) - \rho(z)\big). \tag{58}$$

The equation (58) is called the *balanced set condition.*

Lemma 16.11. *Assume that ρ is balanced, and pick distinct $y, z \in X$. For $0 \leq i, j \leq d$ we have*

$$\sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \rho(w) - \sum_{w\in\Gamma_j(y)\cap\Gamma_i(z)} \rho(w) = r_{i,j}^k\big(\rho(y) - \rho(z)\big),$$

where $(y, z) \in R_k$ and

$$r_{i,j}^k = p_{i,j}^k \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_k^*}.$$

Proof. The left-hand side of (58) is a scalar multiple of $\rho(y) - \rho(z)$; denote the scalar by α. To compute α, take the inner product of $\rho(y)$ with each side of (58). We have

$$\left\langle \rho(y), \sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \rho(w) \right\rangle = \sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \langle \rho(y), \rho(w) \rangle = \sum_{w\in\Gamma_i(y)\cap\Gamma_j(z)} \theta_i^* = p_{i,j}^k \theta_i^*.$$

Similarly

$$\left\langle \rho(y), \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} \rho(w) \right\rangle = p^k_{i,j}\theta^*_j.$$

We also have

$$\langle \rho(y), \rho(y) \rangle = \theta^*_0, \qquad \langle \rho(y), \rho(z) \rangle = \theta^*_k.$$

By these comments,

$$p^k_{i,j}(\theta^*_i - \theta^*_j) = \alpha(\theta^*_0 - \theta^*_k).$$

The result follows. □

We have some comments about the representation diagram Δ_E. This diagram has vertex set $0, 1, 2, \ldots, d$. Since $\mathcal{X}$ is symmetric, the edges in Δ_E are undirected. Some of the vertices might have a loop. Let Δ^R_E denote the diagram obtained from Δ_E by removing the loops. We call Δ^R_E the *reduced representation diagram* for E.

We now state the next main result.

Theorem 16.12. *The following are equivalent:*

(i) *ρ is balanced;*

(ii) *Δ^R_E is a tree.*

We will prove Theorem 16.12 shortly. First we mention a corollary.

Corollary 16.13. *Assume that $\mathcal{X}$ is Q-polynomial with respect to E. Then ρ is balanced.*

Proof. The diagram Δ^R_E is a path and hence a tree. □

To prove Theorem 16.12, we will use the subconstituent algebra. For the rest of this section, fix a vertex $x \in X$. Recall that $T = T(x)$ is generated by $\mathcal{M}$ and $\mathcal{M}^* = \mathcal{M}^*(x)$. Abbreviate $A^* = A^*_1 \in \mathcal{M}^*$. By construction

$$A^* = \sum_{i=0}^{d} \theta^*_i E^*_i.$$

We define a subspace $\mathcal{L}$ of the vector space T:

$$\mathcal{L} = \text{Span}\{MA^*N - NA^*M | M, N \in \mathcal{M}\}.$$

Lemma 16.14. *The set*

$$\{E_iA^*E_j - E_jA^*E_i | 0 \leq i < j \leq d,\ q^1_{i,j} \neq 0\} \tag{59}$$

is a basis for $\mathcal{L}$.