Proof. This result asserts that for subsets y, z of $\{1, 2, \ldots, v\}$ such that

$$|y| = d, \qquad |z| = d, \qquad |y \cap z| = d - \ell,$$

the number of subsets $w \subseteq \{1, 2, \ldots, v\}$ such that

$$w \subseteq y, \qquad w \subseteq z, \qquad |w| = d - i$$

is equal to

$$\binom{d-\ell}{d-i}.$$

This assertion is routinely checked. □

Lecture 20

Lemma 15.5. *For $0 \le i \le d$ the following holds on V:*

$$A_i = \sum_{\ell=0}^{i} \frac{\mathbf{R}^\ell}{\ell!} \frac{\mathbf{L}^\ell}{\ell!} (-1)^{i-\ell} \binom{d-\ell}{d-i}.$$

Proof. Use linear algebra to solve the system of linear equations in Lemma 15.4. □

Proposition 15.6. *For $J(v, d)$ the entries of P are given as follows. For $0 \le i, j \le d$,*

$$P_i(j) = \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{d-j}{\ell} \binom{v-d-j+\ell}{\ell} \binom{d-\ell}{d-i}.$$

Proof. Let $\mathbf{W}$ denote an irreducible T-module with endpoint j. For $0 \le \ell \le d$ the matrix

$$\frac{\mathbf{R}^\ell}{\ell!} \frac{\mathbf{L}^\ell}{\ell!}$$

acts on $\mathbf{E}_d^*\mathbf{W}$ as $\gamma_\ell \mathbf{I}$, where

$$\gamma_\ell = \binom{d-j}{\ell} \binom{v-d-j+\ell}{\ell}.$$

The result follows. □

For $J(v, d)$ the matrix Q satisfies

$$Q_i(j) = \frac{m_i}{k_j} P_j(i) \qquad (0 \le i, j \le d).$$

In order to clarify our formulas, we bring in hypergeometric series. For $a \in \mathbb{C}$ define

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \qquad (n \in \mathbb{N}).$$

We interpret $(a)_0 = 1$. For $n, m \in \mathbb{N}$ we have

$$(-m)_n = \begin{cases} \neq 0 & \text{if } n \leq m; \\ 0 & \text{if } n \geq m+1. \end{cases}$$

For $r, s \in \mathbb{N}$ and complex scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_r; \qquad \beta_1, \beta_2, \ldots, \beta_s$$

the corresponding *hypergeometric series* is

$${}_rF_s\left(\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_r \\ \beta_1, \beta_2, \ldots, \beta_s\end{matrix} \middle| \, z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_r)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_s)_n} \frac{z^n}{n!}.$$

If at least one of $\alpha_1, \alpha_2, \ldots, \alpha_r$ is an integer at most 0, then the above sum has finitely many nonzero summands.

Proposition 15.7. *For $0 \leq i, j \leq d$ we have*

$$\frac{P_i(j)}{k_i} = \frac{Q_j(i)}{m_j} = {}_3F_2\left(\begin{matrix}-i, -j, j-v-1 \\ d-v, -d\end{matrix} \middle| \, 1\right). \tag{57}$$

Proof. Use Proposition 15.6. □

Recall the abbreviation

$$\theta_i^* = Q_1(i) \qquad (0 \leq i \leq d).$$

Corollary 15.8. *For $J(v, d)$ we have*

$$\theta_i^* = v - 1 - \frac{v(v-1)i}{d(v-d)} \qquad (0 \leq i \leq d).$$

Proof. Set $j = 1$ in Proposition 15.7. □

Note that $\{\theta_i^*\}_{i=0}^d$ are mutually distinct.

The following definition is for notational convenience.

Definition 15.9. Define

$$s^* = \frac{v(1-v)}{d(v-d)}.$$

Further define

$$\varphi_i = s^* i(i-d-1)(i+d-v-1) \qquad (1 \leq i \leq d).$$

Note that $\varphi_i \neq 0$ for $1 \leq i \leq d$.

Proposition 15.10. *For $0 \le i, j \le d$ the common value in (57) is equal to*

$$\sum_{n=0}^{\min(i,j)} \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{n-1}^*)(\theta_j - \theta_0)(\theta_j - \theta_1)\cdots(\theta_j - \theta_{n-1})}{\varphi_1\varphi_2\cdots\varphi_n}.$$

Proof. For $0 \le n \le d$ we have

$$\begin{aligned}
(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{n-1}^*) &= (-1)^n s^{*n}(-i)_n,\\
(\theta_j - \theta_0)(\theta_j - \theta_1)\cdots(\theta_j - \theta_{n-1}) &= (-1)^n(-j)_n(j-v-1)_n,\\
\varphi_1\varphi_2\cdots\varphi_n &= s^{*n}(d-v)_n(-d)_n n!
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*)\cdots(\theta_i^* - \theta_{n-1}^*)(\theta_j - \theta_0)(\theta_j - \theta_1)\cdots(\theta_j - \theta_{n-1})}{\varphi_1\varphi_2\cdots\varphi_n}\\
&\quad= \frac{(-i)_n(-j)_n(j-v-1)_n}{(d-v)_n(-d)_n n!}.
\end{aligned}$$

The result follows from this and Proposition 15.7. □

Definition 15.11. For $0 \le i \le d$ define the polynomials $\tau_i, \tau_i^* \in \mathbb{R}[\lambda]$ by

$$\begin{aligned}
\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1)\cdots(\lambda - \theta_{i-1}),\\
\tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*)\cdots(\lambda - \theta_{i-1}^*).
\end{aligned}$$

Each of τ_i, τ_i^* is monic with degree i.

Lemma 15.12. *For $0 \le i, j \le d$ we have*

$$\tau_i(\theta_j) = \begin{cases} 0 & \text{if } j \le i-1;\\ \ne 0 & \text{if } j \ge i.\end{cases} \qquad \tau_i^*(\theta_j^*) = \begin{cases} 0 & \text{if } j \le i-1;\\ \ne 0 & \text{if } j \ge i.\end{cases}$$

Proof. Since $\{\theta_j\}_{j=0}^d$ are mutually distinct and $\{\theta_j^*\}_{j=0}^d$ are mutually distinct. □

Definition 15.13. For $0 \le i \le d$ define the polynomials $v_i, v_i^* \in \mathbb{R}[\lambda]$ by

$$v_i = k_i \sum_{n=0}^{i} \frac{\tau_n^*(\theta_i^*)\tau_n}{\varphi_1\varphi_2\cdots\varphi_n}, \qquad v_i^* = m_i \sum_{n=0}^{i} \frac{\tau_n(\theta_i)\tau_n^*}{\varphi_1\varphi_2\cdots\varphi_n}.$$

Each of v_i, v_i^* has degree i. We call v_i (resp. v_i^*) the i^{th} *Eberlein* (resp. *Hahn*) polynomial.

Proposition 15.14. *For $0 \le i, j \le d$ we have*

$$P_i(j) = v_i(\theta_j), \qquad Q_i(j) = v_i^*(\theta_j^*).$$

Proof. By Proposition 15.10, Lemma 15.12, and Definition 15.13 we obtain

$$P_i(j) = k_i \sum_{n=0}^{\min(i,j)} \frac{\tau_n^*(\theta_i^*)\tau_n(\theta_j)}{\varphi_1\varphi_2\cdots\varphi_n} = k_i \sum_{n=0}^{i} \frac{\tau_n^*(\theta_i^*)\tau_n(\theta_j)}{\varphi_1\varphi_2\cdots\varphi_n} = v_i(\theta_j).$$

The proof of $Q_i(j) = v_i^*(\theta_j^*)$ is similar. □

Corollary 15.15. *The Johnson graph $J(v,d)$ is Q-polynomial with respect to the ordering* $\{E_i\}_{i=0}^d$.

Proof. For $0 \le i \le d$ we displayed a polynomial v_i^* of degree i such that $Q_i(j) = v_i^*(\theta_j^*)$ for $0 \le j \le d$. The result follows by Theorem 12.9. $\square$

Remark 15.16. The polynomials $\{v_i\}_{i=0}^d$ from Definition 15.13 are the same as the polynomials $\{v_i\}_{i=0}^d$ from Definition 11.4. The polynomials $\{v_i^*\}_{i=0}^d$ from Definition 15.13 are the same as the polynomials $\{v_i^*\}_{i=0}^d$ from Definition 12.4.

Next we compute the Krein parameters for $J(v,d)$.

Lemma 15.17. *For $J(v,d)$ we have*

$$
\begin{aligned}
b_i^* &= \frac{v(v-1)}{d(v-d)} \frac{(d-i)(v-i+1)(v-d-i)}{(v-2i)(v-2i+1)} && (0 \le i \le d-1),\\
c_i^* &= \frac{v(v-1)}{d(v-d)} \frac{i(d-i+1)(v-d-i+1)}{(v-2i+1)(v-2i+2)} && (1 \le i \le d),\\
a_i^* &= \frac{(v-1)(v-2d)^2 i(v-i+1)}{d(v-d)(v-2i)(v-2i+2)} && (0 \le i \le d-1),\\
a_d^* &= \frac{(v-1)(v-2d)(v-d+1)}{(v-d)(v-2d+2)}.
\end{aligned}
$$

Proof. Evaluate the 3-term recurrence given in Definition 12.4 using the formulas in Definition 15.13. $\square$

Problem 15.18. Show that for $H(d,q)$ the following holds for $0 \le i, j \le d$:

$$\frac{P_i(j)}{k_i} = \frac{Q_j(i)}{m_j} = {}_2F_1\left(\left.\begin{matrix}-i,-j\\-d\end{matrix}\,\right|\, \frac{q}{q-1}\right).$$