Lemma 14.13. *The matrices R, L, A^* satisfy*

$$A^*L - LA^* = 2L, \qquad A^*R - RA^* = -2R, \qquad LR - RL = A^*.$$

Proof. To verify these equations, for $y, z \in X$ compare the (y,z)-entry on either side. $\square$

Lecture 19

Remark 14.14. The above equations are the defining relations for the Lie algebra $\mathfrak{sl}_2$. We briefly explain the details. The Lie algebra $\mathfrak{sl}_2$ consists of the 2×2 matrices over $\mathbb{C}$ that have trace 0, together with the Lie bracket $[r,s] = rs - sr$. The vector space $\mathfrak{sl}_2$ has a basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie bracket satisfies

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H. \tag{54}$$

By these comments, the standard module V of $H(d,2)$ becomes an $\mathfrak{sl}_2$-module on which E, F, H act as L, R, A^* respectively.

Recall that the standard module V is an orthogonal direct sum of irreducible T-modules.

Definition 14.15. Let W denote an irreducible T-module. Define

$$r = \min\{i | 0 \le i \le d,\ E_i^* W \ne 0\}, \qquad \delta = \big|\{i | 0 \le i \le d,\ E_i^* W \ne 0\}\big| - 1.$$

We call r (resp. δ) the *endpoint* (resp. *diameter*) of W.

Proposition 14.16. *Let W denote an irreducible T-module. The endpoint r of W satisfies*

$$0 \le r \le d/2.$$

The diameter δ of W satisfies

$$\delta = d - 2r.$$

There exists a basis $\{w_i\}_{i=0}^{\delta}$ of W such that

$$\begin{aligned} &w_i \in E_{r+i}^* W && (0 \le i \le \delta), \\ &Rw_i = (i+1)w_{i+1} && (0 \le i \le \delta), \qquad Rw_\delta = 0, \\ &Lw_i = (\delta - i + 1)w_{i-1} && (1 \le i \le \delta), \qquad Lw_0 = 0. \end{aligned}$$

Proof. Pick $0 \neq w_0 \in E^*_r W$. For $0 \leq i \leq d-r$ define

$$w_i = \frac{R^i w_0}{i!}.$$

We have $w_i \in E^*_{r+i}W$, so

$$A^* w_i = (d - 2r - 2i)w_i.$$

By construction,

$$Rw_i = (i+1)w_{i+1} \qquad (0 \leq i \leq d-r),$$

where $w_{d-r+1} = 0$. By the definition of w_0, we have $Lw_0 = 0$. By this and $LR - RL = A^*$ we find by induction on i that

$$Lw_{i+1} = (d - 2r - i)w_i \qquad (0 \leq i \leq d-r).$$

There exists a unique integer s $(0 \leq s \leq d-r)$ such that $w_0, w_1, \ldots, w_s$ are nonzero and $w_{s+1} = 0$. We claim that $s = d - 2r$. To see this, note that

$$Lw_{s+1} = (d - 2r - s)w_s.$$

In the above equation, we have $Lw_{s+1} = L0 = 0$ and $w_s \neq 0$, so $s = d - 2r$. The claim is proved. One readily checks that the vectors $\{w_i\}_{i=0}^{d-2r}$ form a basis for W, and the result follows. □

Definition 14.17. Decompose the standard module V into an orthogonal direct sum of irreducible T-modules. For an integer r $(0 \leq r \leq d/2)$ let $\mathrm{mult}(r)$ denote the number of irreducible T-modules in this decomposition that have endpoint r.

Proposition 14.18. *With the above notation,*

$$\begin{aligned} \mathrm{mult}(0) &= 1, \\ \mathrm{mult}(r) &= \binom{d}{r} - \binom{d}{r-1} \qquad (1 \leq r \leq d/2). \end{aligned}$$

Proof. Using Proposition 14.16 we find that for $0 \leq i \leq d/2$,

$$\sum_{j=0}^{i} \mathrm{mult}(j) = \dim E^*_i V = k_i = \binom{d}{i}.$$

The result follows. □

Recall that

$$A^*L - LA^* = 2L, \qquad A^*R - RA^* = -2R, \qquad LR - RL = A^*. \tag{55}$$

Next we express these relations in terms of A, A^*.

Proposition 14.19. *For the graph $H(d,2)$ the adjacency matrix A and dual adjacency matrix A^* satisfy*

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^*,$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4A.$$

Proof. Recall $A = R + L$. Adding the first two equations in (55), we obtain

$$AA^* - A^*A = 2(R - L).$$

Combining this with $A = R + L$, we obtain

$$R = \frac{AA^* - A^*A + 2A}{4}, \qquad L = \frac{A^*A - AA^* + 2A}{4}.$$

Use these equations to eliminate R, L in the first two relations from (55). The result follows. □

For more information about $H(d,2)$ see

Junie Go. The Terwilliger algebra of the hypercube. Europ. J. Combin. (2002) 399–429.

15 The Johnson scheme $J(v,d)$

In this section we consider the Johnson association scheme $J(v,d)$. We mentioned earlier that $J(v,d)$ is P-polynomial; we view $J(v,d)$ as a distance-regular graph. This graph has valency $k = d(v-d)$ and intersection numbers

$$c_i = i^2, \qquad b_i = (d-i)(v-d-i), \qquad a_i = i(v-2i)$$

for $0 \leq i \leq d$. We will show that $J(v,d)$ is Q-polynomial. As we will see, $J(v,d)$ is not self-dual. Note that

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} = \binom{d}{i}\binom{v-d}{i} \qquad (0 \leq i \leq d).$$

The vertex set X of $J(v,d)$ consists of the d-subsets of $\{1,2,\ldots,v\}$. Consequently, we may identify X with the d^{th} subconstituent of $H(v,2)$ with respect to the vertex $\emptyset$. We will use bold face notation for $H(v,2)$. Thus

$$V = \mathbf{E}^*_d\mathbf{V}.$$

Lemma 15.1. *On V,*

$$\mathbf{RL} - d\mathbf{I} = \mathbf{LR} - (v-d)\mathbf{I}. \tag{56}$$

Proof. We have $\mathbf{LR} - \mathbf{RL} = \mathbf{A}^*$. Also $\mathbf{A}^*$ acts on V as $(v-2d)\mathbf{I}$. The result follows. □

Lemma 15.2. *The following are the same:*

(i) *the adjacency matrix A of $J(v,d)$;*

(ii) *the restriction of either side of* (56) *to $X \times X$.*

Proof. By the definition of adjacency for the graph $J(v,d)$. □

Recall the abbreviation

$$\theta_j = P_1(j) \qquad (0 \le j \le d).$$

For notational convenience, we order the primitive idempotents such that

$$\theta_0 > \theta_1 > \cdots > \theta_d.$$

Lemma 15.3. *For $0 \le j \le d$ we have*

$$E_j V = \mathbf{E}_d^* \mathbf{W}_j,$$

where $\mathbf{W}_j$ is the sum of the irreducible $\mathbf{T}$-modules that have endpoint j. We have

$$\theta_j = (d-j)(v-d-j) - j \qquad (0 \le j \le d).$$

Moreover, $m_0 = 1$ and

$$m_j = \binom{v}{j} - \binom{v}{j-1} \qquad (1 \le j \le d).$$

Proof. Let $\mathbf{W}$ denote an irreducible $\mathbf{T}$-module with endpoint j. Note that $\mathbf{RL} - d\mathbf{I}$ acts on $\mathbf{E}_d^*\mathbf{W}$ as $\alpha_j \mathbf{I}$, where

$$\alpha_j = (d-j)(v-d-j+1) - d = (d-j)(v-d-j) - j.$$

The result follows. □

Adjusting the above formula for m_j, we find

$$m_j = \binom{v}{j} \frac{v-2j+1}{v-j+1} \qquad (0 \le j \le d).$$

Our next goal is to compute $P_i(j)$ for $0 \le i,j \le d$. To do this, we describe A_i in terms of $\mathbf{R}$, $\mathbf{L}$.

Lemma 15.4. *For $0 \le i \le d$ the following holds on V:*

$$\frac{\mathbf{R}^i}{i!} \frac{\mathbf{L}^i}{i!} = \sum_{\ell=0}^{i} A_\ell \binom{d-\ell}{d-i}.$$