Proof. We use the generating function in Lemma 14.3. For $0 \le j \le d$ define

$$G_j(z) = \big(1 - z\big)^j \big(1 + (q-1)z\big)^{d-j}.$$

Let $D = d/dz$ denote the derivative with respect to z. By elementary calculus,

$$\theta_j G_j(z) = (q-1)\big(dz - z^2 D\big) G_j(z) + (q-2) z D G_j(z) + D G_j(z).$$

One routinely checks that

$$\begin{aligned}
\theta_j G_j(z) &= \sum_{i=0}^{d} \theta_j K_i(j) z^i, \\
(q-1)\big(dz - z^2 D\big) G_j(z) &= \sum_{i=0}^{d} b_{i-1} K_{i-1}(j) z^i, \\
(q-2) z D G_j(z) &= \sum_{i=0}^{d} a_i K_i(j) z^i, \\
D G_j(z) &= \sum_{i=0}^{d} c_{i+1} K_{i+1}(j) z^i.
\end{aligned}$$

The result follows. □

Proposition 14.5. *We have*

$$P_i(j) = K_i(j) \qquad\qquad (0 \le i, j \le d).$$

Proof. View j as fixed, and consider the sequences $\{P_i(j)\}_{i=0}^d$, $\{K_i(j)\}_{i=0}^d$. These sequences satisfy the same 3-term recurrence. They also satisfy the same initial condition $P_0(j) = 1 = K_0(j)$. The result follows. □

Lecture 18

Our next goal is to show that $Q_i(j) = K_i(j)$ for $0 \le i, j \le d$.

Lemma 14.6. *For $0 \le i, j \le d$ we have*

$$\frac{K_i(j)}{k_i} = \frac{K_j(i)}{k_j}. \tag{46}$$

Proof. Using (45) we find that each side of (46) is equal to

$$\sum_{\ell} \frac{(-1)^\ell}{(q-1)^\ell} \frac{i!(d-i)!j!(d-j)!}{(i-\ell)!(j-\ell)!\ell!(d-i-j+\ell)!},$$

where the sum is over all nonnegative integers ℓ such that $i + j - d \le \ell \le \min(i, j)$. □

Proposition 14.7. *We have*

$$Q_i(j) = K_i(j) \qquad\qquad (0 \le i, j \le d).$$

Proof. We have

$$\frac{Q_i(j)}{m_i} = \frac{\overline{P_j(i)}}{k_j} = \frac{K_j(i)}{k_j} = \frac{K_i(j)}{k_i} = \frac{K_i(j)}{m_i}.$$

Therefore $Q_i(j) = K_i(j)$. □

Corollary 14.8. *The Hamming scheme $H(d, q)$ is self-dual.*

Proof. We have $P = Q = \overline{Q}$. □

We mention some alternative forms for the Krawtchouk polynomials.

Lemma 14.9. For $0 \le i \le d$ we have

$$K_i = \sum_{\ell=0}^{i} (-1)^\ell q^\ell (q-1)^{i-\ell} \binom{d-\ell}{i-\ell} \binom{\lambda}{\ell}, \tag{47}$$

$$K_i = \sum_{\ell=0}^{i} (-1)^\ell q^{i-\ell} \binom{d-i+\ell}{\ell} \binom{d-\lambda}{i-\ell}. \tag{48}$$

Proof. We will use generating functions. First we consider (47). For $0 \le j \le d$ we have

$$\begin{aligned}
&\sum_{i=0}^{d} z^i \sum_{\ell=0}^{i} (-1)^\ell q^\ell (q-1)^{i-\ell} \binom{d-\ell}{i-\ell} \binom{j}{\ell} \\
&= \sum_{\ell=0}^{j} \sum_{i=\ell}^{d} z^i (-1)^\ell q^\ell (q-1)^{i-\ell} \binom{d-\ell}{i-\ell} \binom{j}{\ell} \\
&= \sum_{\ell=0}^{j} (-1)^\ell q^\ell z^\ell \binom{j}{\ell} \sum_{i=\ell}^{d} z^{i-\ell} (q-1)^{i-\ell} \binom{d-\ell}{i-\ell} \\
&= \sum_{\ell=0}^{j} (-1)^\ell q^\ell z^\ell \binom{j}{\ell} \sum_{r=0}^{d-\ell} z^r (q-1)^r \binom{d-\ell}{r} \qquad\qquad r = i-\ell \\
&= \sum_{\ell=0}^{j} (-1)^\ell q^\ell z^\ell \binom{j}{\ell} \big(1 + (q-1)z\big)^{d-\ell} \\
&= \big(1 + (q-1)z\big)^d \sum_{\ell=0}^{j} (-1)^\ell q^\ell z^\ell \binom{j}{\ell} \big(1 + (q-1)z\big)^{-\ell} \\
&= \big(1 + (q-1)z\big)^d \left(1 - \frac{qz}{1 + (q-1)z}\right)^j \\
&= \big(1 + (q-1)z\big)^d \left(\frac{1-z}{1 + (q-1)z}\right)^j \\
&= (1-z)^j \big(1 + (q-1)z\big)^{d-j}.
\end{aligned}$$

We now consider (48). For $0 \le j \le d$ we have

$$
\begin{aligned}
\sum_{i=0}^{d} z^i \sum_{\ell=0}^{i} (-1)^\ell q^{i-\ell} \binom{d-i+\ell}{\ell} \binom{d-j}{i-\ell}
&= \sum_{r=0}^{d-j} z^r q^r \binom{d-j}{r} \sum_{\ell=0}^{d-r} z^\ell (-1)^\ell \binom{d-r}{\ell} \qquad r = i-\ell \\
&= \sum_{r=0}^{d-j} z^r q^r \binom{d-j}{r} (1-z)^{d-r} \\
&= (1-z)^j \sum_{r=0}^{d-j} \binom{d-j}{r} q^r z^r (1-z)^{d-j-r} \\
&= (1-z)^j \big(1+(q-1)z\big)^{d-j}.
\end{aligned}
$$

□

Remark 14.10. We mention an alternative proof of Proposition 14.5. We use the notation from the proof of Lemma 14.1. For $0 \le i \le d$ we have

$$A_i = \sum F_1 \otimes F_2 \otimes \cdots \otimes F_d, \tag{49}$$

where the sum is over all the sequences $F_1, F_2, \ldots, F_d$ involving i copies of $\mathcal{A}$ and $d-i$ copies of I. The sum (49) has $\binom{d}{i}$ summands. Recall that for $0 \le j \le d$ we have

$$E_j V = \sum U_1 \otimes U_2 \otimes \cdots \otimes U_d, \tag{50}$$

where the sum is over all the sequences $U_1, U_2, \ldots, U_d$ involving j copies of W_1 and $d-j$ copies of W_0. The sum (50) has $\binom{d}{j}$ summands. Fix one of these summands:

$$U_1 \otimes U_2 \otimes \cdots \otimes U_d. \tag{51}$$

We compute the action of A_i on (51). For the moment, pick a summand $F_1 \otimes F_2 \otimes \cdots \otimes F_d$ from (49). Let ℓ denote the number of integers r $(1 \le r \le d)$ such that $F_r = \mathcal{A}$ and $U_r = W_1$. Note that $0 \le \ell \le i$. The summand $F_1 \otimes F_2 \otimes \cdots \otimes F_d$ acts on (51) as $(-1)^\ell (q-1)^{i-\ell}$ times the identity. We call ℓ the *index* of $F_1 \otimes F_2 \otimes \cdots \otimes F_d$ on (51). For $0 \le \ell \le i$ there are exactly $\binom{d-j}{i-\ell}\binom{j}{\ell}$ summands in (49) that have index ℓ on (51). By these comments, A_i acts on (51) as the following scalar multiple of the identity:

$$\sum_{\ell=0}^{i} (-1)^\ell (q-1)^{i-\ell} \binom{d-j}{i-\ell} \binom{j}{\ell}.$$

Consequently

$$P_i(j) = \sum_{\ell=0}^{i} (-1)^\ell (q-1)^{i-\ell} \binom{d-j}{i-\ell} \binom{j}{\ell} \qquad (0 \le i, j \le d).$$

Remark 14.11. The formula (48) has the following combinatorial interpretation. We use the notation from the proof of Lemma 14.1. For $0 \leq i \leq d$ define

$$\Phi_i = \sum H_1 \otimes H_2 \otimes \cdots \otimes H_d, \tag{52}$$

where the sum is over all the sequences $H_1, H_2, \ldots, H_d$ involving i copies of $\mathcal{A} + I$ and $d - i$ copies of I. The sum (52) has $\binom{d}{i}$ summands. By combinatorial counting, we find

$$\Phi_i = \sum_{\ell=0}^{i} \binom{d-\ell}{i-\ell} A_\ell \qquad (0 \leq i \leq d).$$

Solving the above equations, we obtain

$$A_i = \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{d-\ell}{i-\ell} \Phi_\ell \qquad (0 \leq i \leq d). \tag{53}$$

For $0 \leq j, \ell \leq d$ we now compute the action of Φ_ℓ on the eigenspace $E_j V$. The matrix $\mathcal{A} + I$ acts on W_0 (resp. W_1) as q (resp. 0) times the identity. Therefore, Φ_ℓ acts on $E_j V$ as $q^\ell \binom{d-j}{\ell}$ times the identity. By this and (53) we find that for $0 \leq i, j \leq d$,

$$P_i(j) = \sum_{\ell=0}^{i} (-1)^{i-\ell} q^\ell \binom{d-\ell}{i-\ell} \binom{d-j}{\ell}.$$

In the above equation we make a change of variables $\ell \mapsto i - \ell$; this yields

$$P_i(j) = \sum_{\ell=0}^{i} (-1)^\ell q^{i-\ell} \binom{d-i+\ell}{\ell} \binom{d-j}{i-\ell}.$$

By this and $P_i(j) = K_i(j)$ we get (48).

For the rest of this section, we assume that $q = 2$. We view $F = \{0, 1\}$. The Hamming graph $H(d, 2)$ is often called the *binary Hamming graph*, or the *d-cube*, or a *hypercube*. We identify the vertices of $H(d, 2)$ with the subsets of $\{1, 2, \ldots d\}$. Vertices $y, z \in X$ are adjacent whenever one contains the other, and their cardinalities differ by one. The graph $H(d, 2)$ has valency $k = d$ and intersection numbers

$$c_i = i, \qquad b_i = d - i, \qquad a_i = 0$$

for $0 \leq i \leq d$. Moreover

$$k_i = \binom{d}{i} \qquad (0 \leq i \leq d).$$

The adjacency matrix A has eigenvalues

$$\theta_i = d - 2i \qquad (0 \leq i \leq d).$$

Since $H(d,2)$ is self-dual, we have

$$c_i^* = i, \qquad b_i^* = d-i, \qquad a_i^* = 0$$

for $0 \le i \le d$. Moreover

$$m_i = \binom{d}{i} \qquad (0 \le i \le d)$$

and

$$\theta_i^* = d - 2i \qquad (0 \le i \le d).$$

Fix the vertex $x = \emptyset$. For $y \in X$ we have $\partial(x,y) = |y|$.

Our next goal is to describe the subconstituent algebra $T = T(x)$.

Definition 14.12. For $y, z \in X$ we say that z *covers* y whenever $y \subseteq z$ and $|y| + 1 = |z|$. define

$$R = \sum_{i=0}^{d-1} E_{i+1}^* A E_i^*, \qquad L = \sum_{i=1}^{d} E_{i-1}^* A E_i^*.$$

For $y \in X$ we have

$$R\hat{y} = \sum_{z \text{ covers } y} \hat{z}, \qquad L\hat{y} = \sum_{y \text{ covers } z} \hat{z}.$$

Note that

$$A = R + L, \qquad R^t = L.$$

Recall the dual adjacency matrix $A^* = A_1^*$ with respect to x. We have

$$A^* = \sum_{i=0}^{d} (d-2i) E_i^*.$$

Lemma 14.13. *The matrices R, L, A^* satisfy*

$$A^*L - LA^* = 2L, \qquad A^*R - RA^* = -2R, \qquad LR - RL = A^*.$$

Proof. To verify these equations, for $y, z \in X$ compare the (y,z)-entry on either side. □

The above equations are the defining relations for the Lie algebra $\mathfrak{sl}_2$. We now explain the details. The Lie algebra $\mathfrak{sl}_2$ consists of the 2×2 matrices over $\mathbb{C}$ that have trace 0, together with the Lie bracket $[r,s] = rs - sr$. The vector space $\mathfrak{sl}_2$ has a basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie bracket satisfies

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = H. \tag{54}$$

Lemma 14.14. *The standard module V of $H(d,2)$ becomes an $\mathfrak{sl}_2$-module on which E, F, H act as L, R, A^* respectively.*

Proof. Compare Lemma 14.13 with (54). □