13 The conjugacy class association scheme for a finite abelian group

In this section we consider the conjugacy class association scheme for a finite abelian group G. Our goal is to show that this association scheme is self-dual.

Recall that any finite abelian group is a direct sum of cyclic groups. Write

$$G = (\mathbb{Z}/n_1\mathbb{Z}) \oplus (\mathbb{Z}/n_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/n_r\mathbb{Z}).$$

The group operation is expressed additively:

$$\begin{aligned} G \times G &\to G \\ (i,j) &\mapsto i+j \end{aligned}$$

For $1 \leq i \leq r$ let $\omega_i \in \mathbb{C}$ denote a primitive n_i^{th} root of unity. Thus $\omega_i^{n_i} = 1$, and $\omega_i^j \neq 1$ for $1 \leq j \leq n_i - 1$. Note that $\overline{\omega_i} = \omega_i^{-1}$ for $1 \leq i \leq r$.

We denote the group association scheme by $\mathcal{X} = (G, \{R_i\}_{i \in G})$. The associate matrices of $\mathcal{X}$ satisfy

$$\begin{aligned} A_i A_j &= A_{i+j} \qquad && (i,j \in G), \\ A_i^t &= A_{-i} \qquad && (i \in G). \end{aligned}$$

Recall the eigenmatrices P, Q of $\mathcal{X}$.

Lemma 13.1. *The matrices P, Q are described as follows after permuting the primitive idempotents as necessary. For $i = (i_1, i_2, \ldots, i_r) \in G$ and $j = (j_1, j_2, \ldots, j_r) \in G$ we have*

$$P_i(j) = \omega_1^{i_1 j_1} \omega_2^{i_2 j_2} \cdots \omega_r^{i_r j_r}, \qquad Q_i(j) = \omega_1^{-i_1 j_1} \omega_2^{-i_2 j_2} \cdots \omega_r^{-i_r j_r}.$$

Proof. Define

$$p_i(j) = \omega_1^{i_1 j_1} \omega_2^{i_2 j_2} \cdots \omega_r^{i_r j_r}, \qquad q_i(j) = \omega_1^{-i_1 j_1} \omega_2^{-i_2 j_2} \cdots \omega_r^{-i_r j_r}.$$

Note that $p_i(j) q_i(j) = 1$ and $\overline{p_i(j)} = q_i(j)$. For $a, b, i, j \in G$ we have

$$\begin{aligned} &p_i(0) = 1, \quad p_0(j) = 1, \quad p_i(a)p_i(b) = p_i(a+b), \quad p_a(j)p_b(j) = p_{a+b}(j), \\ &p_i(j) = p_j(i), \quad p_{-i}(j) = q_i(j) = p_i(-j), \\ &q_i(0) = 1, \quad q_0(j) = 1, \quad q_i(a)q_i(b) = q_i(a+b), \quad q_a(j)q_b(j) = q_{a+b}(j), \\ &q_i(j) = q_j(i), \quad q_{-i}(j) = p_i(j) = q_i(-j). \end{aligned}$$

It suffices to show that the matrices

$$E_i = |G|^{-1} \sum_{j \in G} q_i(j) A_j \qquad (i \in G)$$

are the primitive idempotents of $\mathcal{X}$, and that $A_r E_i = p_r(i) E_i$ for $i, r \in G$. For $i, r \in G$ we have

$$\begin{aligned} A_r E_i &= |G|^{-1} \sum_{j \in G} q_i(j) A_r A_j \\ &= |G|^{-1} \sum_{j \in G} q_i(j) A_{r+j} \\ &= |G|^{-1} \sum_{j \in G} q_i(j - r) A_j \\ &= |G|^{-1} \sum_{j \in G} q_i(j) q_i(-r) A_j \\ &= |G|^{-1} \sum_{j \in G} q_i(j) p_i(r) A_j \\ &= p_i(r) E_i \\ &= p_r(i) E_i. \end{aligned}$$

So far, we have shown that E_i is a scalar multiple of a primitive idempotent of $\mathcal{X}$. The scalar is equal to 1, because $\mathrm{tr}(E_i) = 1$ and every primitive idempotent has trace 1. The result follows. □

Proposition 13.2. *We have* $P = \overline{Q}$.

Proof. By Lemma 13.1 we have $P_i(j) = \overline{Q_i(j)}$ for $i, j \in G$. □

Corollary 13.3. *The conjugacy class association scheme* $\mathcal{X}$ *is self-dual.*

Proof. By Proposition 7.6 and Proposition 13.2. □

14 The Hamming association scheme $H(d, q)$

In this section we consider the Hamming association scheme $H(d, q)$. We saw earlier that $H(d, q)$ is P-polynomial. Our next goal is to show that $H(d, q)$ is Q-polynomial. We will do this by showing that $H(d, q)$ is self-dual. Recall that $H(d, q)$ has valency $k = (q-1)d$ and intersection numbers

$$c_i = i, \qquad b_i = (q-1)(d-i), \qquad a_i = (q-2)i$$

for $0 \leq i \leq d$. We have

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} = (q-1)^i \binom{d}{i} \qquad (0 \leq i \leq d).$$

Recall the abbreviation

$$\theta_j = P_1(j) \qquad (0 \leq j \leq d).$$

For notational convenience, we order the primitive idempotents such that

$$\theta_0 > \theta_1 > \cdots > \theta_d.$$

Lemma 14.1. *For $0 \le j \le d$ we have*

$$\theta_j = (q-1)(d-j) - j, \qquad m_j = k_j.$$

Proof. The vertex set X of $H(d,q)$ is given by

$$X = F \times F \times \cdots \times F \qquad (d \text{ copies}),$$

where $|F| = q$. View F as the vertex set of the complete graph K_q. Let $\mathcal{A}$ denote the adjacency matrix of K_q. The matrix $\mathcal{A}$ has eigenvalues $q-1$ (with multiplicity 1) and -1 (with multiplicity $q-1$). Let $W = \mathbb{C}^F$ denote the standard module for K_q. Let W_0 (resp. W_1) denote the eigenspace of $\mathcal{A}$ with eigenvalue $q-1$ (resp. -1). The dimension of W_0 (resp. W_1) is 1 (resp. $q-1$). The sum $W = W_0 + W_1$ is direct. Recall the standard module V of $H(d,q)$. We view

$$V = W \otimes W \otimes \cdots \otimes W \qquad (d \text{ factors}).$$

From this point of view, the adjacency matrix $A = A_1$ of $H(d,q)$ satisfies

$$A = \sum_{i=1}^{d} I \otimes \cdots \otimes I \otimes \mathcal{A} \otimes I \otimes \cdots \otimes I,$$

where $\mathcal{A}$ is the i^{th} factor. We have

$$\begin{aligned} V &= (W_0 + W_1) \otimes (W_0 + W_1) \otimes \cdots \otimes (W_0 + W_1) \\ &= \sum U_1 \otimes U_2 \otimes \cdots \otimes U_d, \end{aligned}$$

where the sum is over all sequences $U_1, U_2, \ldots, U_d$ such that U_i is one of W_0, W_1 for $1 \le i \le d$. On each summand $U_1 \otimes U_2 \otimes \cdots \otimes U_d$ the matrix A acts as $(q-1)(d-j) - j$ times the identity, where

$$j = \big|\{i \mid 1 \le i \le d,\ U_i = W_1\}\big|.$$

Consequently

$$\theta_j = (q-1)(d-j) - j \qquad (0 \le j \le d).$$

Moreover, for $0 \le j \le d$ we have

$$E_j V = \sum U_1 \otimes U_2 \otimes \cdots \otimes U_d,$$

where the sum is over all sequences $U_1, U_2, \ldots, U_d$ involving j copies of W_1 and $d-j$ copies of W_0. This sum has $\binom{d}{j}$ summands, and each summand has dimension $(q-1)^j$. Therefore

$$m_j = \dim E_j V = (q-1)^j \binom{d}{j} = k_j.$$

□

Lemma 14.2. *For $0 \le i, j \le d$ we have*

$$\theta_j P_i(j) = b_{i-1} P_{i-1}(j) + a_i P_i(j) + c_{i+1} P_{i+1}(j),$$

where $P_{-1}(j) = 0$ and $P_{d+1}(j) = 0$.

Proof. We have

$$AA_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1},$$

where $A_{-1} = 0$ and $A_{d+1} = 0$. In the above equation multiply each side by E_j, and evaluate the result. □

Recall the polynomial algebra $\mathbb{R}[\lambda]$. For $0 \le i \le d$ define the polynomial $K_i \in \mathbb{R}[\lambda]$ by

$$K_i = \sum_{\ell=0}^{i} (-1)^\ell (q-1)^{i-\ell} \binom{d-\lambda}{i-\ell} \binom{\lambda}{\ell}. \qquad (45)$$

Note that K_i has degree i. We call K_i the i^{th} *Krawtchouk polynomial* with parameters d, q. For example

$$K_0 = 1, \qquad\qquad K_1 = (q-1)(d-\lambda) - \lambda.$$

We are going to show that $P_i(j) = K_i(j)$ for $0 \le i, j \le d$. It is convenient to use generating functions.

Lemma 14.3. Let z denote an indeterminate. Then

$$\sum_{i=0}^{d} K_i(j) z^i = \big(1-z\big)^j \big(1 + (q-1)z\big)^{d-j} \qquad (0 \le j \le d).$$

Proof. Consider the right-hand side of the above equation. For $0 \le i \le d$ compute the coefficient of z^i using the binomial theorem. Evaluate this coefficient using (45). □

Lemma 14.4. *For $0 \le i, j \le d$ we have*

$$\theta_j K_i(j) = b_{i-1} K_{i-1}(j) + a_i K_i(j) + c_{i+1} K_{i+1}(j),$$

where $K_{-1} = 0$ and $K_{d+1} = 0$.

Proof. We use the generating function in Lemma 14.3. For $0 \le j \le d$ define

$$G_j(z) = \big(1-z\big)^j \big(1 + (q-1)z\big)^{d-j}.$$

Let $D = d/dz$ denote the derivative with respect to z. By elementary calculus,

$$\theta_j G_j(z) = (q-1)\big(dz - z^2 D\big) G_j(z) + (q-2) z D G_j(z) + D G_j(z).$$

One routinely checks that

$$\begin{aligned}
\theta_j G_j(z) &= \sum_{i=0}^{d} \theta_j K_i(j) z^i, \\
(q-1)\big(dz - z^2 D\big) G_j(z) &= \sum_{i=0}^{d} b_{i-1} K_{i-1}(j) z^i, \\
(q-2) z D G_j(z) &= \sum_{i=0}^{d} a_i K_i(j) z^i, \\
D G_j(z) &= \sum_{i=0}^{d} c_{i+1} K_{i+1}(j) z^i.
\end{aligned}$$

The result follows. □

Proposition 14.5. *We have*

$$P_i(j) = K_i(j) \qquad\qquad (0 \le i, j \le d).$$

Proof. View j as fixed, and consider the sequences $\{P_i(j)\}_{i=0}^d$, $\{K_i(j)\}_{i=0}^d$. These sequences satisfy the same 3-term recurrence. They also satisfy the same initial condition $P_0(j) = 1 = K_0(j)$. The result follows. □

Our next goal is to show that $Q_i(j) = K_i(j)$ for $0 \le i, j \le d$.

Lemma 14.6. *For* $0 \le i, j \le d$ *we have*

$$\frac{K_i(j)}{k_i} = \frac{K_j(i)}{k_j}. \tag{46}$$

Proof. Using (45) we find that each side of (46) is equal to

$$\sum_{\ell} \frac{(-1)^\ell}{(q-1)^\ell} \frac{i!(d-i)!j!(d-j)!}{(i-\ell)!(j-\ell)!\ell!(d-i-j+\ell)!},$$

where the sum is over all nonnegative integers ℓ such that $i + j - d \le \ell \le \min(i, j)$. □

Proposition 14.7. *We have*

$$Q_i(j) = K_i(j) \qquad\qquad (0 \le i, j \le d).$$

Proof. We have

$$\frac{Q_i(j)}{m_i} = \frac{\overline{P_j(i)}}{k_j} = \frac{K_j(i)}{k_j} = \frac{K_i(j)}{k_i} = \frac{K_i(j)}{m_i}.$$

Therefore $Q_i(j) = K_i(j)$. □