Theorem 11.9. *For the symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ the following are equivalent:*

(i) *the ordering $\{R_i\}_{i=0}^d$ is P-polynomial;*

(ii) *the first intersection matrix B_1 is irreducible tridiagonal;*

(iii) *there exist polynomials $\{v_i\}_{i=0}^d$ in $\mathbb{R}[\lambda]$ such that v_i has degree i and $A_i = v_i(A)$ for $0 \le i \le d$, where $A = A_1$;*

(iv) *there exist polynomials $\{v_i\}_{i=0}^d$ in $\mathbb{R}[\lambda]$ such that v_i has degree i $(0 \le i \le d)$ and*

$$P_i(j) = v_i(\theta_j) \qquad (0 \le i, j \le d),$$

where $\theta_j = P_1(j)$ for $0 \le j \le d$.

Proof. (i) $\Rightarrow$ (ii) By Lemma 11.2.
(ii) $\Rightarrow$ (iii) In the proof of Lemma 11.5, we only used the fact that B_1 is irreducible tridiagonal.
(iii) $\Leftrightarrow$ (iv) Use

$$A_i = \sum_{j=0}^{d} P_i(j) E_j \qquad (0 \le i \le d)$$

and $A = \sum_{j=0}^{d} \theta_j E_j$.
(iii) $\Rightarrow$ (i) Recall that

$$k_\ell p^\ell_{i,j} = k_i p^i_{j,\ell} = k_j p^j_{\ell,i} \qquad (0 \le i, j, \ell \le d).$$

It suffices to show that for $0 \le i, j \le d$ with $i + j \le d$,

$$i + j = \max\{\ell | 0 \le \ell \le d,\ p^\ell_{i,j} > 0\}. \tag{42}$$

In the equation

$$A_i A_j = \sum_{\ell=0}^{d} p^\ell_{i,j} A_\ell,$$

we view each side as a polynomial in A. By comparing the degrees we routinely obtain (42). □

Lecture 16

Next, we explain how P-polynomial association schemes are related to distance-regular graphs.

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and adjacency relation $\mathcal{R}$. Vertices x, y are *adjacent* whenever $(x, y) \in \mathcal{R}$.

To avoid trivialities, we assume that $|X| \geq 2$. Let ∂ denote the path-length distance function for Γ, and define $d = \max\{\partial(x,y)|x, y \in X\}$. We call d the *diameter* of Γ. For $x \in X$ and an integer $i \geq 0$ define $\Gamma_i(x) = \{y \in X | \partial(x,y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$ we say that Γ is *regular with valency* k whenever $|\Gamma(x)| = k$ for all $x \in X$. We say that Γ is *distance-regular* whenever for $0 \leq i \leq d$ and $x, y \in X$ with $\partial(x,y) = i$, the constants

$$a_i = |\Gamma_i(x) \cap \Gamma(y)|, \qquad b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \qquad c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$$

depend only on i and not on the choice of x, y. Assume that Γ is distance-regular. By construction $a_0 = 0$, $b_d = 0$, $c_0 = 0$, $c_1 = 1$. Moreover

$$c_i > 0 \quad (1 \leq i \leq d), \qquad b_i > 0 \quad (0 \leq i \leq d-1).$$

The graph Γ is regular with valency $k = b_0$. Moreover,

$$c_i + a_i + b_i = k \qquad (0 \leq i \leq d).$$

Theorem 11.10. *Let $\Gamma = (X, \mathcal{R})$ denote a distance-regular graph with diameter d. For $0 \leq i \leq d$ define*

$$R_i = \{(x,y)|x, y \in X, \ \partial(x,y) = i\}.$$

Then $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme, and the ordering $\{R_i\}_{i=0}^d$ is P-polynomial.

Proof. For $0 \leq i \leq d$ define a matrix $A_i \in M_X(\mathbb{C})$ with (y,z)-entry

$$(A_i)_{y,z} = \begin{cases} 1, & \text{if } \partial(y,z) = i; \\ 0, & \text{if } \partial(y,z) \neq i \end{cases} \qquad (y, z \in X).$$

The matrix A_i is symmetric. Note that $A_0 = I$. Abbreviate $A = A_1$. The distance-regularity of Γ implies that

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \qquad (1 \leq i \leq d-1), \tag{43}$$

$$AA_d = b_{d-1}A_{d-1} + a_dA_d. \tag{44}$$

Let $\mathcal{M}$ denote the subalgebra of $M_X(\mathbb{C})$ generated by A. The algebra $\mathcal{M}$ is commutative. By (43), (44) the matrices $\{A_i\}_{i=0}^d$ form a basis for $\mathcal{M}$. Consequently $\mathcal{M}$ is closed under Hadamard multiplication. The algebra $\mathcal{M}$ contains $J = \sum_{i=0}^d A_i$. By these comments and Proposition 2.4, we see that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme. The ordering $\{R_i\}_{i=0}^d$ is P-polynomial because the distance function ∂ satisfies the triangle inequality. □

Theorem 11.11. *Let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denote a symmetric association scheme such that $\{R_i\}_{i=0}^d$ is P-polynomial. Then the graph (X, R_1) is distance-regular with diameter d. Moreover*

$$R_i = \{(x,y)|x, y \in X, \ \partial(x,y) = i\} \qquad (0 \leq i \leq d).$$

Proof. Routine consequence of Lemma 11.3. □

Problem 11.12. Assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is P-polynomial with respect to the ordering $\{R_i\}_{i=0}^d$. Show that

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad (0 \leq i \leq d).$$

Problem 11.13. Show that the Hamming scheme $H(d, q)$ is P-polynomial, with valency $k = (q-1)d$ and intersection numbers

$$c_i = i, \qquad b_i = (q-1)(d-i), \qquad a_i = (q-2)i$$

for $0 \leq i \leq d$.

Problem 11.14. Show that the Johnson scheme $J(v, d)$ is P-polynomial, with valency $k = d(v-d)$ and intersection numbers

$$c_i = i^2, \qquad b_i = (d-i)(v-d-i), \qquad a_i = i(v-2i)$$

for $0 \leq i \leq d$.

12 Q-polynomial association schemes

In this section, we continue to discuss a symmetric association $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. We assume that $d \geq 1$.

Definition 12.1. The ordering $\{E_i\}_{i=0}^d$ is called *Q-polynomial* whenever the following hold for $0 \leq i, j, k \leq d$:

(i) $q_{i,j}^k = 0$ if one of i, j, k is greater than the sum of the other two;

(ii) $q_{i,j}^k \neq 0$ if one of i, j, k is equal to the sum of the other two.

We say that $\mathcal{X}$ is *Q-polynomial* whenever there exists a Q-polynomial ordering of the primitive idempotents.

Lemma 12.2. *Assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. Then the first dual intersection matrix has the form*

$$B_1^* = \begin{pmatrix} a_0^* & c_1^* & & & & \mathbf{0} \\ b_0^* & a_1^* & c_2^* & & & \\ & b_1^* & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & c_d^* \\ \mathbf{0} & & & & b_{d-1}^* & a_d^* \end{pmatrix},$$

where we abbreviate

$$c_i^* = q_{1,i-1}^i (1 \leq i \leq d), \qquad a_i^* = q_{1,i}^i (0 \leq i \leq d), \qquad b_i^* = q_{1,i+1}^i (1 \leq i \leq d-1).$$

Moreover $\{c_i^*\}_{i=1}^d$ and $\{b_i^*\}_{i=0}^{d-1}$ are nonzero.

Proof. By the definition of the dual intersection matrices. □

Until further notice, assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial. We fix $x \in X$, and consider the subconstituent algebra $T = T(x)$. We abbreviate $A^* = A_1^*$. Note that $a_0^* = 0$ and $c_1^* = 1$.

Lemma 12.3. *We have*

$$A^* A_i^* = b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^* \qquad (1 \leq i \leq d-1),$$
$$A^* A_d^* = b_{d-1}^* A_{d-1}^* + a_d^* A_d^*.$$

Proof. This is $A_i^* A_j^* = \sum_{k=0}^d q_{i,j}^k A_k^*$ with $j = 1$. □

Definition 12.4. We define some polynomials $\{v_i^*\}_{i=0}^{d+1}$ in $\mathbb{R}[\lambda]$ such that

$$v_0^* = 1, \qquad v_1^* = \lambda,$$
$$\lambda v_i^* = b_{i-1}^* v_{i-1}^* + a_i^* v_i^* + c_{i+1}^* v_{i+1}^* \qquad (1 \leq i \leq d),$$

where $c_{d+1}^* = 1$.

Lemma 12.5. *The following* (i)–(iv) *hold:*

(i) $\deg v_i^* = i \quad (0 \leq i \leq d+1)$;

(ii) *the coefficient of* λ^i *in* v_i^* *is* $(c_1^* c_2^* \cdots c_i^*)^{-1} \quad (0 \leq i \leq d+1)$;

(iii) $v_i^*(A^*) = A_i^* \quad (0 \leq i \leq d)$;

(iv) $v_{d+1}^*(A^*) = 0$.

Proof. Similar to the proof of Lemma 11.5. □

Corollary 12.6. *The following hold:*

(i) *the algebra* $\mathcal{M}^*$ *is generated by* A^*;

(ii) *the minimal polynomial of* A^* *is* $c_1^* c_2^* \cdots c_d^* v_{d+1}^*$.

Proof. Similar to the proof of Lemma 11.6. □

Recall that

$$A_i^* = \sum_{j=0}^d Q_i(j) E_j^* \qquad (0 \leq i \leq d).$$

Define

$$\theta_j^* = Q_1(j) \qquad (0 \leq j \leq d).$$

Note that

$$A^* = \sum_{j=0}^d \theta_j^* E_j^*.$$

Lemma 12.7. *The following* (i)–(iii) *hold:*

(i) *the scalars* $\{\theta^*_j\}_{j=0}^d$ *are mutually distinct, and these are the roots of the polynomial* v^*_{d+1};

(ii) *the eigenspaces of* A^* *are subconstituents* $\{E^*_jV\}_{j=0}^d$;

(iii) *for* $0 \le j \le d$, θ^*_j *is the eigenvalue of* A^* *for* E^*_jV.

Proof. Similar to the proof of Lemma 11.7. □

Lemma 12.8. *We have*

$$Q_i(j) = v^*_i(\theta^*_j) \qquad\qquad (0 \le i, j \le d).$$

Proof. We have

$$\sum_{j=0}^{d} Q_i(j)E^*_j = A^*_i = v^*_i(A^*) = \sum_{j=0}^{d} v_i(\theta^*_j)E^*_j.$$

□

We have been describing some features of the Q-polynomial association scheme $\mathcal{X}$. Next, we use these features to characterize the Q-polynomial property. Going forward, we no longer assume that the ordering $\{E_i\}_{i=0}^d$ is Q-polynomial.

Theorem 12.9. *For the symmetric association scheme* $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ *the following are equivalent:*

(i) *the ordering* $\{E_i\}_{i=0}^d$ *is* Q*-polynomial;*

(ii) *the first dual intersection matrix* B^*_1 *is irreducible tridiagonal;*

(iii) *there exist polynomials* $\{v^*_i\}_{i=0}^d$ *in* $\mathbb{R}[\lambda]$ *such that* v^*_i *has degree* i *and* $A^*_i = v^*_i(A^*)$ *for* $0 \le i \le d$, *where* $A^* = A^*_1$;

(iv) *there exist polynomials* $\{v^*_i\}_{i=0}^d$ *in* $\mathbb{R}[\lambda]$ *such that* v^*_i *has degree* i $(0 \le i \le d)$ *and*

$$Q_i(j) = v^*_i(\theta_j) \qquad\qquad (0 \le i, j \le d),$$

where $\theta^*_j = Q_1(j)$ *for* $0 \le j \le d$.

Proof. Similar to the proof of Theorem 11.9. □

As we will see, both the Hamming scheme $H(d, q)$ and the Johnson scheme $J(v, d)$ are Q-polynomial.

Problem 12.10. Assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$. Show that

$$m_i = \frac{b^*_0 b^*_1 \cdots b^*_{i-1}}{c^*_1 c^*_2 \cdots c^*_i} \qquad\qquad (0 \le i \le d).$$