Lecture 15

Proposition 10.17. $\tilde{\mathcal{M}}^\circ_\Lambda$ *is the Bose-Mesner algebra of a commutative association scheme with vertex set* $\tilde{X}$, *associate matrices* $\{D_i\}_{i=0}^t$, *and primitive idempotents* $\{k_\Omega \tilde{E}_i\}_{i=0}^t$.

Proof. By Proposition 2.4 and Lemmas 10.12, 10.16. □

Definition 10.18. The association scheme in Proposiiton 10.17 is called the *quotient association scheme* of $\mathcal{X}$ induced by the relation R_Ω. We denote this association scheme by $\mathcal{Q}$.

Lemma 10.19. *The association schemes* $\mathcal{Q}$ *and* $\mathcal{X}$ *are related as follows:*

(i) $q^k_{i,j}(\mathcal{Q}) = q^k_{i,j}(\mathcal{X}) \qquad (0 \le i, j, k \le t)$;

(ii) $m_i(\mathcal{Q}) = m_i(\mathcal{X}) \qquad (0 \le i \le t)$;

(iii) $|\tilde{X}| = \sum_{i=0}^t m_i$, *where* $m_i = m_i(\mathcal{Q}) = m_i(\mathcal{X})$.

Proof. (i) For $0 \le i, j \le t$ we have

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q^k_{i,j}(\mathcal{X}) E_k$$
$$= |X|^{-1} \sum_{k=0}^{t} q^k_{i,j}(\mathcal{X}) E_k.$$

In the above equation we apply the map $A \mapsto \tilde{A}$ to each side; this yields

$$\tilde{E}_i \circ \tilde{E}_j = |X|^{-1} \sum_{k=0}^{t} q^k_{i,j}(\mathcal{X}) \tilde{E}_k.$$

We also have

$$\left(k_\Omega \tilde{E}_i\right) \circ \left(k_\Omega \tilde{E}_j\right) = |\tilde{X}|^{-1} \sum_{k=0}^{t} q^k_{i,j}(\mathcal{Q}) k_\Omega \tilde{E}_i.$$

Comparing the above equations using $k_\Omega |\tilde{X}| = |X|$, we get the result.
(ii) Since $m_i = q^0_{i,\hat{i}}$.
(iii) Apply Lemma 4.3(iii) to the association scheme $\mathcal{Q}$. □

Corollary 10.20. *For the relation* R_Ω,

$$|X| = \left(\sum_{i=0}^{s} k_i\right)\left(\sum_{j=0}^{t} m_j\right).$$

Proof. By Lemma 10.19(iii) and since

$$|\tilde{X}| = k_\Omega^{-1}|X|, \qquad k_\Omega = \sum_{i=0}^{s} k_i.$$

□

Recall the eigenmatrices P, Q for $\mathcal{X}$. Let $\tilde{P}$ and $\tilde{Q}$ denote the eigenmatrices for $\mathcal{Q}$.

Proposition 10.21. *The following* (i)–(iv) *hold.*

(i) *For $0 \leq i \leq t$ the submatrix $Q|_{\Omega_i \times \Lambda}$ has all rows identical.*

(ii) *For $0 \leq i, j \leq t$ the (i, j)-entry of $\tilde{Q}$ is equal to the (α, j)-entry of Q, where $\alpha \in \Omega_i$.*

(iii) *For $0 \leq j \leq t$ the submatrix $P|_{\{t+1,\ldots,d\} \times \Omega_j}$ has row sum 0.*

(iv) *For $0 \leq i, j \leq t$ the (i, j)-entry of $\tilde{P}$ is equal to k_Ω^{-1} times the i^{th} row sum of $P|_{\Lambda \times \Omega_j}$.*

Proof. Similar to the proof of Proposition 10.7. □

11 Distance-regular graphs and P-polynomial association schemes

In this section, we restrict our attention to symmetric association schemes.

Throughout this section, $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denotes a symmetic association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$. To avoid trivialities, we assume that $d \geq 1$. Since the matrices in $\mathcal{M}$ are symmetric, we have

$$i' = i, \qquad \hat{i} = i \qquad (0 \leq i \leq d).$$

Consequently

$$P_i(j) \in \mathbb{R}, \qquad Q_i(j) \in \mathbb{R} \qquad (0 \leq i, j \leq d).$$

Definition 11.1. The ordering $\{R_i\}_{i=0}^d$ is called *P-polynomial* whenever the following hold for $0 \leq i, j, k \leq d$:

(i) $p_{i,j}^k = 0$ if one of i, j, k is greater than the sum of the other two;

(ii) $p_{i,j}^k \neq 0$ if one of i, j, k is equal to the sum of the other two.

We say that $\mathcal{X}$ is *P-polynomial* whenever there exists a P-polynomial ordering of the relations.

Lemma 11.2. *Assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. Then the first intersection matrix has the form*

$$B_1 = \begin{pmatrix} a_0 & c_1 & & & & \mathbf{0} \\ b_0 & a_1 & c_2 & & & \\ & b_1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & c_d \\ \mathbf{0} & & & & b_{d-1} & a_d \end{pmatrix},$$

where we abbreviate

$$c_i = p^i_{1,i-1} (1 \le i \le d), \qquad a_i = p^i_{1,i} (0 \le i \le d), \qquad b_i = p^i_{1,i+1} (1 \le i \le d-1).$$

Moreover $\{c_i\}_{i=1}^d$ and $\{b_i\}_{i=0}^{d-1}$ are nonzero.

Proof. By the definition of the intersection matrices. □

Until further notice, assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial. We abbreviate $A = A_1$. Note that $a_0 = 0$ and $c_1 = 1$.

Lemma 11.3. *We have*

$$\begin{aligned} AA_i &= b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \qquad && (1 \le i \le d-1), \\ AA_d &= b_{d-1}A_{d-1} + a_dA_d. && \end{aligned}$$

Proof. This is $A_iA_j = \sum_{k=0}^d p^k_{i,j} A_k$ with $j = 1$. □

Let λ denote an indeterminate. Let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra of polynomials in λ that have all coefficients in $\mathbb{R}$.

Definition 11.4. We define some polynomials $\{v_i\}_{i=0}^{d+1}$ in $\mathbb{R}[\lambda]$ such that

$$\begin{aligned} v_0 &= 1, \qquad v_1 = \lambda, && \\ \lambda v_i &= b_{i-1}v_{i-1} + a_iv_i + c_{i+1}v_{i+1} \qquad && (1 \le i \le d), \end{aligned}$$

where $c_{d+1} = 1$.

Lemma 11.5. *The following* (i)–(iv) *hold:*

(i) $\deg v_i = i \quad (0 \le i \le d+1)$;

(ii) *the coefficient of λ^i in v_i is $(c_1c_2\cdots c_i)^{-1} \quad (0 \le i \le d+1)$;*

(iii) $v_i(A) = A_i \quad (0 \le i \le d)$;

(iv) $v_{d+1}(A) = 0$.

Proof. (i), (ii) By Definition 11.4.
(iii), (iv) Compare Lemma 11.3 and Definition 11.4. □

Corollary 11.6. *The following hold:*

(i) *the algebra $\mathcal{M}$ is generated by A;*

(ii) *the minimal polynomial of A is $c_1 c_2 \cdots c_d v_{d+1}$.*

Proof. By Lemma 11.5 and since $\{A_i\}_{i=0}^d$ is a basis for $\mathcal{M}$. □

Recall that

$$A_i = \sum_{j=0}^{d} P_i(j) E_j \qquad (0 \le i \le d).$$

Define

$$\theta_j = P_1(j) \qquad (0 \le j \le d).$$

Note that

$$A = \sum_{j=0}^{d} \theta_j E_j. \tag{41}$$

Lemma 11.7. *The following* (i)–(iii) *hold:*

(i) *the scalars $\{\theta_j\}_{j=0}^d$ are mutually distinct, and these are the roots of the polynomial v_{d+1};*

(ii) *the eigenspaces of A are $\{E_j V\}_{j=0}^d$;*

(iii) *for $0 \le j \le d$, θ_j is the eigenvalue of A for $E_j V$.*

Proof. (i) The roots of v_{d+1} are mutually distinct by Corollary 11.6(ii) and since A is diagonalizable. These roots are $\{\theta_j\}_{i=0}^d$ by (41) .
(ii), (iii) By (41). □

Lemma 11.8. *We have*

$$P_i(j) = v_i(\theta_j) \qquad (0 \le i, j \le d).$$

Proof. We have

$$\sum_{j=0}^{d} P_i(j) E_j = A_i = v_i(A) = \sum_{j=0}^{d} v_i(\theta_j) E_j.$$

□

We have been describing some features of the P-polynomial association scheme $\mathcal{X}$. Next, we use these features to characterize the P-polynomial property. Going forward, we no longer assume that the ordering $\{R_i\}_{i=0}^d$ is P-polynomial.

We make some definitions. A matrix $B \in M_{d+1}(\mathbb{R})$ is called *tridiagonal* whenever

$$B_{i,j} = 0 \quad \text{if} \quad |i-j| > 1 \qquad (0 \le i, j \le d).$$

Assume that B is tridiagonal. Then B is called *irreducible* whenever

$$B_{i,i-1} \neq 0, \qquad B_{i-1,i} \neq 0 \qquad (1 \le i \le d).$$

Theorem 11.9. *For the symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ the following are equivalent:*

(i) *the ordering $\{R_i\}_{i=0}^d$ is P-polynomial;*

(ii) *the first intersection matrix B_1 is irreducible tridiagonal;*

(iii) *there exist polynomials $\{v_i\}_{i=0}^d$ in $\mathbb{R}[\lambda]$ such that v_i has degree i and $A_i = v_i(A)$ for $0 \leq i \leq d$, where $A = A_1$;*

(iv) *there exist polynomials $\{v_i\}_{i=0}^d$ in $\mathbb{R}[\lambda]$ such that v_i has degree i $(0 \leq i \leq d)$ and*

$$P_i(j) = v_i(\theta_j) \qquad (0 \leq i, j \leq d),$$

where $\theta_j = P_1(j)$ for $0 \leq j \leq d$.

Proof. (i) ⇒ (ii) By Lemma 11.2.
(ii) ⇒ (iii) In the proof of Lemma 11.5, we only used the fact that B_1 is irreducible tridiagonal.
(iii) ⇔ (iv) Use

$$A_i = \sum_{j=0}^{d} P_i(j) E_j \qquad (0 \leq i \leq d)$$

and $A = \sum_{j=0}^d \theta_j E_j$.
(iii) ⇒ (i) Recall that

$$k_\ell p^\ell_{i,j} = k_i p^i_{j,\ell} = k_j p^j_{\ell,i} \qquad (0 \leq i, j, \ell \leq d).$$

It suffices to show that for $0 \leq i, j \leq d$ with $i + j \leq d$,

$$i + j = \max\{\ell | 0 \leq \ell \leq d,\ p^\ell_{i,j} > 0\}. \tag{42}$$

In the equation

$$A_i A_j = \sum_{\ell=0}^{d} p^\ell_{i,j} A_\ell,$$

we view each side as a polynomial in A. By comparing the degrees we routinely obtain (42). □

Next, we explain how P-polynomial association schemes are related to distance-regular graphs.

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and adjacency relation $\mathcal{R}$. Vertices x, y are *adjacent* whenever $(x, y) \in \mathcal{R}$. To avoid trivialities, we assume that $|X| \geq 2$. Let ∂ denote the path-length distance function for Γ, and define $d = \max\{\partial(x, y) | x, y \in X\}$. We call d the *diameter* of Γ. For $x \in X$ and an integer $i \geq 0$ define $\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an