(iii) *The map*

$$\mathcal{M}_\Omega \to \mathcal{M}_\Omega|_{Y\times Y}$$
$$A \mapsto A|_{Y\times Y}$$

is an algebra isomorphism with respect to matrix multiplication and Hadamard multiplication.

(iv) *The above map sends $k_\Omega^{-1}A_\Omega$ to the trivial primitive idempotent for $\mathcal{Y}$.*

Proof. (i) By construction.
(ii) By (i) above.
(iii) The map is a bijection because it sends the basis $\{A_i\}_{i=0}^s$ of $\mathcal{M}_\Omega$ to the basis $\{A_i|_{Y\times Y}\}_{i=0}^s$ of $\mathcal{M}_\Omega|_{Y\times Y}$. The map is an algebra homomorphism by the block-diagonal nature of the matrices in $\mathcal{M}_\Omega$.
(iv) By the comments above (39). □

The algebra $\mathcal{M}_\Omega$ is closed under the conjugate-transpose map. By Lemma 3.3 the algebra $\mathcal{M}_\Omega$ has a basis of primitive idempotents. One of these primitive idempotents is $k_\Omega^{-1}A_\Omega$, in view of Lemma 10.5(iv).

Lemma 10.6. *For the subalgebra $\mathcal{M}_\Omega$ the primitive idempotents have the form $\{E_{\Lambda_i}\}_{i=0}^s$ such that:*

(i) *$\{\Lambda_i\}_{i=0}^s$ is a partition of $\{0, 1, \dots, d\}$ into nonempty sets;*

(ii) *$E_{\Lambda_i} = \sum_{j\in\Lambda_i} E_j \qquad (0 \le i \le s)$;*

(iii) *$0 \in \Lambda_0$;*

(iv) *$E_{\Lambda_0} = k_\Omega^{-1}A_\Omega$.*

Proof. (i), (ii) Each primitive idempotent E of $\mathcal{M}_\Omega$ is a linear combination of $\{E_j\}_{j=0}^d$. In this linear combination, each coefficient is zero or one because $E^2 = E$. The result is a routine consequence of this.
(iii) There exists i $(0 \le i \le s)$ such that $0 \in \Lambda_i$. After relabelling, we may assume that $i = 0$.
(iv) There exists i $(0 \le i \le s)$ such that $E_{\Lambda_i} = k_\Omega^{-1}A_\Omega$. We have $i = 0$ by (iii) and $A_\Omega J \ne 0$. □

Lecture 14

Recall the eigenmatrices P, Q for $\mathcal{X}$. Let P_Ω, Q_Ω denote the eigenmatrices for $\mathcal{Y}$.

Proposition 10.7. *The following* (i)–(iv) *hold.*

(i) *For $0 \le i \le s$ the submatrix $P|_{\Lambda_i\times\Omega}$ has all rows identical.*

(ii) *For $0 \le i, j \le s$ the (i,j)-entry of P_Ω is equal to the (α, j)-entry of P, where $\alpha \in \Lambda_i$.*

(iii) *For* $0 \leq j \leq s$ *the submatrix* $Q|_{\{s+1,\dots,d\}\times\Lambda_j}$ *has row sum 0.*

(iv) *For* $0 \leq i, j \leq s$ *the* (i,j)*-entry of* Q_Ω *is equal to* $|Y|/|X|$ *times the* i^{th} *row sum of* $Q|_{\Omega\times\Lambda_j}$.

Proof. (i), (ii) For $0 \leq j \leq s$ we have $A_j = \sum_{i=0}^{d} P_j(i)E_i$. In this equation the right-hand side is a linear combination of $\{E_{\Lambda_i}\}_{i=0}^{s}$. The result follows.
(iii), (iv) For $0 \leq j \leq s$ we have

$$E_{\Lambda_j} = \sum_{\alpha\in\Lambda_j} E_\alpha = |X|^{-1}\sum_{\alpha\in\Lambda_j}\sum_{i=0}^{d} Q_\alpha(i)A_i = |X|^{-1}\sum_{i=0}^{d} A_i \sum_{\alpha\in\Lambda_j} Q_\alpha(i).$$

The matrix E_{Λ_j} is a linear combination of $\{A_i\}_{i=0}^{s}$. The result follows. □

We have been discussing the subscheme $\mathcal{Y}$ of $\mathcal{X}$ induced by an equivalence class Y of R_Ω. Next we discuss the quotient association scheme induced by R_Ω. For the quotient association scheme the vertex set consists of the equivalence classes of R_Ω.

For notational convenience, we abbreviate $\Lambda = \Lambda_0$. We have $0 \in \Lambda$. Write $t+1 = |\Lambda|$. Permuting $\{E_i\}_{i=0}^{d}$ if necessary, we may assume without loss of generality that

$$\Lambda = \{0, 1, \dots, t\}.$$

By construction,

$$k_\Omega^{-1} A_\Omega = \sum_{j=0}^{t} E_j = E_\Lambda. \tag{40}$$

Note that

$$E_j E_\Lambda = E_j \qquad (0 \leq j \leq t).$$

Lemma 10.8. *The following hold:*

(i) *if* $0 \leq i, j \leq t$ *and* $q_{i,j}^{k} > 0$ *then* $0 \leq k \leq t$ $\quad (0 \leq i, j, k \leq d)$;

(ii) *for* $0 \leq i \leq t$ *we have* $0 \leq \hat{i} \leq t$.

Proof. (i) In the equation $A_\Omega \circ A_\Omega = A_\Omega$, use (40) to write each side as a linear combination of the primitive idempotents of $\mathcal{X}$.
(ii) By (40) and since A_Ω is symmetric. □

Lemma 10.9. *The matrices* $\{E_i\}_{i=0}^{t}$ *form a basis for a subalgebra* $\mathcal{M}_\Lambda^\circ$ *of* $\mathcal{M}^\circ$ *that is closed under matrix multiplication, complex conjugation, and the transpose map. With respect to matrix multiplication,* $\mathcal{M}_\Lambda^\circ$ *is a commutative algebra with multiplicative identity* E_Λ.

Proof. By Lemma 10.8 and the construction. □

Lemma 10.10. *The subalgebra* $\mathcal{M}_\Lambda^\circ$ *has a basis* $\{A_{\Omega_i}\}_{i=0}^{t}$ *such that:*

(i) $\{\Omega_i\}_{i=0}^t$ *is a partition of* $\{0, 1, \ldots, d\}$ *into nonempty sets;*

(ii) $A_{\Omega_i} = \sum_{j \in \Omega_i} A_j \qquad (0 \leq i \leq t)$;

(iii) $A_\Omega A_{\Omega_i} = A_{\Omega_i} A_\Omega = k_\Omega A_{\Omega_i} \qquad (0 \leq i \leq t)$;

(iv) $\Omega_0 = \Omega$.

Proof. (i), (ii) Because $\mathcal{M}_\Lambda^\circ$ is closed under Hadamard multiplication and contains $J = |X| E_0$.
(iii) Since $E_\Lambda = k_\Omega^{-1} A_\Omega$ is the multiplicative identity for $\mathcal{M}_\Lambda^\circ$.
(iv) Permuting $\{\Omega_i\}_{i=0}^t$ if necessary, we may assume that $0 \in \Omega_0$. Since $A_\Omega \in \mathcal{M}_\Lambda^\circ$, A_Ω is a linear combination of $\{A_{\Omega_i}\}_{i=0}^t$. In this linear combination each coefficient is 0 or 1, since A_Ω has all entries 0 or 1. So Ω is a union of some of $\{\Omega_i\}_{i=0}^t$. We have $0 \in \Omega$ and $0 \in \Omega_0$, so $\Omega_0 \subseteq \Omega$. We have

$$A_\Omega A_{\Omega_0} = k_\Omega A_{\Omega_0}.$$

Since $0 \in \Omega_0$, the product $A_\Omega A_0$ will contribute to $A_\Omega A_{\Omega_0}$. But $A_\Omega A_0 = A_\Omega I = A_\Omega$, so A_Ω will contribute to $A_\Omega A_{\Omega_0}$. Therefore $\Omega_0 = \Omega$. □

Recall the equivalence classes $X_1, X_2, \ldots, X_r$ of R_Ω. The subsets $\{X_i\}_{i=1}^r$ partition X.

Lemma 10.11. *For* $0 \leq i \leq t$ *consider the matrix* A_{Ω_i}.

(i) *For* $1 \leq \alpha, \beta \leq r$ *the submatrix* $A_{\Omega_i}|_{X_\alpha \times X_\beta}$ *has all entries 0 or all entries 1.*

(ii) *Define an* $r \times r$ *matrix* D_i *with* (α, β)*-entry*

$$D_i(\alpha, \beta) = \begin{cases} 1 & \text{if } A_{\Omega_i}|_{X_\alpha \times X_\beta} \text{ has all entries 1;} \\ 0 & \text{if } A_{\Omega_i}|_{X_\alpha \times X_\beta} \text{ has all entries 0} \end{cases} \qquad (1 \leq \alpha, \beta \leq r).$$

Then $A_{\Omega_i} = D_i \otimes J$, *where* J *is* $k_\Omega \times k_\Omega$.

Proof. (i) The matrix A_Ω is block diagonal, with each diagonal block a copy of J. Every entry of A_{Ω_i} is 0 or 1. We have $A_\Omega A_{\Omega_i} = A_{\Omega_i} A_\Omega = k_\Omega A_{\Omega_i}$. The result follows by matrix multiplication.
(ii) This is a reformulation of (i) above. □

We have a comment.

Lemma 10.12. *The following hold:*

(i) $D_0 = I$;

(ii) *the matrix* $\sum_{i=0}^t D_i$ *has all entries 1.*

Proof. (i) The matrix A_Ω is block diagonal, with each diagonal block a copy of J. Therefore $A_\Omega = I \otimes J$, so $D_0 = I$.
(ii) The matrix $\sum_{i=0}^d A_i$ has all entries 1, and

$$\sum_{i=0}^d A_i = \sum_{i=0}^t A_{\Omega_i} = \sum_{i=0}^t D_i \otimes J.$$

□

Define the set

$$\tilde{X} = \{X_1, X_2, \ldots, X_r\}.$$

We view $D_i \in M_{\tilde{X}}(\mathbb{C})$ for $0 \leq i \leq t$. We are going to show that $\{D_i\}_{i=0}^t$ are the associate matrices for a commutative association scheme with vertex set $\tilde{X}$. This association scheme is the quotient scheme that was mentioned earlier.

Lemma 10.13. *For $A \in \mathcal{M}_\Lambda^\circ$ there exists a unique $\tilde{A} \in M_{\tilde{X}}(\mathbb{C})$ such that $A = \tilde{A} \otimes J$.*

Proof. By Lemma 10.11 and since $\{A_{\Omega_i}\}_{i=0}^t$ is a basis for $\mathcal{M}_\Lambda^\circ$. □

Lemma 10.14. *The following* (i)–(vii) *hold for $A, B \in \mathcal{M}_\Lambda^\circ$ and $\alpha \in \mathbb{C}$:*

(i) $\widetilde{A+B} = \tilde{A} + \tilde{B}$;

(ii) $\widetilde{\alpha A} = \alpha \tilde{A}$;

(iii) $\widetilde{A \circ B} = \tilde{A} \circ \tilde{B}$;

(iv) $\widetilde{AB} = k_\Omega \tilde{A}\tilde{B}$;

(v) $\tilde{A} = 0$ iff $A = 0$;

(vi) $\widetilde{A^t} = \tilde{A}^t$;

(vii) $\overline{\tilde{A}} = \tilde{\overline{A}}$.

Proof. These are readily checked using Lemma 10.13. □

Definition 10.15. *Define the vector space $\tilde{\mathcal{M}}_\Lambda^\circ = \{\tilde{A} | A \in \mathcal{M}_\Lambda^\circ\}$.*

Lemma 10.16. *The vector space $\tilde{\mathcal{M}}_\Lambda^\circ$ is closed under Hadamard multiplication, matrix multiplication, complex conjugation, and the transpose map. Moreover:*

(i) *The map*

$$\begin{aligned} \mathcal{M}_\Lambda^\circ &\to \tilde{\mathcal{M}}_\Lambda^\circ \\ A &\mapsto \tilde{A} \end{aligned}$$

is an isomorphism of algebras with respect to Hadamard multiplication.

(ii) *The map*

$$\begin{aligned} \mathcal{M}_\Lambda^\circ &\to \tilde{\mathcal{M}}_\Lambda^\circ \\ A &\mapsto k_\Omega \tilde{A} \end{aligned}$$

is an isomorphism of algebras with respect to matrix multiplication.

Proof. Immediate from Lemma 10.14. □