We show that $C_1 = C$. Suppose $C_1 \subsetneq C$. Since C is connected, there exists $r \in C_1$ and $s \in C \backslash C_1$ such that $s \to r$. We examine the r-coordinate in $B_i^t v = k_i v$; this gives

$$k_i = \sum_{j=0}^{d} (B_i^t)_{r,j} v_j = \sum_{j=0}^{d} (B_i)_{j,r} v_j = \sum_{j=0}^{d} p_{i,j}^r v_j. \tag{36}$$

For $0 \leq j \leq d$ we have $p_{i,j}^r = 0$ if $j \notin C$ and $|v_j| \leq 1$ if $j \in C$; therefore $p_{i,j}^r |v_j| \leq p_{i,j}^r$. Consequently

$$k_i = \left| \sum_{j=0}^{d} p_{i,j}^r v_j \right| \leq \sum_{j=0}^{d} p_{i,j}^r |v_j| \leq \sum_{j=0}^{d} p_{i,j}^r = k_i. \tag{37}$$

Combining (36), (37) we obtain $v_j = 1$ for all $j \in C$ such that $j \to r$. This fails for $j = s$, so we have a contradiction. Therefore $C_1 = C$, so $v_r = 1$ for $r \in C$. In particular $v_r = v_s$ for $r, s \in C$.
(b) $\Rightarrow$ (a) This holds because B_i^t has constant row sum k_i.
We have shown that (a), (b) are equivalent. Consequently $\dim W = m$, and the result follows.
(ii) Similar to the proof of (i) above. □

Lecture 13

Proposition 9.18. *The following are equivalent:*

(i) *there exists $i \in \{1, 2, \ldots, d\}$ such that Δ_{A_i} is disconnected;*

(ii) *there exists $j \in \{1, 2, \ldots, d\}$ such that Δ_{E_j} is disconnected.*

Proof. For $1 \leq i, j \leq d$ we have

$$\frac{P_i(j)}{k_i} = \frac{\overline{Q_j(i)}}{m_j}.$$

Therefore, $P_i(j) = k_i$ if and only if $Q_j(i) = m_j$. The result follows from this and Proposition 9.17. □

Proposition 9.19. *The following are equivalent:*

(i) *the association scheme $\mathfrak{X}$ is primitive;*

(ii) *the distribution diagram Δ_{A_i} is connected for $1 \leq i \leq d$;*

(iii) *the representation diagram Δ_{E_j} is connected for $1 \leq j \leq d$.*

Proof. By Definition 9.4, Corollary 9.13, and Proposition 9.18. □

Problem 9.20. For a finite group G, show that the following are equivalent:

(i) the conjugacy–class association scheme for G is primitive;

(ii) G is simple.

10 Subschemes and quotient schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

Throughout this section, we assume that $\mathcal{X}$ is imprimitive. By Proposition 9.14 there exists a subset $\{0\} \subsetneq \Omega \subsetneq \{0, 1, \dots, d\}$ such that R_Ω is an equivalence relation. Recall that

$$R_\Omega = \cup_{k \in \Omega} R_k.$$

Write $s+1 = |\Omega|$. Permuting the relations $\{R_i\}_{i=0}^d$ if necessary, we may assume without loss of generality that

$$\Omega = \{0, 1, \dots, s\}.$$

Lemma 10.1. *The following hold:*

(i) *if $0 \le i, j \le s$ and $p_{i,j}^k > 0$ then $0 \le k \le s$* $\qquad (0 \le i, j, k \le d)$;

(ii) *for $0 \le i \le s$ we have $0 \le i' \le s$.*

Proof. (i) The relation R_Ω is transitive.
(ii) The relation R_Ω is symmetric. □

Lemma 10.2. *Let $Y \subseteq X$ denote an equivalence class of R_Ω. Then:*

(i) *$\mathcal{Y} = (Y, \{R_i|_{Y\times Y}\}_{i=0}^s)$ is a commutative association scheme;*

(ii) *$p_{i,j}^k(\mathcal{Y}) = p_{i,j}^k(\mathcal{X}) \qquad (0 \le i, j, k \le s)$;*

(iii) *$k_i(\mathcal{Y}) = k_i(\mathcal{X}) \qquad (0 \le i \le s)$;*

(iv) *$|Y| = \sum_{i=0}^s k_i$ where $k_i = k_i(\mathcal{X}) = k_i(\mathcal{Y})$.*

Proof. (i) We check that $\mathcal{Y}$ satisfies the four conditions in Definition 1.1.

- The trivial relation for $\mathcal{Y}$ is $R_0|_{Y\times Y} = \{(y,y) | y \in Y\}$.
- The relations $\{R_i|_{Y\times Y}\}_{i=0}^s$ partition $Y \times Y$, because for $(x,y) \in Y \times Y$ we have $(x,y) \in R_\Omega$, so there exists $i \in \Omega = \{0, 1, \dots, s\}$ such that $(x,y) \in R_i$. This i is unique by construction.
- For $0 \le i \le s$ we have $0 \le i' \le s$ and

$$\left(R_i|_{Y\times Y}\right)^t = R_{i'}|_{Y\times Y}.$$

- For $0 \le i, j, k \le s$ and $x, y \in Y$ with $(x,y) \in R_k$,

$$\begin{aligned} p_{i,j}^k(\mathcal{X}) &= \left|\{z \in X | (x,z) \in R_i,\ (z,y) \in R_j\}\right| \\ &= \left|\{z \in Y | (x,z) \in R_i|_{Y\times Y},\ (z,y) \in R_j|_{Y\times Y}\}\right| \\ &= p_{i,j}^k(\mathcal{Y}). \end{aligned}$$

(ii) This was obtained in the proof of (i) above.
(iii) Since $k_i = p^0_{i,i'}$.
(iv) Clear. □

Definition 10.3. The association scheme $\mathcal{Y}$ in Lemma 10.2 is called the *subscheme of* $\mathcal{X}$ *induced on* Y.

Consider the subscheme $\mathcal{Y} = (Y, \{R_i|_{Y\times Y}\}_{i=0}^s)$ from Lemma 10.2. Our next general goal is to describe how the Bose-Mesner algebra of $\mathcal{Y}$ is related to $\mathcal{M}$.

For notational convenience, define

$$k_\Omega = \sum_{i=0}^{s} k_i, \qquad A_\Omega = \sum_{i=0}^{s} A_i.$$

Note that

$$A_i \circ A_\Omega = A_i \qquad (0 \le i \le s).$$

Moreover

$$A_\Omega \circ A_\Omega = A_\Omega. \tag{38}$$

Lemma 10.4. *The matrices* $\{A_i\}_{i=0}^s$ *form a basis for a subalgebra* $\mathcal{M}_\Omega$ *of* $\mathcal{M}$ *that is closed under Hadamard multiplication, complex conjugation, and the transpose map. With respect to Hadamard multiplication,* $\mathcal{M}_\Omega$ *is an algebra with multiplicative identity* A_Ω.

Proof. By Lemma 10.1 and the construction. □

By construction, the algebra $\mathcal{M}_\Omega$ in Lemma 10.4 is commutative.

Let $X_1, X_2, \ldots, X_r$ denote an ordering of the equivalence classes of R_Ω. We have $|X_i| = k_\Omega$ for $1 \le i \le r$. The sets $\{X_i\}_{i=1}^r$ partition of X, so

$$r = k_\Omega^{-1}|X|.$$

Relative to the partition $\{X_i\}_{i=1}^r$, every matrix in $\mathcal{M}_\Omega$ is block-diagonal, with each diagonal block of dimension $k_\Omega \times k_\Omega$. For example A_Ω is block diagonal, with each diagonal block a copy of J with dimension $k_\Omega \times k_\Omega$. We have $J^2 = k_\Omega J$. Therefore

$$(A_\Omega)^2 = k_\Omega A_\Omega. \tag{39}$$

Lemma 10.5. *Let* $Y \subseteq X$ *denote an equivalence class of* R_Ω*, and consider the association scheme* $\mathcal{Y} = (Y, \{R_i|_{Y\times Y}\}_{i=0}^s)$ *from Lemma 10.2.*

(i) *The associate matrices of* $\mathcal{Y}$ *are* $\{A_i|_{Y\times Y}\}_{i=0}^s$.

(ii) *The Bose-Mesner algebra of* $\mathcal{Y}$ *is* $\mathcal{M}_\Omega|_{Y\times Y}$.

(iii) *The map*

$$\mathcal{M}_\Omega \to \mathcal{M}_\Omega|_{Y\times Y}$$
$$A \mapsto A|_{Y\times Y}$$

is an algebra isomorphism with respect to matrix multiplication and Hadamard multiplication.

(iv) *The above map sends $k_\Omega^{-1}A_\Omega$ to the trivial primitive idempotent for $\mathcal{Y}$.*

Proof. (i) By construction.
(ii) By (i) above.
(iii) The map is a bijection because it sends the basis $\{A_i\}_{i=0}^s$ of $\mathcal{M}_\Omega$ to the basis $\{A_i|_{Y\times Y}\}_{i=0}^s$ of $\mathcal{M}_\Omega|_{Y\times Y}$. The map is an algebra homomorphism by the block-diagonal nature of the matrices in $\mathcal{M}_\Omega$.
(iv) By the comments above (39). □

The algebra $\mathcal{M}_\Omega$ is closed under the conjugate-transpose map. By Lemma 3.3 the algebra $\mathcal{M}_\Omega$ has a basis of primitive idempotents. One of these primitive idempotents is $k_\Omega^{-1}A_\Omega$, in view of Lemma 10.5(iv).

Lemma 10.6. *For the subalgebra $\mathcal{M}_\Omega$ the primitive idempotents have the form $\{E_{\Lambda_i}\}_{i=0}^s$ such that:*

(i) *$\{\Lambda_i\}_{i=0}^s$ is a partition of $\{0,1,\dots,d\}$ into nonempty sets;*

(ii) *$E_{\Lambda_i} = \sum_{j\in\Lambda_i} E_j \qquad (0 \le i \le s)$;*

(iii) *$0 \in \Lambda_0$;*

(iv) *$E_{\Lambda_0} = k_\Omega^{-1}A_\Omega$.*

Proof. (i), (ii) Each primitive idempotent E of $\mathcal{M}_\Omega$ is a linear combination of $\{E_j\}_{j=0}^d$. In this linear combination, each coefficient is zero or one because $E^2 = E$. The result is a routine consequence of this.
(iii) There exists i $(0 \le i \le s)$ such that $0 \in \Lambda_i$. After relabelling, we may assume that $i = 0$.
(iv) There exists i $(0 \le i \le s)$ such that $E_{\Lambda_i} = k_\Omega^{-1}A_\Omega$. We have $i = 0$ by (iii) and $A_\Omega J \ne 0$. □

Recall the eigenmatrices P, Q for $\mathcal{X}$. Let P_Ω, Q_Ω denote the eigenmatrices for $\mathcal{Y}$.

Proposition 10.7. *The following* (i)–(iv) *hold.*

(i) *For $0 \le i \le s$ the submatrix $P|_{\Lambda_i\times\Omega}$ has all rows identical.*

(ii) *For $0 \le i, j \le s$ the (i,j)-entry of P_Ω is equal to the (α, j)-entry of P, where $\alpha \in \Lambda_i$.*

(iii) *For $0 \le j \le s$ the submatrix $Q|_{\{s+1,\dots,d\}\times\Lambda_j}$ has row sum 0.*