Lecture 12

Proposition 9.14. *The following are equivalent:*

(i) *the association scheme $\mathcal{X}$ is imprimitive;*

(ii) *there exists a subset $\{0\} \subsetneq \Omega \subsetneq \{0, 1, \ldots, d\}$ such that R_Ω is an equivalence relation.*

Proof. (i) $\Rightarrow$ (ii) There exists a relation R_i $(1 \leq i \leq d)$ such that the graph (X, R_i) is not connected. Consider the corresponding set Ω from Lemma 9.10. By Lemma 9.8 the relation R_Ω is an equivalence relation. By construction $\{0\} \subsetneq \Omega \subsetneq \{0, 1, \ldots, d\}$.
(ii) $\Rightarrow$ (i) There exists $i \in \Omega$ with $i \neq 0$. Each equivalence class of R_Ω is a disjoint union of connected components for (X, R_i). The relation R_Ω has more than one equivalence class, so (X, R_i) is not connected. Therefore $\mathcal{X}$ is imprimitive. $\square$

Earlier we used the intersection numbers to define the distribution diagrams of $\mathcal{X}$. Next we use the Krein parameters to define the representation diagrams of $\mathcal{X}$.

Definition 9.15. For $1 \leq i \leq d$ we define a directed graph Δ_{E_i} with vertex set $\{0, 1, \ldots, d\}$; vertices j, k satisfy $j \to k$ whenever $q^k_{i,j} > 0$. Note that a vertex j of Δ_{E_i} has a loop $j \to j$ whenever $q^j_{i,j} > 0$. We call Δ_{E_i} the *E_i-representation diagram for $\mathcal{X}$.*

Lemma 9.16. *We refer to the representation diagram Δ_{E_i} from Definition 9.15. For $0 \leq a, b \leq d$ the following are equivalent:*

(i) *there exists a path from a to b;*

(ii) *there exists a path from b to a.*

Proof. (i) $\Rightarrow$ (ii) Fix $x \in X$, and consider the dual Bose-Mesner algebra $\mathcal{M}^* = \mathcal{M}^*(x)$. Since A^*_i is nonzero and diagonalizable, there exists $t \in \mathbb{N}$ such that $\operatorname{tr}\big((A^*_{\hat{i}})^{t+1}\big) \neq 0$. Recall that $\{A^*_j\}_{j=0}^d$ is a basis for $\mathcal{M}^*$. Also recall that $A^*_0 = I$. For $1 \leq j \leq d$ we have

$$\operatorname{tr}(A^*_j) = \operatorname{tr}\sum_{\ell=0}^{d} Q_j(\ell) E^*_\ell = \sum_{\ell=0}^{d} Q_j(\ell) k_\ell = m_j \sum_{\ell=0}^{d} \overline{P_\ell(j)} = m_j \sum_{\ell=0}^{d} P_\ell(0)\overline{P_\ell(j)} k_\ell^{-1} = 0.$$

Write $(A^*_i)^t = \sum_{j=0}^d \alpha_j A^*_j$ and $(A^*_{\hat{i}})^{t+1} = \sum_{j=0}^d \beta_j A^*_j$. So $A^*_{\hat{i}} \sum_{j=0}^d \alpha_j A^*_j = \sum_{j=0}^d \beta_j A^*_j$. We have $\beta_0 \neq 0$ since the trace of $(A^*_{\hat{i}})^{t+1}$ is nonzero. Observe that

$$\beta_0 = \sum_{j=0}^{d} \alpha_j q^0_{\hat{i},j} = \sum_{j=0}^{d} \alpha_j \delta_{\hat{i},j} m_j = \alpha_{\hat{i}} m_{\hat{i}}.$$

By these comments $\alpha_{\hat{i}} \neq 0$. Without loss, we may assume that $a \to b$ in Δ_{E_i}. So $q^b_{i,a} \neq 0$. Observe that $q^a_{b,\hat{i}} \neq 0$, so $E_b A^*_{\hat{i}} E_a \neq 0$. Therefore

$$E_b (A^*_i)^t E_a = \sum_{j=0}^{d} \alpha_j E_b A^*_j E_a = \alpha_{\hat{i}} E_b A^*_{\hat{i}} E_a + \text{orthogonal terms} \neq 0.$$

Observe that

$$
\begin{aligned}
E_b(A_i^*)^t E_a &= E_b A_i^* \left(\sum_{b_1=0}^{d} E_{b_1} \right) A_i^* \left(\sum_{b_2=0}^{d} E_{b_2} \right) A_i^* \cdots A_i^* \left(\sum_{b_{t-1}=0}^{d} E_{b_{t-1}} \right) A_i^* E_a \\
&= \sum E_b A_i^* E_{b_1} A_i^* E_{b_2} A_i^* \cdots A_i^* E_{b_{t-1}} A_i^* E_a,
\end{aligned}
$$

where the sum is over all paths $b, b_1, b_2, \ldots, b_{t-1}, a$ in Δ_{E_i}. Such a path exists because $E_b(A_i^*)^t E_a \neq 0$. We have shown that there exists a path in Δ_{E_i} from b to a.
(ii) $\Rightarrow$ (i) By symmetry. □

To motivate the next result, we make some observations. Pick an integer i $(1 \leq i \leq d)$. Recall that

$$
\begin{aligned}
&P_i(0) = k_i, && Q_i(0) = m_i, && \\
&|P_i(j)| \leq k_i, && |Q_i(j)| \leq m_i && (0 \leq j \leq d).
\end{aligned}
$$

The intersection matrix B_i has all entries real and nonnegative. It is diagonalizable, and its characteristic polynomial has roots $\{P_i(j)\}_{j=0}^d$. The dual intersection matrix B_i^* has all entries real and nonnegative. It is diagonalizable, and its characteristic polynomial has roots $\{Q_i(j)\}_{j=0}^d$. Recall the distribution diagram Δ_{A_i} and the representation diagram Δ_{E_i}. The following result is a special case of the Frobenius theory for nonnegative matrices.

Proposition 9.17. *With the above notation,*

(i) *the number of connected components of Δ_{A_i} is equal to*

$$
|\{j|0 \leq j \leq d,\ P_i(j) = k_i\}|; \tag{35}
$$

(ii) *the number of connected components of Δ_{E_i} is equal to*

$$
|\{j|0 \leq j \leq d,\ Q_i(j) = m_i\}|.
$$

Proof. (i) Let W denote the k_i-eigenspace for B_i^t. The dimension of W is equal to (35). Let m denote the number of connected components for Δ_{A_i}. We will show that $\dim W = m$. For $0 \leq r, s \leq d$ define $r \sim s$ whenever r, s are in the same connected component of Δ_{A_i}. Let $v = (v_0, v_1, \ldots, v_d)^t \in \mathbb{C}^{d+1}$. We show that the following are equivalent:

(a) $v \in W$;

(b) $v_r = v_s$ if $r \sim s$ $(0 \leq r, s \leq d)$.

(a) $\Rightarrow$ (b) Let $C \subseteq \{0, 1, \ldots, d\}$ denote a connected component of Δ_{A_i}. We show that $v_r = v_s$ for $r, s \in C$. First assume that $v_r = 0$ for $r \in C$. Then certainly $v_r = v_s$ for $r, s \in C$. Next assume that $\{v_r\}_{r \in C}$ are not all 0. Multiplying v by a nonzero scalar if necessary, we may assume that $|v_r| \leq 1$ for $r \in C$, and also $v_r = 1$ for some $r \in C$. Define $C_1 = \{r \in C | v_r = 1\}$.

We show that $C_1 = C$. Suppose $C_1 \subsetneq C$. Since C is connected, there exists $r \in C_1$ and $s \in C \backslash C_1$ such that $s \to r$. We examine the r-coordinate in $B_i^t v = k_i v$; this gives

$$k_i = \sum_{j=0}^{d} (B_i^t)_{r,j} v_j = \sum_{j=0}^{d} (B_i)_{j,r} v_j = \sum_{j=0}^{d} p_{i,j}^r v_j. \tag{36}$$

For $0 \le j \le d$ we have $p_{i,j}^r = 0$ if $j \notin C$ and $|v_j| \le 1$ if $j \in C$; therefore $p_{i,j}^r |v_j| \le p_{i,j}^r$. Consequently

$$k_i = \left| \sum_{j=0}^{d} p_{i,j}^r v_j \right| \le \sum_{j=0}^{d} p_{i,j}^r |v_j| \le \sum_{j=0}^{d} p_{i,j}^r = k_i. \tag{37}$$

Combining (36), (37) we obtain $v_j = 1$ for all $j \in C$ such that $j \to r$. This fails for $j = s$, so we have a contradiction. Therefore $C_1 = C$, so $v_r = 1$ for $r \in C$. In particular $v_r = v_s$ for $r, s \in C$.
(b) $\Rightarrow$ (a) This holds because B_i^t has constant row sum k_i.
We have shown that (a), (b) are equivalent. Consequently $\dim W = m$, and the result follows.
(ii) Similar to the proof of (i) above. □

Proposition 9.18. *The following are equivalent:*

(i) *there exists $i \in \{1, 2, \ldots, d\}$ such that Δ_{A_i} is disconnected;*

(ii) *there exists $j \in \{1, 2, \ldots, d\}$ such that Δ_{E_j} is disconnected.*

Proof. For $1 \le i, j \le d$ we have

$$\frac{P_i(j)}{k_i} = \frac{\overline{Q_j(i)}}{m_j}.$$

Therefore, $P_i(j) = k_i$ if and only if $Q_j(i) = m_j$. The result follows from this and Proposition 9.17. □

Proposition 9.19. *The following are equivalent:*

(i) *the association scheme $\mathcal{X}$ is primitive;*

(ii) *the distribution diagram Δ_{A_i} is connected for $1 \le i \le d$;*

(iii) *the representation diagram Δ_{E_j} is connected for $1 \le j \le d$.*

Proof. By Definition 9.4, Corollary 9.13, and Proposition 9.18. □

Problem 9.20. For a finite group G, show that the following are equivalent:

(i) the conjugacy–class association scheme for G is primitive;

(ii) G is simple.