Lecture 11

9 Primitive association schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

We will define a condition on $\mathcal{X}$ called primitivity. To motivate this condition, assume for the moment that $\mathcal{X}$ is the conjugacy-class association scheme for a finite group G. Then $\mathcal{X}$ is primitive if and only if G is simple.

Definition 9.1. For $1 \leq i \leq d$ we view the pair (X, R_i) as a directed graph with vertex set X; vertices x, y satisfy $x \to y$ whenever $(x, y) \in R_i$.

Definition 9.2. For an integer $\ell \geq 0$, a *path of length* ℓ in a directed graph is a sequence of vertices $\{x_i\}_{i=0}^{\ell}$ such that $x_{i-1} \to x_i$ for $1 \leq i \leq \ell$. This path goes *from* x_0 *to* x_ℓ. For example, there is a path of length zero from any vertex to itself. A directed graph is said to be *connected* whenever for all vertices x, y there is a path from x to y.

Lemma 9.3. Consider the graph (X, R_i) from Definition 9.1. For $x, y \in X$ and $\ell \in \mathbb{N}$ the following are equal:

(i) the number of paths of length ℓ from x to y;

(ii) the (x, y)-entry of A_i^ℓ.

Proof. Routine. □

Definition 9.4. The association scheme $\mathcal{X}$ is called *primitive* whenever the directed graph (X, R_i) is connected for $1 \leq i \leq d$. We say that $\mathcal{X}$ is *imprimitive* whenever $\mathcal{X}$ is not primitive.

As we investigate primitivity, the following notation will be useful. For a subset $\Omega \subseteq \{0, 1, \ldots, d\}$ define the relation

$$R_\Omega = \cup_{k \in \Omega} R_k. \tag{34}$$

We consider the case in which R_Ω is an equivalence relation. This happens if $\Omega = \{0\}$ or $\Omega = \{0, 1, \ldots, d\}$. We are going to show that $\mathcal{X}$ is imprimitive iff there exists $\{0\} \subsetneq \Omega \subsetneq \{0, 1, \ldots, d\}$ such that R_Ω is an equivalence relation.

Lemma 9.5. *We refer to the graph (X, R_i) from Definition 9.1. For $x \in X$ the set*

$$\Gamma^{(i)}(x) = \{y \in X \mid \text{there exists a path from } x \text{ to } y\}$$

is described as follows:

(i) *there exists a subset* $\Omega \subseteq \{0, 1, \ldots, d\}$ *such that*

$$\Gamma^{(i)}(x) = \cup_{k \in \Omega} \Gamma_k(x).$$

(ii) Ω *is the minimal subset of* $\{0, 1, \ldots, d\}$ *such that* (a) $0 \in \Omega$; (b) *for* $0 \leq j, k \leq d$, *if* $j \in \Omega$ *and* $p^k_{i,j} > 0$ *then* $k \in \Omega$.

(iii) Ω *is independent of* x.

(iv) $|\Gamma^{(i)}(x)|$ *is independent of* x.

Proof. (i) Observe that

$$\Gamma^{(i)}(x) = \{y \in X | \exists \ell \in \mathbb{N} \text{ such that } (A_i^\ell)_{x,y} > 0\}.$$

For $\ell \in \mathbb{N}$ the matrix A_i^ℓ is a linear combination of the associate matrices. The result follows.
(ii) By the definition of the intersection numbers.
(iii) By (ii) above.
(iv) By (i) and (iii) above. □

Corollary 9.6. *We refer to the graph* (X, R_i) *from Definition 9.1. For* $x, y \in X$ *the following are equivalent:*

(i) *there exists a path from* x *to* y;

(ii) *there exists a path from* y *to* x.

Proof. (i) ⇒ (ii) We have $y \in \Gamma^{(i)}(x)$. By construction $\Gamma^{(i)}(y) \subseteq \Gamma^{(i)}(x)$. In this inclusion, the two sets have the same size, so $\Gamma^{(i)}(x) = \Gamma^{(i)}(y)$. By this and since $x \in \Gamma^{(i)}(x)$ we see that $x \in \Gamma^{(i)}(y)$. Consequently there exists a path from y to x.
(ii) ⇒ (i) By symmetry. □

Corollary 9.7. *We refer to the graph* (X, R_i) *from Definition 9.1, and the corresponding set* Ω *from Lemma 9.5. Then* $k \in \Omega$ *implies* $k' \in \Omega$ *for* $0 \leq k \leq d$.

Proof. Assume that $k \in \Omega$. Pick $x, y \in X$ with $(x, y) \in R_k$. By assumption, there exists a path from x to y. So there exists a path from y to x. We have $(y, x) \in R_{k'}$. By these comments $k' \in \Omega$. □

Lemma 9.8. *We refer to the graph* (X, R_i) *from Definition 9.1, and the corresponding set* Ω *from Lemma 9.5.*

(i) *The relation* R_Ω *from* (34) *is an equivalence relation;*

(ii) *for* $x \in X$ *the set* $\Gamma^{(i)}(x)$ *is the equivalence class of* R_Ω *that contains* x.

Proof. By Lemma 9.5 and Corollary 9.7. □

We have been discussing the directed graph (X, R_i). We now introduce a directed graph with vertex set $\{0, 1, \ldots, d\}$.

Definition 9.9. For $1 \leq i \leq d$ we define a directed graph Δ_{A_i} with vertex set $\{0, 1, \ldots, d\}$; vertices j, k satisfy $j \to k$ whenever $p^k_{i,j} > 0$. Note that a vertex j of Δ_{A_i} has a loop $j \to j$ whenever $p^j_{i,j} > 0$. We call Δ_{A_i} the *A_i-distribution diagram for $\mathfrak{X}$.*

The graph Δ_{A_i} is related to the graph (X, R_i) as follows.

Lemma 9.10. *With the above notation, the following are equivalent for $0 \leq a, b \leq d$:*

(i) *there exists a path in Δ_{A_i} from a to b;*

(ii) *for all $(x, y) \in R_a$ there exists $z \in \Gamma_b(x)$ such that $z \in \Gamma^{(i)}(y)$;*

(iii) *for all $(x, z) \in R_b$ there exists $y \in \Gamma_a(x)$ such that $z \in \Gamma^{(i)}(y)$;*

(iv) *there exists $(x, y) \in R_a$ and there exists $z \in \Gamma_b(x)$ such that $z \in \Gamma^{(i)}(y)$.*

Proof. (i) $\Rightarrow$ (ii) Call the path $\{a_j\}_{j=0}^{\ell}$. We have $a_0 = a$ and $a_\ell = b$. There exists a path $\{y_j\}_{j=0}^{\ell}$ in (X, R_i) such that $y_0 = y$ and $y_j \in \Gamma_{a_j}(x)$ for $0 \leq j \leq \ell$. Define $z = y_\ell$. By construction $z \in \Gamma_b(x)$ and $z \in \Gamma^{(i)}(y)$.
(ii) $\Rightarrow$ (iv) Clear.
(i) $\Rightarrow$ (iii) Similar to the proof of (i) $\Rightarrow$ (ii).
(iii) $\Rightarrow$ (iv) Clear.
(iv) $\Rightarrow$ (i) There exists a path in (X, R_i) from y to z. Call the path $\{y_j\}_{j=0}^{\ell}$. We have $y_0 = y$ and $y_\ell = z$. For $0 \leq j \leq \ell$ define $a_j \in \{0, 1, \ldots, d\}$ such that $y_j \in \Gamma_{a_j}(x)$. Note that $a_0 = a$ and $a_\ell = b$. By construction the sequence $\{a_j\}_{j=0}^{\ell}$ is a path in Δ_{A_i} from a to b. □

Corollary 9.11. *We refer to the distribution diagram Δ_{A_i} from Definition 9.9. For $0 \leq a, b \leq d$ the following are equivalent:*

(i) *there exists a path from a to b;*

(ii) *there exists a path from b to a.*

Proof. By Corollary 9.6 and Lemma 9.10(i),(iv). □

Lemma 9.12. *We refer to the distribution diagram Δ_{A_i} from Definition 9.9. The following sets are equal:*

(i) *the connected component of Δ_{A_i} that contains 0;*

(ii) *the set Ω from Lemma 9.5.*

Proof. By Lemma 9.10 with $a = 0$. □

Corollary 9.13. *For $1 \leq i \leq d$ the following are equivalent:*

(i) *the graph Δ_{A_i} is connected;*

(ii) *the graph (X, R_i) is connected.*

Proof. By Lemma 9.12 and the construction. □