Proof. For $0 \le i, j \le d$ we have $E_i E_j = \delta_{i,j} E_i$. In this equation we apply Ψ to each side; this yields

$$\delta_{i,j} \Psi(E_i) = \Psi(E_i E_j) = \Psi(E_i) \circ \Psi(E_j).$$

By these comments, the sequence $\{\Psi(E_i)\}_{i=0}^d$ is a permutation of the sequence $\{A_i\}_{i=0}^d$. The result follows. □

Lecture 10

Lemma 7.5. *Assume that $\mathcal{X}$ has a duality Ψ such that $\Psi(E_i) = A_i$ for $0 \le i \le d$. Then* (i)–(iv) *hold below:*

(i) $\Psi(A_i) = |X| E_i^t \qquad (0 \le i \le d)$;

(ii) $i' = \hat{i} \qquad (0 \le i \le d)$;

(iii) $P = \overline{Q}$;

(iv) $p_{i,j}^k = q_{i,j}^k \qquad (0 \le i, j, k \le d)$.

Proof. (i) We have

$$|X| E_i^t = \Psi(\Psi(E_i)) = \Psi(A_i).$$

(ii) We have

$$A_{\hat{i}} = \Psi(E_{\hat{i}}) = \Psi(E_i^t) = \left(\Psi(E_i)\right)^t = A_i^t = A_{i'}.$$

(iii) For $0 \le i \le d$ we have $A_i = \sum_{j=0}^d P_i(j) E_j$. In this equation we apply Ψ to each side; this yields

$$|X| E_i^t = \sum_{j=0}^d P_i(j) A_j.$$

We may now argue

$$\sum_{j=0}^d P_i(j) A_j = |X| E_i^t = |X| \overline{E_i} = \sum_{j=0}^d \overline{Q_i(j)} A_j$$

Therefore $P_i(j) = \overline{Q_i(j)}$ for $0 \le i, j \le d$. Consequently $P = \overline{Q}$.
(iv) We have

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^d q_{i,j}^k E_k \qquad (0 \le i, j \le d).$$

In this equation, we apply Ψ to each side and evaluate the result; this yields

$$A_i A_j = \sum_{k=0}^{d} q_{i,j}^k A_k \qquad (0 \le i, j \le d).$$

Recall that

$$A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k \qquad (0 \le i, j \le d).$$

Therefore $p_{i,j}^k = q_{i,j}^k$ for $0 \le i, j, k \le d$. □

Next, we give a characterization of the self-dual condition.

Proposition 7.6. *The following are equivalent:*

(i) *$\mathcal{X}$ is self-dual;*

(ii) *there exists an ordering $\{R_i\}_{i=0}^d$ of the relations such that $P = \overline{Q}$.*

Proof. (i) ⇒ (ii) By Lemmas 7.4, 7.5.
(ii) ⇒ (i) Without loss, we may assume that $P = \overline{Q}$. The vector space $\mathcal{M}$ has a basis $\{A_i\}_{i=0}^d$ and a basis $\{E_i\}_{i=0}^d$. There exists a $\mathbb{C}$-linear bijection $\Psi : \mathcal{M} \to \mathcal{M}$ such that $\Psi(E_i) = A_i$ for $0 \le i \le d$. For $0 \le i \le d$ we have

$$A_i = \sum_{j=0}^{d} P_i(j) E_j$$

and also

$$|X| E_i^t = |X| \overline{E_i} = \sum_{j=0}^{d} \overline{Q_i(j)} A_j = \sum_{j=0}^{d} P_i(j) A_j.$$

Comparing the above two equations, we find that $\Psi(A_i) = |X| E_i^t$ for $0 \le i \le d$. By these comments Ψ satisfies the two conditions in Definition 7.1. Therefore Ψ is a duality, and $\mathcal{X}$ is self-dual. □

Problem 7.7. Let G denote a finite abelian group. Consider the conjugacy-class association scheme for G. Show that this association scheme has a duality.

Problem 7.8. Recall the Hamming association scheme $H(d, q)$. Let us take $d = 1$. The scheme $K_q = H(1, q)$ is called complete. Show that for K_q,

$$P = \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix} = Q.$$

In particular, K_q is self-dual.

Shortly, we will see that $H(d, q)$ is self-dual for all $d, q \ge 1$.

8 Fusion for commutative association schemes

Throughout this section, we assume that $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is a commutative association scheme with Bose-Mesner algebra $\mathcal{M}$, associate matrices $\{A_i\}_{i=0}^d$, and primitive idempotents $\{E_i\}_{i=0}^d$.

Our next goal is to construct new commutative association schemes by "fusing together" some of the relations $\{R_i\}_{i=0}^d$.

Definition 8.1. A *fusion scheme* for $\mathcal{X}$ is an association scheme $\tilde{\mathcal{X}} = (X, \{\tilde{R}_i\}_{i=0}^{\tilde{d}})$ such that

$$\tilde{R}_i = \cup_{j\in\Lambda_i} R_j \qquad (0 \le i \le \tilde{d}),$$

where $\{\Lambda_i\}_{i=0}^{\tilde{d}}$ is a partition of $\{0, 1, \ldots, d\}$ into nonempty subsets with $\Lambda_0 = \{0\}$.

By construction, a fusion scheme for $\mathcal{X}$ is commutative.

Let $\{\Lambda_i\}_{i=0}^{\tilde{d}}$ denote any partition of $\{0, 1, \ldots, d\}$ into nonempty subsets, such that $\Lambda_0 = \{0\}$. In general, the corresponding fusion scheme does not exist. However, in the following case it does exist.

Example 8.2. Define $\tilde{d} = 1$. Define $\Lambda_0 = \{0\}$ and $\Lambda_1 = \{1, 2, \ldots, d\}$. Then $\tilde{\mathcal{X}} = (X, \{\tilde{R}_i\}_{i=0}^{\tilde{d}})$ is the complete association scheme.

Let $\{\Lambda_i\}_{i=0}^{\tilde{d}}$ denote any partition of $\{0, 1, \ldots, d\}$ into nonempty subsets, such that $\Lambda_0 = \{0\}$. In the next result, we give necessary and sufficient conditions for the corresponding fusion scheme to exist. Recall the first eigenmatrix P for $\mathcal{X}$.

Proposition 8.3. *For the above partition $\{\Lambda_i\}_{i=0}^{\tilde{d}}$, the corresponding fusion scheme $\tilde{\mathcal{X}} = (X, \{\tilde{R}_i\}_{i=0}^{\tilde{d}})$ exists iff* (i), (ii) *hold below:*

(i) *the sequence $\{\tilde{R}_i^t\}_{i=0}^{\tilde{d}}$ is a permutation of the sequence $\{\tilde{R}_i\}_{i=0}^{\tilde{d}}$;*

(ii) *there exists a partition $\{F_i\}_{i=0}^{\tilde{d}}$ of $\{0, 1, \ldots, d\}$ into nonempty subsets such that $P|_{F_i\times\Lambda_j}$ has constant row sum for $0 \le i, j \le \tilde{d}$. By definition $P|_{F_i\times\Lambda_j}$ is the submatrix of P with row set F_i and column set Λ_j.*

Assume that (i), (ii) *hold. Then after permuting $\{F_i\}_{i=0}^{\tilde{d}}$ as necessary, we have:*

(a) *$F_0 = \{0\}$;*

(b) *the associate matrices of $\tilde{\mathcal{X}}$ are*

$$\tilde{A}_i = \sum_{j\in\Lambda_i} A_j \qquad (0 \le i \le \tilde{d}); \tag{31}$$

(c) *the primitive idempotents of $\tilde{\mathcal{X}}$ are*

$$\tilde{E}_i = \sum_{j\in F_i} E_j \qquad (0 \le i \le \tilde{d}); \tag{32}$$

(d) *for the first eigenmatrix* $\tilde{P}$ *of* $\tilde{\mathcal{X}}$ *and for* $0 \le i, j \le \tilde{d}$, *the submatrix* $P|_{F_i \times \Lambda_j}$ *has constant row sum* $\tilde{P}_j(i)$*;*

(e) *the sequence* $\{\tilde{A}_i^t\}_{i=0}^{\tilde{d}}$ *is a permutation of the sequence* $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$*;*

(f) *the sequence* $\{\tilde{E}_i^t\}_{i=0}^{\tilde{d}}$ *is a permutation of the sequence* $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$*.*

Proof. First assume that the association scheme $\tilde{\mathcal{X}}$ exists. Then (b) holds. The Bose-Mesner algebra $\tilde{\mathcal{M}}$ of $\tilde{\mathcal{X}}$ has basis $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$. The vector space $\tilde{\mathcal{M}}$ is closed under the transpose map, so (e) holds. Let $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ denote the primitive idempotents of $\tilde{\mathcal{X}}$. The matrices $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ form a basis for $\tilde{\mathcal{M}}$. For $0 \le i \le \tilde{d}$ the matrix $\tilde{E}_i$ is a linear combination of $\{E_\ell\}_{\ell=0}^{d}$. In this linear combination, the coefficients are all in $\{0, 1\}$ because $(\tilde{E}_i)^2 = \tilde{E}_i$. Therefore, there exists a subset $F_i \subseteq \{0, 1, \ldots, d\}$ such that $\tilde{E}_i = \sum_{j \in F_i} E_j$. Recall that $I = \sum_{\ell=0}^{d} E_\ell$, and $E_r E_s = \delta_{r,s} E_r$ for $0 \le r, s \le d$. Consequently the subsets $\{F_i\}_{i=0}^{\tilde{d}}$ partition $\{0, 1, \ldots, d\}$. This gives (c). Let $\tilde{P}$ denote the first eigenmatrix of $\tilde{\mathcal{X}}$. By construction

$$\tilde{A}_j = \sum_{i=0}^{\tilde{d}} \tilde{P}_j(i) \tilde{E}_i \qquad (0 \le j \le \tilde{d}). \tag{33}$$

In this equation, we evaluate the left-hand side using (31) and the right-hand side using (32). The result shows that the submatrix $P|_{F_i \times \Lambda_j}$ has constant row sum $\tilde{P}_j(i)$ for $0 \le i, j \le \tilde{d}$. This gives (d). Note that $J \in \tilde{\mathcal{M}}$, since $J = \sum_{i=0}^{d} A_i = \sum_{i=0}^{\tilde{d}} \tilde{A}_i$. This gives (a). Item (f) holds because $\tilde{\mathcal{M}}$ is closed under the transpose map and $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ is a basis for $\tilde{\mathcal{M}}$. Items (i) and (ii) are implied by (e) and (d), respectively.

We are done in one logical direction. Next we reverse the logical direction. Assume that (i), (ii) hold. We show that the association scheme $\tilde{\mathcal{X}}$ exists. For $0 \le i \le \tilde{d}$ define $\tilde{A}_i$ as in (31), and note that $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$ are linearly independent. For $0 \le i \le \tilde{d}$ define $\tilde{E}_i$ as in (32), and note that $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ are linearly independent. For $0 \le i, j \le \tilde{d}$ let $\tilde{P}_j(i)$ denote the common row sum for $P|_{F_i \times \Lambda_j}$. Then (33) holds. Consequently $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$ and $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ are both bases for the same vector space $\tilde{\mathcal{M}}$. We show that $\tilde{\mathcal{M}}$ is a Bose-Mesner algebra. To do this, we show that $\tilde{\mathcal{M}}$ satisfies the conditions (i)–(v) in Proposition 2.4.

- We have $I = \tilde{A}_0 \in \tilde{\mathcal{M}}$. Also $J \in \tilde{\mathcal{M}}$ since $J = \sum_{i=0}^{d} A_i = \sum_{i=0}^{\tilde{d}} \tilde{A}_i$.
- $\tilde{\mathcal{M}}$ is closed under matrix multiplication, because the basis $\{\tilde{E}_i\}_{i=0}^{\tilde{d}}$ satisfies $\tilde{E}_i \tilde{E}_j = \delta_{i,j} \tilde{E}_i$ for $0 \le i, j \le \tilde{d}$.
- $\tilde{\mathcal{M}}$ is closed under Hadamard multiplication, because the basis $\{\tilde{A}_i\}_{i=0}^{\tilde{d}}$ satisfies $\tilde{A}_i \circ \tilde{A}_j = \delta_{i,j} \tilde{A}_i$ for $0 \le i, j \le \tilde{d}$.
- $\tilde{\mathcal{M}}$ is closed under the transpose map by assumption (i).
- $\tilde{\mathcal{M}}$ is homogeneous, because $\tilde{\mathcal{M}} \subseteq \mathcal{M}$ and $\mathcal{M}$ is homogeneous.

We have shown that $\tilde{\mathcal{M}}$ is a Bose-Mesner algebra, and consequently the association scheme $\tilde{\mathcal{X}}$ exists. $\square$