

Math 846

Lecture 9

We continue to discuss a DRG $\Gamma = (X, E)$ with $\text{diam } D \geq 1$.

Fix $x \in X$ and write $T = T(x)$.

Let W denote an irred T -module.

Define

$$\underset{\text{"r"}}{\text{endpoint}}(W) = \min \{ i \mid 0 \leq i \leq D, E_i^* W \neq 0 \}$$

$$\underset{\text{"d"}}{\text{diameter}}(W) = \left| \{ i \mid 0 \leq i \leq D, E_i^* W \neq 0 \} \right| - 1$$

Then

$$\bullet \quad 0 \leq r \leq r + d \leq D$$

$$\bullet \quad \text{For } 0 \leq i \leq D$$

$$E_i^* W \neq 0 \text{ iff } r \leq i \leq r + d$$

Next assume the ordering $\{E_i\}_{i=0}^D$ is Q -polynomial.

$$W = \sum_{i=0}^D E_i W = \overset{E_i V \cap W}{\underset{\text{might be 0}}{E_i W}} \quad (\text{orthog dir sum})$$

$$\text{For } 0 \leq i \leq D,$$

$$A^* E_i W \subseteq E_{i-1} W + E_i W + E_{i+1} W$$

For $0 \leq i < j \leq D$ suppose

$$E_i W \neq 0 \text{ and } E_j W \neq 0$$

then $E_l W \neq 0$ for $i \leq l \leq j$

Define $\overset{t}{\text{dual endpt}}(W) = \min\{i / 0 \leq i \leq D, E_i W \neq 0\}$
 $\overset{\delta}{\text{dual diameter}}(W) = |\{i / 0 \leq i \leq D, E_i W \neq 0\}| - 1$

then

$$0 \leq t \leq t + \delta \leq D$$

$$\text{For } 0 \leq i \leq D$$

$$E_i W \neq 0 \text{ iff } t \leq i \leq t + \delta$$

until further notice

$$\Gamma = H(D, 2)$$

Fix $x \in X$ and write $T = T(x)$

Thm 62. Let W denote an irred T -module with end r and diameter d . Then:

(i) $0 \leq r \leq d/2$

(ii) $d = D - 2r$

(iii) W has a basis $\{w_i\}_{i=0}^d$ st

$$w_i \in E_{r+i}^* W \quad (0 \leq i \leq d)$$

$$Rw_i = (i+1)w_{i+1} \quad (0 \leq i \leq d-1), \quad Rw_d = 0$$

$$Lw_i = (d-i+1)w_{i-1} \quad (1 \leq i \leq d), \quad Lw_0 = 0$$

p f Pick $0 \neq w \in E_r^* W$

So $Lw = 0$

define $w_i = \frac{R^i w}{i!} \quad i \in \mathbb{N}$

So $Rw_i = (i+1)w_{i+1} \quad i \in \mathbb{N}$

Obs

$$w_i \in E_{r+i}^* W \quad 0 \leq i \leq d-r$$

$$w_i = 0 \quad i > d-r$$

so $A^* w_i = (d-2r-2i) w_{i+1} \quad i \in \mathbb{N}$

claim $L w_i = (d-2r-i) w_{i+1} \quad i \geq 1$

pf of By induction on i

case $i=1$:

$$\begin{aligned} 0 &= (LR - RL - A^*) w_0 \\ &= \underbrace{LR w_0}_{L w_1} - \underbrace{RL w_0}_0 - \underbrace{A^* w_0}_{(d-2r) w_0} \end{aligned}$$

so $L w_1 = (d-2r) w_0 \quad \checkmark$

Case $i \geq 1$:

$$\begin{aligned} 0 &= (LR - RL - A^*) w_{i+1} \\ &= i L w_i - (i+1)(d-2r-i+2) w_{i+1} - (d-2r-2i+2) w_{i+1} \end{aligned}$$

$$iLw_i = i(d-2r-i+1)w_i$$

$$\text{so } Lw_i = (d-2r-i+1)w_i$$

claim proved $\checkmark$

obs $\text{Span}\{w_i\}_{i \in \mathbb{N}}$ is invar under R, L, A^*

so $\text{Span}\{w_i\}_{i \in \mathbb{N}}$ is a T -module

By irred

$$W = \text{Span}\{w_i\}_{i \in \mathbb{N}}$$

$\forall i \in \mathbb{N}$

$$w_i = 0 \rightarrow w_{i+1} = 0$$

so

$$w_i \neq 0 \quad 0 \leq i \leq d$$

$$w_i = 0 \quad i \geq d+1$$

so

$\{w_i\}_{i=0}^d$ is basis for W

By claim with $i = d+1$

$$0 = (d-2r-d)w_d \neq 0$$

$$\text{so } d = d-2r, \quad d \geq 0 \text{ so } r \leq d/2$$

□

DEF 63 Let W denote an irred T -module.

By a standard basis for W we mean a basis for W from Th 59 (iii)

Note The standard basis for W is unique up to mult of each basis vector by the same non-zero scalar

LEM 64 Let W denote an irred T -module with standard basis $\{w_i\}_{i=0}^d$.

then $\{w_i\}_{i=0}^d$ are mutually orthogonal and

$$\|w_i\|^2 = \binom{d}{i} \|w_0\|^2 \quad (0 \leq i \leq d) \quad (*)$$

pf First assertion holds since the $\{E_j^* V\}_{j=0}^0$ are mutually orthogonal

check (*):

$$\langle R w_{j-1}, w_j \rangle = \langle w_{j-1}, \bar{R}^t w_j \rangle$$

"

$$j \langle w_{j-1}, w_j \rangle$$

$$\langle w_{j-1}, L w_j \rangle$$

"

"

$$j \|w_j\|^2$$

$$(d-j+1) \langle w_{j-1}, w_j \rangle$$

"

$$(d-j+1) \|w_{j-1}\|^2$$

(*) follows from this and induction on i

□

LEM 65 Let W denote an irred T -module with diameter d . Then

$$(i) \dim W = d+1$$

$$(ii) d-d \text{ is even}$$

p f By Th 62

□

LEM 66 Let W denote an irred. T -module
 with dimension d . With respect to a standard
 basis for W the matrices representing A, A^* are

$$A = \begin{pmatrix} 0 & d & & & 0 \\ 1 & 0 & d-1 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & \\ 0 & & & & 0 & 1 \\ & & & & d & 0 \end{pmatrix}$$

$$A^* = \text{diag}(d, d-2, d-4, \dots, -d)$$

pf By Th 62 and $A = R + L$

□

Th 67 Let W denote an unred T -module with endpt r and diam d (Recall $d = D - 2r$).

(i) the dual endpt of W is r

(ii) the dual diameter of W is d .

(iii) W has a basis $\{w_i^*\}_{i=0}^d$ s.t.

$$w_i^* \in E_{r+i} W \quad (0 \leq i \leq d)$$

$$R^* w_i^* = (i+1) w_{i+1}^* \quad (0 \leq i \leq d-1), \quad R^* w_d^* = 0$$

$$L^* w_i^* = (d-i+1) w_{i-1}^* \quad (1 \leq i \leq d), \quad L^* w_0^* = 0$$

p.f. Mimic the p.f. of Th 62, Lem 65

after swapping

$$R \leftrightarrow R^*$$

$$L \leftrightarrow L^*$$

$$A \leftrightarrow A^*$$

□

DEF 68 let W denote an irred. T -module.

By a dual standard basis for W we mean
a basis for W from Thm 67 (iii)

Note The dual standard basis for W is
unique up to mult of each basis vector by the
same non zero scalar.

LEM 69 let W denote an irred T -module
with dual standard basis $\{w_i^*\}_{i=0}^d$

Then $\{w_i^*\}_{i=0}^d$ are mutually orthogonal and

$$\|w_i^*\|^2 = \binom{d}{i} \|w_0^*\|^2 \quad (0 \leq i \leq d)$$

pf Similar to Lem 64

LEM 70 let W denote an irred T -module
 with dimension d . With respect to a dual
 standard basis for W the matrices representing
 A and A^* are

$$A: \text{diag}(d, d-2, d-4, \dots, -d)$$

$$A^*: \begin{pmatrix} \circ & d & & & \circ \\ 1 & \circ & d-2 & & \\ & 2 & \circ & \ddots & \\ \circ & & & \ddots & \circ \\ & & & & \circ & 1 \\ & & & & d & \circ \end{pmatrix}$$

pf By thm 67 and $A^* = R^* + L^*$

□

LEM 71 let W, W' denote irred T -modules.

Then the following are equivalent (TRUE):

(i) W, W' are isomorphic

(ii) W, W' have same endpoint

(iii) W, W' have same diameter

(iv) W, W' have same dimension

pf use thm 62

□

For $0 \leq r \leq D/2$ we abbreviate

$\text{mult}(r)$

for the multiplicity of an irred T -module with endpoint r .

LEM 72 For $0 \leq r \leq d/2$

$$(p_r) = \sum_{i=0}^r \text{mult}(i)$$

pf write

$$V = \sum_W W \quad (\text{orthog dir sum of mod } T\text{-modules}) \quad *$$

Apply E_r^*

$$E_r^* V = \sum_W E_r^* W$$

$$\uparrow \begin{cases} \dim 1 & \text{if } \text{endpt } W \leq r \\ 0 & \text{if } \text{endpt } W > r \end{cases}$$

$$\begin{aligned} (p_r) = \dim E_r^* V &= \# \text{ summands in } * \text{ with endpt } \leq r \\ &= \sum_{i=0}^r \text{mult}(i) \end{aligned} \quad \square$$

cor 73 We have

$$(i) \text{mult}(0) = 1$$

$$(ii) \text{Fn } 1 \leq r \leq 0/2$$

$$\text{mult}(r) = \binom{0}{r} - \binom{0}{r-1}$$

pt By lem 72

□