

Math 846

Lecture 8

We continue to discuss the DRG $\Gamma = H(0, 2)$ $n \geq 1$.

Fix $x \in X$ and write $T = T(x)$

Recall $A = \sum_{i=0}^p \theta_i E_i$ $A^* = \sum_{i=0}^p \theta_i^* E_i^*$

$$\theta_i^* = \theta_i^* = 1 - 2i \quad (0 \leq i \leq p)$$

LEM 53 For $0 \leq i, j \leq p$

$$E_i A^* E_j = 0 \quad \text{if } |i-j| \neq 1$$

pf Assume $|i-j| \neq 1$

$$0 = A^2 A^* - 2 A A^* A + A^* A^2 - 4 A^*$$

Multiply each term on left by E_i and right by E_j

$$0 = E_i A^* E_j \left(\underbrace{\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4}_{\substack{= \\ (\theta_i - \theta_j)^2 - 4 \\ = \\ (\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) \\ = \\ 4(1-i-1)(1-i+1) \\ \neq 0 \quad \neq 0}} \right)$$

So $E_i A^* E_j = 0$

□

By Lem 53

$$A^* E_i V \subseteq E_{i+1} V + E_{i-1} V \quad (1 \leq i \leq D-1)$$

$$A^* E_0 V \subseteq E_1 V$$

$$A^* E_D V \subseteq E_{D-1} V$$

Next goal: introduce

dual raising matrix R^*

dual lowering matrix L^*

DEF 54 Define $R^* = R^*(x) \in \text{Mat}_X(\mathbb{F})$ by

$$R^* = \sum_{i=0}^{D-1} E_{i+1} A^* E_i$$

Define $L^* = L^*(x) \in \text{Mat}_X(\mathbb{F})$ by

$$L^* = \sum_{i=1}^D E_{i-1} A^* E_i$$

By construction

$$R^* E_i V \subseteq E_i V \quad (0 \leq i \leq n-1)$$

$$R^* E_0 V = 0$$

$$L^* E_i V \subseteq E_i V \quad (1 \leq i \leq n)$$

$$L^* E_0 V = 0$$

$$\overline{R^*} = R^*, \quad \overline{L^*} = L^*, \quad (R^*)^G = L^*$$

LEM 55

$$A^* = R^* + L^*$$

pf

$$A^* = I A^* I$$

$$= \left(\sum_{i=0}^p E_i \right) A^* \left(\sum_{j=0}^p E_j \right)$$

$$= \sum_{\substack{0 \leq i, j \leq p \\ |i-j|=1}} E_i A^* E_j$$

$$= R^* + L^*$$

□

LEM 56 We have

$$(i) \quad L^* = \frac{AA^* - A^*A + 2A^*}{4}$$

$$(ii) \quad R^* = \frac{A^*A - AA^* + 2A^*}{4}$$

pt (i)

$$\frac{AA^* - A^*A + 2A^*}{4} = I \frac{AA^* - A^*A + 2A^*}{4} I$$

$$= \sum_{i=0}^D \sum_{j=0}^D E_i \frac{AA^* - A^*A + 2A^*}{4} E_j$$

$$= \sum_{0 \leq i, j \leq D} \underbrace{E_i A^* E_j}_{\parallel} \underbrace{\frac{\theta_i - \theta_j + 2}{4}}_{\parallel}$$

0 if $|i-j| \neq 1$ $\frac{j-i+1}{2} = \begin{cases} 0 & \text{if } j=i \\ 1 & \text{if } j=i+1 \end{cases}$

$$= \sum_{j=1}^D E_{j-1} A^* E_j$$

$$= L^*$$

(ii) Sim.

□

LEM 57

We have

$$(i) \quad AL^* - L^*A = 2L^*$$

$$(ii) \quad AR^* - R^*A = -2R^*$$

$$(iii) \quad L^*R^* - R^*L^* = A$$

pf (i) Elim L^* using Lem 56 (i) and compare with Thm 52

$$(ii) \quad \dots R^* \dots \text{Lem 56 (ii)} \dots$$

$$(iii) \quad L^*R^* - R^*L^* = (L^* + R^*)R^* - R^*(L^* + R^*)$$

$$= A^*R^* - R^*A^*$$

$$= A^* \frac{A^*A - AA^* + 2A^*}{4} - \frac{A^*A - AA^* + 2A^*}{4} A^*$$

$$= \frac{A^{*2}A - 2A^*AA^* + AA^{*2}}{4}$$

$$= \frac{4A}{4} = A$$

□

Next we consider the representation theory of T .
 Until further notice, Γ is any DRG with $D \geq 1$.

- A T -module is a subspace $W \subseteq V$ s.t.
 $TW \subseteq W$.

- obs if W, W' are T -modules then so is $W \cap W'$

- For any subspace $W \subseteq V$ define

$$W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W\}$$

"orthogonal complement of W in V "

so $V = W + W^\perp$ (orthog direct sum)

- If W is a T -module then $W^\perp$ is a T -module

check: $\forall B \in T$ show $BW^\perp \subseteq W^\perp$:

$$\forall w \in W \ \forall u \in W^\perp$$

$$\langle w, Bu \rangle = \langle \underbrace{Bw}_{\in W}, \underbrace{u}_{\in W^\perp} \rangle = 0 \quad \text{ok.}$$

- A T -module W is irreducible whenever
 $W \neq 0$ and $\nexists$ T -module U st
 $U \subseteq W$ and $U \neq 0$ and $U \neq W$.

LEM 58 A nmo T -module W is an
 orthogonal direct sum of irred T -modules.

pf By induction on $\dim W$
 W contains an irred T -module U .

Assume $U \neq W$ else done.

$$W = U + W \cap U^\perp \quad (\text{orth dir sum})$$

$\uparrow$
 T -module

$$\dim(W \cap U^\perp) < \dim W$$

By ind $W \cap U^\perp$ is orth dir sum of irred T -modules

Result follows

□

DEF 59 Let W, W' denote fixed T -modules. By an isomorphism of T -modules from W to W' we mean an $\mathbb{F}$ -linear

bijection $\sigma: W \rightarrow W'$ s.t.

$$\forall B \in T,$$

$$\sigma B = B\sigma \quad \text{on } W$$

We say W, W' are isomorphic whenever

$\exists$ isomorphism of T -modules from W to W'

LEM 60 Let U, W denote irred T -modules
that are not isomorphic. Then

$$\langle U, W \rangle = 0$$

pf We have

$$V = W + W^\perp \quad (\text{orthog dir sum of irred } T\text{-modules})$$

$$\exists \phi \in \text{Mat}_X(F) \text{ s.t.}$$

$$\phi = \text{id on } W$$

$$\phi = 0 \text{ on } W^\perp$$

so

$$W = \phi V$$

$$W^\perp = \ker(\phi)$$

$$\forall B \in T$$

$$B\phi = \phi B$$

consider the restriction

$$\phi|_U : U \rightarrow W$$

$\phi U =$ T -module contained in W

so

$$\phi U = 0 \quad \text{or} \quad \phi U = W$$

Also,

$\ker(\phi|_U) =$ T -module contained in U

so

$$\ker(\phi|_U) = 0 \quad \text{or} \quad \ker(\phi|_U) = U$$

so either

$$\phi U = 0$$

or

$$\phi|_U : U \rightarrow W \quad \text{is big} \quad \left(\begin{array}{l} \text{not allowed} \\ \text{since } U, W \text{ not iso} \end{array} \right)$$

so

$$\phi U = 0$$

so

$$U \subseteq \ker(\phi) = W^\perp$$

□

Def 61 let W denote an irred. T -module.

Consider

$$\text{Span} \{ W' \mid W' \text{ is irred } T\text{-module isomorphic to } W \} \quad (\star)$$

Obs $(\star)$ is T -module

Decompose $(\star)$ into an alg direct sum of irred T -modules (all of which are iso W by Lem 60)

let $\text{mult}(W)$ denote the number of summands in the above sum (this number is indep of decomp)

Call $\text{mult}(W)$ the multiplicity of W in V

We seek to describe a given irred T -module W

Obs

$$W = \sum_{i=0}^D E_i^* W \quad \text{("orthog dir sum")}$$

$\nwarrow$
might be 0

$= E_i^* V \cap W$

We seek to know which summands are 0

For $0 \leq i \leq D$

$$A E_i^* W \subseteq E_{i-1}^* W + E_i^* W + E_{i+1}^* W$$

For $0 \leq i < j \leq D$ suppose

$$E_i^* W \neq 0 \quad E_j^* W \neq 0$$

then for $i \leq l \leq j$

$$E_l^* W \neq 0$$

else

$$E_0^* W + E_1^* W + \dots + E_l^* W$$

(*)

is A -invar (hence M -invar)

(*) is M^* -invar by constr.

(*) is T -module

(*) is non 0 since it contains $E_i^* W \neq 0$

(*) $\neq W$ since (*) does not contain $E_j^* W \neq 0$
contradicts W is irreducible