

Math 846

Lecture 7

We continue to consider the distance-regular

graph $\Gamma = (X, E)$ with Q -polynomial

ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents. (021)

Fix $x \in X$ and write $T = T(x)$

LEM 43. For $0 \leq i \leq D$

$$(i) \quad A_i x^\wedge = E_i^* \mathbb{1}$$

$$\left(\mathbb{1} = \sum_{y \in X} \hat{y} \right)$$

$$(ii) \quad A_i^* \mathbb{1} = |X| E_i x^\wedge$$

pf (i) Both sides equal

$$\sum_{y \in \Gamma_i(x)} \hat{y}$$

(ii) For $y \in X$ compare y -coord of LHS, RHS

$$\text{LHS: } y\text{-coord} = (A_i^* / y) = |X| (E_i / y)$$

$$\text{RHS: } y\text{-coord} = |X| (E_i / y) = |X| (E_i / y)$$

□

Note For $B \in \text{Mat}_X(\mathbb{F})$ and $u, v \in V$

$$\langle Bu, v \rangle = \langle u, B^t v \rangle$$

Prop 44

For $0 \leq i, j \leq p$

$$u_i(\theta_j) = u_i^*(\theta_i^*)$$

"Askey-Wilson
duality"

pf We evaluate

$$\langle A_i \hat{x}, A_j^* \mathbb{1} \rangle$$

in 2 ways.

View I:

$$\langle A_i \hat{x}, A_j^* \mathbb{1} \rangle$$

$$= \langle v_i(A) \hat{x}, |X| E_j \hat{x} \rangle$$

↳

$$= |X| \langle \hat{x}, v_i(A) E_j \hat{x} \rangle$$

$$= |X| v_i(\theta_j) \underbrace{\langle \hat{x}, E_j \hat{x} \rangle}_{\parallel}$$

$$(E_j)_{x,x} = |X|^{-1} m_j$$

$$= m_j v_i(\theta_j)$$

Now II:

$$\langle A_i^* x, A_j^* \mathbb{1} \rangle$$

$$= \langle E_i^* \mathbb{1}, V_j^*(A^*) \mathbb{1} \rangle$$

$\hookleftarrow$

$$= \langle V_j^*(A^*) E_i^* \mathbb{1}, \mathbb{1} \rangle$$

$$= V_j^*(\theta_i^*) \underbrace{\langle E_i^* \mathbb{1}, \mathbb{1} \rangle}_{\text{" sum of all entries in } E_i^* \text{"}}$$

" k_i

$$= k_i V_j^*(\theta_i^*)$$

So far,

$$m_j v_i(\theta_j) = k_i V_j^*(\theta_i^*)$$

$$\text{So } u_i(\theta_j) = \frac{v_i(\theta_j)}{k_i} = \frac{V_j^*(\theta_i^*)}{m_j} = u_j^*(\theta_i^*)$$

□

Comments

We have

$$V = \sum_{i=0}^p E_i^* V \quad (\text{orth dir sum})$$

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By triple product relation (TPR),

$$E_i^* A E_j^* = 0 \quad \text{if } |i-j| > 1 \quad (0 \leq i, j \leq p)$$

So for $0 \leq i \leq p$

$$A E_i^* V \leq E_{i-1}^* V + E_i^* V + E_{i+1}^* V$$

where $E_{-1}^* = 0$ and $E_{p+1}^* = 0$

By TPR and since we assume Q poly

$$E_i A^* E_j = 0 \quad \text{if } |i-j| > 1 \quad (0 \leq i, j \leq p)$$

So for $0 \leq i \leq p$

$$A^* E_i V \leq E_{i-1} V + E_i V + E_{i+1} V$$

where $E_{-1} = 0$ and $E_{p+1} = 0$

These comments show that A and A^*
act on each others eigenspaces in a
"block-tridiagonal" fashion.

We return our attention to $\Gamma \in H(0, 2)$

Shortly we will show that the ordering $\{E_i\}_{i=0}^p$

is $\mathbb{Q}$ -polynomial. For now, define

$$\theta_i^* = m_i u_i(\theta_i) \quad (0 \leq i \leq p)$$

$$A^* = \sum_{i=0}^p \theta_i^* E_i^*$$

LEM 45 For $\Gamma \in H(0, 2)$

$$\theta_i^* = p - 2i \quad (0 \leq i \leq p)$$

pf By construction

$$\theta_0^* = m_1 = \begin{pmatrix} p \\ 1 \end{pmatrix} = p.$$

$$\theta_1^* = m_1 u_1(\theta_1) = m_1 \frac{\theta_1}{h} = \theta_1 = p - 2$$

For $1 \leq i \leq p-1$

$$\begin{aligned} \theta_i^* \theta_i^* &= c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* \\ &= i \theta_{i-1}^* + 0 + (p-i) \theta_{i+1}^* \end{aligned}$$

The sequence $\{p - 2i\}_{i=0}^p$ satisfies the same rec and int. conditions. Result follows. $\square$

Ex. 46

$$F_a \quad \Gamma = H(qz)$$

$$\bullet \quad F_n \quad 0 \leq i, j \leq n$$

$$u_i(\theta_1) = \sum_{n=0}^{\infty} \frac{(-i)_n (-j)_n}{(-n)_n n!} z^n \quad *$$

where

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1) \quad n \in \mathbb{N}$$

So

$$u_i(\theta_1) = u_j(\theta_1)$$

Note the series * is called

$${}_2F_1 \left(\begin{matrix} -i & -j \\ -n \end{matrix} \middle| z \right)$$

" hypergeometric series "

$$\bullet \quad F_n \quad 0 \leq i, j \leq n$$

$$v_i(\theta_1) = \begin{pmatrix} 0 \\ i \end{pmatrix} u_i(\theta_1)$$

Ex 47 For $\Gamma = H(0, 2)$

$$q_{ij}^h = p_{ij}^h \quad (0 \leq h, i, j \leq 0)$$

In particular

$$c_i^x = c_i = i' \quad 0 \leq i \leq 0$$

$$b_i^x = b_i = 0 - i'$$

$$a_i^x = a_i = 0$$

$$, \quad \text{For } 0 \leq i \leq 0,$$

$$u_i^x = u_i, \quad v_i^x = v_i$$

Ex 48 For $\Gamma = H(0, 2)$ we have the

generating function

$$\sum_{i=0}^0 v_i(\theta) \lambda^{i'} = (1-\lambda)^2 (1+\lambda)^{0-2} \quad (0 \leq i \leq 0)$$

Until further notice assume

$$\Gamma = H(0, 2)$$

$$0 \leq 1$$

Fix $x \in X$ and write $T = T(x)$

Recall

$$A = \sum_{i=0}^p \theta_i E_i$$

$$\theta_i = 0 - z_i$$

$$A^* = \sum_{i=0}^p \theta_i^* E_i^*$$

$$\theta_i^* = 0 - z_i^*$$

Note that A^* generates M^* since $\{\theta_i^*\}_{i=0}^p$
are mutually distinct.

So T is generated by A, A^*

Next goal: describe how A, A^* are related.

Def 49 Define $R = R(x) \in \text{Mat}_X(\mathbb{F})$ by

$$R = \sum_{i=0}^{n-1} E_{i,n}^* A E_i^* \quad \text{"raising matrix"}$$

Define $L = L(x) \in \text{Mat}_X(\mathbb{F})$ by

$$L = \sum_{i=1}^D E_{i,n}^* A E_i^* \quad \text{"lowering matrix"}$$

By construction, $\int_X \gamma \in X$

$$R_y^{\wedge} = \sum_{z \in \Gamma(y)} \hat{z}$$

$$\partial(x, z) = \partial(x, y) + 1$$

$$L_y^{\wedge} = \sum_{z \in \Gamma(y)} \hat{z}$$

$$\partial(x, z) = \partial(x, y) - 1$$

Obs

$$R E_i^* V \leq E_{i,n}^* V \quad (0 \leq i \leq n-1)$$

$$R E_0^* V = 0$$

$$L E_i^* V \subseteq E_{i-1}^* V \quad (1 \leq i \leq n)$$

$$L E_0^* V = 0$$

obs

$$\bar{R} = R, \quad \bar{L} = L, \quad R^t = L$$

$$A = R + L$$

LEM 50 We have

$$(i) \quad A^* L - L A^* = 2L$$

$$(ii) \quad A^* R - R A^* = -2R$$

$$(iii) \quad LR - RL = A^*$$

pf Pick $y \in X$ and apply each side to $\hat{y}$.
Write $i = \partial(x, y)$

$$(i) \quad L \hat{y} \in E_{i-1}^* V \text{ so } A^* L \hat{y} = \theta_{i-1}^* L \hat{y}$$

$$\hat{y} \in E_i^* V \text{ so } A^* \hat{y} = \theta_i^* \hat{y}. \text{ So } L A^* \hat{y} = \theta_i^* L \hat{y}$$

$$\text{Require } \theta_{i-1}^* - \theta_i^* = 2 \quad \text{OK}$$

" " " "

$i-2 \leq i \leq i$ $i-2 \leq i$

(ii), (iii) Similar

□

LEM 51 We have

$$(i) \quad L = \frac{A^*A - AA^* + 2A}{4}$$

$$(ii) \quad R = \frac{AA^* - A^*A + 2A}{4}$$

pf We have

$$A = L + R \quad (*)$$

$$\begin{aligned} A^*A - AA^* &= A^*L - LA^* + A^*R - RA^* \\ &= 2L - 2R \end{aligned} \quad (**)$$

Solve $(*)$, $(**)$ for L, R . □

Thm 52 We have

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^*$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4A$$

pf Elim L, R in Lem 50 using Lem 51. □