

Math 846

Lecture 6

We continue to discuss a distance-regular graph $\Gamma = (X, E)$ with diameter $D \geq 1$.

Fix $x \in X$ and write $T = T(x)$

[Finish pt of Prop 35]

$E_X 36$ For our DRG $\Gamma = (X, E)$

$$\sum_{l=0}^D u_l(e_i) u_l(e_j) k_l = \delta_{ij} m_i^{-1} |X| \quad (0 \leq i, j \leq D)$$

$$\sum_{l=0}^D u_l(e_l) u_l(e_l) m_l = \delta_{ii} k_i^{-1} |X| \quad (0 \leq i, j \leq D)$$

"orthogonality relations of 1st and 2nd kind"

Ex 37 For a BRG $P = (X, E)$

$$p_{ij}^h = |X|^{-1} k_i k_j \sum_{\ell=0}^h u_i(e_\ell) u_j(e_\ell) u_h(e_\ell) m_\ell$$

$$q_{ij}^h = |X|^{-1} m_i m_j \sum_{\ell=0}^h u_\ell(e_i) u_\ell(e_j) u_\ell(e_h) k_\ell$$

Let $0 \leq h, i, j \leq n$. By Prop 35

$$q_{ij}^h \geq 0$$

"Krein condition"

Also

$$E_h^* A_i E_j^* = 0 \quad \text{iff} \quad p_{ij}^h = 0$$

$$E_h A_i^* E_j = 0 \quad \text{iff} \quad q_{ij}^h = 0$$

"triple product relations"

— 0 —

Up to now we assumed the eigenvalues of P satisfy

$$k = \theta_0 > \theta_1 > \dots > \theta_n$$

This ordering was just for notational convenience.

Going forward, we allow an arbitrary ordering

$$\{\theta_i\}_{i=0}^n \quad \text{such that} \quad \theta_0 = k,$$

DEF 38 The given ordering $\{E_i\}_{i=0}^p$ of the primitive idempotents is called $\mathbb{Q}$ -polynomial whenever for $0 \leq h, i, j \leq p$

• $q_{ij}^h = 0$ if one of h, i, j is greater than the sum of the other two

• $q_{ij}^h \neq 0$ if one of h, i, j is equal to the sum of the other two.

Until further notice, assume Γ is $\mathbb{Q}$ -poly wrt $\{E_i\}_{i=0}^p$

Write

$$c_i^* = q_{1, i-1}^i \quad (1 \leq i \leq p), \quad c_0^* = 0$$

$$a_i^* = q_{1, i}^i \quad (0 \leq i \leq p)$$

$$b_i^* = q_{1, i+1}^i \quad (0 \leq i \leq p-1), \quad b_p^* = 0$$

Obs $c_i^* \neq 0 \quad (1 \leq i \leq p)$

$b_i^* \neq 0 \quad (0 \leq i \leq p-1)$

Ex 39 We have

$$c_i^* + a_i^* + b_i^* = m_i \quad (0 \leq i \leq n)$$

$$a_0^* = 0,$$

$$c_1^* = 1$$

$$m_i = q_{ii}^0 \quad (0 \leq i \leq n)$$

$$m_i = \frac{b_0^* b_1^* \cdots b_{i-1}^*}{c_1^* c_2^* \cdots c_i^*} \quad (0 \leq i \leq n)$$

— 0 —

Def 40

We abbreviate

$$A^* = A_1^*$$

We call A^* the dual adjacency matrix
of Γ wrt x and the ordering $\{E_i\}_{i=0}^p$

Recall from Prop 2.2

$$A_i^* = m_i \sum_{j=0}^D u_j(\theta_i) E_j^*$$

$$E_i = |X|^{-1} m_i \sum_{j=0}^D u_j(\theta_i) A_j$$

Abbreviate

$$\theta_i^* = m_i u_i(\theta_i) \quad (0 \leq i \leq D)$$

So

$$A^* = \sum_{i=0}^D \theta_i^* E_i^*$$

$$E_i = |X|^{-1} \sum_{j=0}^D \theta_j^* A_j$$

We will show that $\{\theta_i^*\}_{i=0}^D$ are mutually distinct

We call θ_i^* the i th dual eigenvalue of T

wrt $\{E_i\}_{i=0}^D$

We recursively define some polynomials

$$\{v_i^*\}_{i=0}^{D+1} \text{ in } F[\lambda]$$

$$v_0^* = 1,$$

$$v_1^* = \lambda$$

$$\lambda v_i^* = b_{i-1}^* v_{i-1}^* + a_i^* v_i^* + c_{i+1}^* v_{i+1}^* \quad (1 \leq i \leq D-1)$$

$$\lambda v_D^* = b_{D-1}^* v_{D-1}^* + a_D^* v_D^* + v_{D+1}^*$$

obs for $0 \leq i \leq D$

v_i^* has degree i and leading coef $\frac{1}{c_1^* \cdots c_i^*}$

v_{D+1}^* has degree $D+1$ and leading coef $\frac{1}{c_1^* \cdots c_D^*}$

Ex 4)

We have

$$(i) \quad A_i^* = V_i^* (A^*) \quad 0 \leq i \leq n$$

$$(ii) \quad \text{the minpoly of } A^* \text{ is } c_1^* c_2^* \dots c_n^* V_0^*$$

$$(iii) \quad A^* \text{ generates } M^*$$

$$(iv) \quad \{\theta_i^*\}_{i=0}^n \text{ are mutually distinct}$$

$$(v) \quad V_0^* = \frac{(\lambda - \theta_0^*)(\lambda - \theta_1^*) \dots (\lambda - \theta_n^*)}{c_1^* c_2^* \dots c_n^*}$$

We recursively define some polynomials

$$\{u_i^*\}_{i=0}^{d+1} \text{ in } F[\lambda];$$

$$u_0^* = 1, \quad u_1^* = \frac{\lambda}{m_1}$$

$$\lambda u_i^* = c_i^* u_{i-1}^* + a_i^* u_i^* + b_i^* u_{i+1}^* \quad (1 \leq i \leq d-1)$$

$$\lambda u_d^* = c_d^* u_{d-1}^* + a_d^* u_d^* + u_{d+1}^*$$

obs for $0 \leq i \leq d$

u_i^* has degree i and leading coef $\frac{1}{b_0^* b_1^* \cdots b_{i-1}^*}$

u_{d+1}^* has degree $d+1$ and leading coef $\frac{1}{b_0^* b_1^* \cdots b_d^*}$

→ Ex 42 we have

$$u_i^* = \frac{v_i^*}{m_i} \quad (0 \leq i \leq n)$$

$$u_{DIT}^* = \frac{v_{DIT}^*}{m_D} = \frac{(\lambda - \theta_0^*)(\lambda - \theta_1^*) \dots (\lambda - \theta_{n-1}^*)}{b_0^* b_1^* \dots b_{n-1}^*}$$

$$u_i^*(\theta_0^*) = 1 \quad (0 \leq i \leq n)$$

$$v_i^*(\theta_0^*) = m_i \quad (0 \leq i \leq n)$$