

Math 846

Lecture 5

We continue to discuss a distance-regular graph

$$\Gamma = (X, E) \text{ with diam } D \geq 1$$

We fix $x \in X$ and write $M^* = M^*(x)$

LEM 25 $\exists$ $\mathbb{F}$ -linear bijection

$$p: M \rightarrow M^*$$

that sends

$$A_i \rightarrow E_i^* \quad (0 \leq i \leq D)$$

p.f. Since $\{A_i\}_{i=0}^D$ and $\{E_i^*\}_{i=0}^D$ are bases for M and M^* , resp. □

LEM 26 For $B \in M$

$p(B)$ is the (diagonal) matrix in M^*

with (y, y) -entry B_{xy} (y $\in X$)

p.f. By Lem 25 □

$$B = \begin{pmatrix} \boxed{a \ b \ c \ \dots} \\ * \end{pmatrix}$$

$$p(B) = \begin{pmatrix} \boxed{a \ b \ c \ \dots} & \circ & \\ \circ & & \end{pmatrix}$$

Obs $\forall B, C \in M$

$$\rho(B \circ C) = \rho(B) \rho(C)$$

"via ρ , entrywise product $\circ$ in M becomes matrix mult in M^* "

DEF 27 For $0 \leq i \leq D$ define $A_i^* = A_i^*(x) \in M^*$ by

$$A_i^* = |X| \rho(E_i)$$

We call A_i^* the dual distance matrix of Γ

wrt x .

By constructn

$\{A_i^*\}_{i=0}^D$ form a basis for M^*

Obs

$$\begin{aligned} A_0^* &= |X| \rho(E_0) = |X| \rho(|X|^{-1} J) \\ &= \rho(J) = I \end{aligned}$$

$$\sum_{i=0}^D A_i^* = |X| \sum_{i=0}^D \rho(E_i) = |X| \rho(I) = |X| E_0^*$$

LEM 28 For $0 \leq i, j \leq D$

$$A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^*$$

pf Recall

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h$$

Apply ρ to each side and use previous comments. $\square$

Prop 2.9 For $0 \leq j \leq D$

$$(i) \quad E_j^* = |X|^{-1} \sum_{i=0}^D v_j(\theta_i) A_i^*$$

$$(ii) \quad A_j^* = m_j \sum_{i=0}^D u_i(\theta_j) E_i^*$$

pf Apply the iso $\rho: M \rightarrow M^*$ to the equations in Prop 2.2.

DEF 30 Let $T = T(x)$ denote the subalgebra of $\text{Mat}_x(\mathbb{F})$ generated by M and M^* . We call T the subconstituent algebra of Γ wrt x .

Note that T is noncommutative in general.

Shortly we will find some relations in T .

Note

$$(i) \quad \forall B \in M, \\ B^t \in M \text{ and } \overline{B} \in M$$

$$(ii) \quad \forall B \in M^*, \\ B^t \in M^* \text{ and } \overline{B^*} \in M^*$$

$$(iii) \quad \forall B \in T, \\ B^t \in T \text{ and } \overline{B} \in T$$

1) We introduce a form $\langle , \rangle$ on $\text{Mat}_X(\mathbb{F})$

such that for $B, C \in \text{Mat}_X(\mathbb{F})$

$$\langle B, C \rangle = \text{trace}(B^t \overline{C})$$

$\langle , \rangle$ is Hermitian if $\mathbb{F} = \mathbb{C}$ and symmetric
if $\mathbb{F} = \mathbb{R}$.

For $B \in \text{Mat}_X(\mathbb{F})$ write

$$\|B\|^2 = \langle B, B \rangle$$

obs

$$\|B\|^2 \in \mathbb{R}$$

$$\|B\|^2 \geq 0$$

$$\|B\|^2 = 0 \quad \text{iff} \quad B = 0.$$

(1) LEM 31. For $0 \leq i, j \leq D$

$$(i) \quad \langle A_i, A_j \rangle = \delta_{ij} k_i |X|$$

$$(ii) \quad \langle E_i, E_j \rangle = \delta_{ij} m_i$$

pf (i) $\langle A_i, A_j \rangle = \text{tr}(A_i A_j)$
 $= \text{tr} \sum_{h=0}^D p_{ij}^h A_h$

$$= p_{ij}^0 \text{tr} A_0$$

"I"

$$\delta_{ij} k_i$$

$$= \delta_{ij} k_i |X|$$

$$(ii) \quad \langle E_i, E_j \rangle = \text{tr}(E_i^t \overline{E_j})$$

$$= \text{tr}(E_i E_j)$$

$$= \text{tr} \delta_{ij} E_i$$

$$= \delta_{ij} m_i$$

□

Recall the $\mathbb{F}$ -linear bij $\rho: M \rightarrow M^*$

LEM 32 For $B, C \in M$

$$\langle \rho(B), \rho(C) \rangle = |X|^{-1} \langle B, C \rangle$$

$$\begin{aligned} \text{pf. } \langle \rho(B), \rho(C) \rangle &= \text{tr}(\rho(B)^t \rho(C)) \\ &= \text{tr}(\rho(B) \rho(C)) \\ &= \sum_{y \in X} \rho(B)_{yy} \rho(C)_{yy} \\ &= \sum_{y \in X} B_{yy} \bar{C}_{yy} \\ &= \sum_{y \in X} B_{yy} (\bar{C}^t)_{yy} \\ &= (B \bar{C}^t)_{xx} \\ &= |X|^{-1} \text{tr}(B \bar{C}^t) \end{aligned}$$

since matrices in M have
constant diagonal

$$\begin{aligned} &= |X|^{-1} \text{tr}(\bar{C} B^t) \\ &= |X|^{-1} \text{tr}(B^t \bar{C}) \\ &= |X|^{-1} \langle B, C \rangle \end{aligned}$$

M is commutative

□

LEM 33 For $0 \leq i, j \leq p$

$$(i) \quad \langle A_i^x, A_j^x \rangle = \delta_{ij} m_i |X|$$

$$(ii) \quad \langle E_i^x, E_j^x \rangle = \delta_{ij} k_i$$

pf (i) By Lem 32

$$\langle A_i^x, A_j^x \rangle = \langle |X| p(E_i), |X| p(E_j) \rangle$$

$$= |X| \langle E_i, E_j \rangle$$

$$= \delta_{ij} m_i |X|$$

$$(ii) \quad \langle E_i^x, E_j^x \rangle = \langle p(A_i), p(A_j) \rangle$$

$$= |X|^{-1} \langle A_i, A_j \rangle$$

$$= \delta_{ij} k_i$$

□

Prop 35

(i) The matrices

$$E_h^* A_i E_j^* \quad 0 \leq h, i, j \leq D$$

are mutually orthogonal and

$$\| E_h^* A_i E_j^* \|^2 = k_h p_{ij}^h$$

(ii) The matrices

$$E_h A_i^* E_j \quad 0 \leq h, i, j \leq D$$

are mutually orthogonal and

$$\| E_h A_i^* E_j \|^2 = m_h q_{ij}^h$$

pt (i) For $0 \leq h, i, j, r, s, t \leq D$

$$\langle E_h^* A_i E_j^*, E_r^* A_s E_t^* \rangle$$

$$= \text{tr} \left((E_h^* A_i E_j^*)^t \overline{E_r^* A_s E_t^*} \right)$$

$$= \text{tr} \left(E_j^* A_i \underbrace{E_h^* E_r^*}_{\text{shr } E_r^*} A_s E_t^* \right)$$

$$\text{tr}(BC) = \text{tr}(CB)$$

$$= \text{shr}_r \text{shr}_t \text{tr} (A_i E_h^* A_s E_j^*)$$

$$\text{tr} (A_i E_h^* A_s E_j^*)$$

$$= \sum_{y \in X} (A_i E_h^* A_s E_j^*)_{yy}$$

$$= \sum_{y \in X} \sum_{z \in X} (A_i)_{y,z} (E_h^*)_{z,z} (A_s)_{z,y} (E_j^*)_{yy}$$

$$= \sum_{y, z \in X} 1 = \sum_{i \in I} k_h p_{ij}^h$$

$$\partial(y, z) = i'$$

$$z \in \Gamma_h(x)$$

$$\partial(y, z) = \lambda$$

$$y \in \Gamma_g^{\lambda}(x)$$

pf (ii) For $0 \leq h, i, j, r, s, t \leq p$

$$\langle E_h A_i^* E_j, E_r A_s^* E_t \rangle$$

$$= \text{tr} \left((E_h A_i^* E_j)^t \overline{E_r A_s^* E_t} \right)$$

$$= \text{tr} \left(E_j A_i^* \underbrace{E_h}_{\delta_{hr}} E_r A_s^* E_t \right)$$

$\hookrightarrow$

$$= \delta_{hr} \delta_{jt} \text{tr} (A_i^* E_h A_s^* E_j)$$

$$\text{tr} (A_i^* E_h A_s^* E_j)$$

$$= \sum_{y \in X} (A_i^* E_h A_s^* E_j)_{y,y}$$

$$= \sum_{y, z \in X} \underbrace{(A_i^*)_{y,y}}_{\parallel} (E_h)_{y,z} \underbrace{(A_s^*)_{z,z}}_{\parallel} \underbrace{(E_j)_{z,y}}_{\parallel} \\ \parallel \quad \parallel \quad \parallel \\ |X| (E_i)_{x,y} \quad |X| (E_s)_{x,z} \quad |X| (E_j)_{z,x}$$

$$= |X|^2 \sum_{y, z \in X} (E_i)_{x,y} (E_h \circ E_j)_{y,z} (E_s)_{z,x}$$

$$= |X|^2 \left(E_i (E_h \otimes E_j) E_a \right)_{x,x}$$

$$= |X| \operatorname{tr} \left(E_i \underbrace{(E_h \otimes E_j)}_{\parallel} E_a \right)$$

$$|X|^2 \sum_{l=0}^D g_{hj}^l E_l$$

$$= \operatorname{tr} \sum_{l=0}^D g_{hj}^l \underbrace{E_i E_l E_a}_{\parallel}$$

$\parallel$
0 unless $i=l=a$

$$= \delta_{ia} g_{hj}^i \operatorname{tr} (E_i)$$

$$= \delta_{ia} g_{hj}^i m_i$$

$$= \delta_{ia} m_h g_{ij}^h$$

□