

Math 846

Lecture 41

$\mathbb{F}$ arb

$V = n$ mo f.d. vector space over $\mathbb{F}$

Fix TD system on V :

$$\Phi = (A; \{E_i\}_{i=0}^d; A^{\vee}; \{E_i^{\vee}\}_{i=0}^d)$$

Recall the shape vector $\{p_i\}_{i=0}^d$ of Φ :

$$p_i = \dim(E_i V) = \dim(E_i^{\vee} V) \quad (0 \leq i \leq d)$$

$$p_i = p_{d-i} \quad (0 \leq i \leq d)$$

Φ is called sharp whenever

$$p_0 = 1$$

It turns out that Φ is sharp if $\mathbb{F}$ is algebraically closed.

Next goal: classify Φ up to iso in the sharp case.

Recall the eigenvalue sequence $\{\theta_i\}_{i=0}^d \in \mathbb{F}$
 and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d \in \mathbb{F}$.

So

$$\theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad \text{if } i \neq j \quad (0 \leq i, j \leq d)$$

Also

$$\frac{\theta_{i-2} - \theta_{i-1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i-1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Recall the polynomials

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1})$$

$$\eta_i = (\lambda - \theta_{i1})(\lambda - \theta_{i2}) \cdots (\lambda - \theta_{i-d+i})$$

$0 \leq i \leq d$

$$\tau_i^* = (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*)$$

$$\eta_i^* = (\lambda - \theta_{i1}^*)(\lambda - \theta_{i2}^*) \cdots (\lambda - \theta_{i-d+i}^*)$$

We recall the $\mathbb{F}$ -split decomp of V .

Define

$$U_i = (E_0^* V + E_1^* V + \dots + E_i^* V) \cap (E_i V + E_{i+1} V + \dots + E_d V) \quad (0 \leq i \leq d)$$

So $U_0 = E_0^* V$ and $U_d = E_d V$

We saw

$$V = \sum_{i=0}^d U_i \quad (\text{dir sum})$$

Recall

$$(A - \theta_i I) U_i \subseteq U_{i+1} \quad (0 \leq i \leq d-1)$$

$$(A - \theta_d I) U_d = 0$$

$$(A^* - \theta_i^* I) U_i \subseteq U_{i-1} \quad (1 \leq i \leq d)$$

$$(A^* - \theta_0^* I) U_0 = 0$$

Recall the raising map $R \in \text{End}(V)$

satisfies

$$R = A - \theta_i I \quad \text{on } U_i \quad (0 \leq i \leq d)$$

so $R U_i \subseteq U_{i+1} \quad (0 \leq i \leq d-1), \quad R U_d = 0$

Recall the Lowering map $L \in \text{End}(V)$

satisfies

$$L = A^* - \theta_i^* I \quad \text{on } U_i^* \quad (0 \leq i \leq d)$$

so $L U_i^* \subseteq U_{i-1}^* \quad (1 \leq i \leq d) \quad L U_0^* = 0$

Recall for $0 \leq i \leq d/2$

$$R^{d-2i} : U_i \rightarrow U_{d-i} \quad \text{is a bijection}$$

$$L^{d-2i} : U_{d-i} \rightarrow U_i \quad \text{is a bijection.}$$

In particular

$$R^d : U_0 \rightarrow U_d \quad \text{is a bijection}$$

$$L^d : U_d \rightarrow U_0 \quad \text{is a bijection.}$$

For $0 \leq i \leq d$ the subspace U_0 is invariant under

$$L^i R^i$$

Let z_i denote the corresp eigenvalue. So

$$L^i R^i = z_i I \quad \text{on } U_0$$

So on U_0 ,

$$(A - \theta_1^* I) \cdots (A - \theta_{i-1}^* I) (A - \theta_i^* I) (A - \theta_{i+1}^* I) \cdots (A - \theta_d^* I) = z_i I$$

Applying E_0^* to each side and simplifying, we get

$$E_0^* \gamma_i(A) E_0^* = \frac{z_i E_0^*}{(\theta_0^* - \theta_1^*) (\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}$$

We call the sequence $\{z_i\}_{i=0}^d$ the split sequence

for Φ . Note $z_0 = 1$.

Note also

$$z_d \neq 0$$

Since $R^d: U_0 \rightarrow U_d$ and $L^d: U_d \rightarrow U_0$

are bijections.

We now apply the above results to $\Phi^{\Downarrow}$

$$\Phi^{\Downarrow} = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

We have

$$Z_d^{\Downarrow} \neq 0$$

where

$$E_0^* \gamma_d(A) E_0^* = \frac{Z_d^{\Downarrow} E_0^*}{(e_0^* - e_1^*)(e_0^* - e_2^*) \cdots (e_0^* - e_{d-1}^*)}$$

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We have

$$Z_d^{\Downarrow} = \sum_{i=0}^d \gamma_{d-i}(\theta_0) \gamma_{d-i}^*(\theta_0^*) Z_i$$

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Note that

$$\gamma_d = \sum_{i=0}^d \gamma_{d-i}(\theta_0) \tau_i$$

○ We have

$$\frac{z_d^{\downarrow} E_0^{\downarrow}}{(\theta_0^{\downarrow} - \theta_1^{\downarrow}) \dots (\theta_0^{\downarrow} - \theta_d^{\downarrow})} = E_0^{\downarrow} \gamma_d(A) E_0^{\downarrow}$$

$$= \sum_{i=0}^d \gamma_{d-i}(\theta_0) E_0^{\downarrow} \gamma_i(A) E_0^{\downarrow}$$

$$= \sum_{i=0}^d \frac{\gamma_{d-i}(\theta_0) z_i E_0^{\downarrow}}{(\theta_0^{\downarrow} - \theta_1^{\downarrow}) \dots (\theta_0^{\downarrow} - \theta_i^{\downarrow})}$$

$$= E_0^{\downarrow} \sum_{i=0}^d \frac{\gamma_{d-i}(\theta_0) z_i (\theta_0^{\downarrow} - \theta_{i+1}^{\downarrow}) \dots (\theta_0^{\downarrow} - \theta_d^{\downarrow})}{(\theta_0^{\downarrow} - \theta_1^{\downarrow}) \dots (\theta_0^{\downarrow} - \theta_i^{\downarrow})}$$

$$= E_0^{\downarrow} \sum_{i=0}^d \frac{\gamma_{d-i}(\theta_0) \gamma_{d-i}^{\downarrow}(\theta_0^{\downarrow}) z_i}{(\theta_0^{\downarrow} - \theta_1^{\downarrow}) \dots (\theta_0^{\downarrow} - \theta_i^{\downarrow})}$$

the result follows.

□

Def 76 We call the sequence

$$\left(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{z_i\}_{i=0}^d \right)$$

the parameter array of Φ

We are now ready to state the classification theorem.

THM 77 (Ito + Namura + Ter, 2009)

Assume $\mathbb{F}$ arb and $d \in \mathbb{N}$.

Let

$$\left(\{ \theta_i \}_{i=0}^d, \{ \theta_i^x \}_{i=0}^d, \{ z_i \}_{i=0}^d \right) \quad (*)$$

denote a sequence of scalars in $\mathbb{F}$.

Then there exists a sharp tridiagonal system Φ over $\mathbb{F}$ with parameter array $(*)$ iff

$$(i) \quad \theta_i \neq \theta_j, \quad \theta_i^x \neq \theta_j^x \text{ if } i \neq j \quad (0 \leq i, j \leq d)$$

(ii) the scalars

$$\frac{\theta_{i-2} - \theta_{i+1}^x}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^x - \theta_{i+1}}{\theta_{i-1}^x - \theta_i^x}$$

are equal and indep of i for $2 \leq i \leq d-1$

(iii) $z_0 = 1$ and $z_d \neq 0$ and

$$0 = \sum_{i=0}^d \gamma_{d-i}(\theta_0) \gamma_{d-i}^x(\theta_0^x) z_i$$

Suppose (i) - (iii). Then Φ is unique up to iso of tridiagonal systems.

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the (lengthy) proof of thm 7.7
can be found in

T. Ito, K. Nomura, P. Terwilliger

A classification of sharp tridiagonal pairs.

Linear Algebra Appl. 435 (2011) 1857-1884.